Critical exponents from field theory

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We present a detailed study of the methods of summation based on Borel transformation and conformal mapping, which we have used to calculate critical exponents of the n-vector model through field theoretical methods. In particular we discuss the sensitivity of the results to various changes in the summation procedure.

I. INTRODUCTION

We want, in this article, to explain in some detail how we have calculated, using field theoretical methods, the values of the critical exponents of ferromagnetic systems given in a previous publication[1] and to discuss thoroughly the sensitivity of the results to various changes in the summation procedure.

We shall remind the reader of the theoretical basis of this calculation.

Strong arguments[2,3] have been given which indicate that the long-distance properties of systems (for example ferromagnetic systems, fluids, binary mixtures...) in the neighborhood of a second-order phase transition, can be described by a continuous Euclidean field theory with an action $A(\phi)$

$$A(\phi) = \int d^d x \left[\frac{1}{2} (\partial_\mu \vec{\phi})^2 + \frac{1}{2} m^2 \vec{\phi}^2 + \frac{1}{4!} g (\vec{\phi}^2)^2 \right]$$

$$+ \text{(counter terms)} , \qquad (1)$$

where $\vec{\phi}$ is an n-component vector field and d the dimension of space.

The long-distance properties of this field theory can then be studied through renormalization-group methods. The one-particle irreducible correlation functions $\Gamma^{(L,N)}(q_i; p_j)$ satisfy the renormalization-group equation[4]:

$$\delta(\Sigma q_i + p_j) \Gamma^{(L,N)}(q_i; p_j) = \int e^{i(q_i y_i + p_j x_j)} d\vec{x} d\vec{y} \langle \phi^2(y_1) \cdots \phi^2(y_L) \phi(x_1) \cdots \phi(x_N) \rangle_{1PI}$$

$$(2)$$

$$\left[m \frac{\partial}{\partial m} + W(g) \frac{\partial}{\partial g} - \frac{N}{2} \eta(g) - L \left(\frac{1}{\nu(g)} - 2 \right) \right] \Gamma^{(L,N)}(q_i; p_j) = m^2 [2 - \eta(g)] \Gamma^{(L+1,N)}(0, q_i; p_j) .$$

The power-law behavior of correlation functions at the critical point is governed by the infrared zero g^* of the renormalization-group function $W(g)$ and the values $\eta(g^*)$ and $\nu(g^*)$. The infrared zero g^* of $W(g)$ is defined by

$$W(g^*) = 0 ,$$

$$W'(g^*) > 0 . \qquad (3)$$

The critical exponents then are given by

$$\eta = \eta(g^*), \quad \nu = \nu(g^*) ,$$

$$\gamma = \nu(2 - \eta), \quad \beta = \frac{1}{2}\nu(d - 2 + \eta) , \qquad (4)$$

$$\omega = W'(g^*) .$$

γ and ν are the critical exponents which govern the behavior near the critical temperature T_c of the magnetic susceptibility χ and of the correlation length ξ, respectively, such that

$$\chi \sim |T - T_c|^{-\gamma}, \quad \xi \sim |T - T_c|^{-\nu} .$$

The exponent η gives the large-distance behavior at T_c of the spin-spin correlation function $G(x)$ $\sim x^{2-d-\eta}$, and ω governs the leading corrections to scaling.

A last remark: Notations concerning critical properties will throughout the article follow those of Ref. 3.

The renormalization-group functions $W(g), \eta(g)$, and $\nu(g)$ can be calculated as power series of g.

In the neighborhood of four dimensions, the $W(g)$ has an infrared stable zero close to zero so that perturbative methods are applicable. Setting

$$d = 4 - \epsilon \ , \tag{5}$$

it is possible to calculate the critical exponents as power series in ϵ.[5] Calculations have been done up to order ϵ^4.[6,7] As these series are only at best asymptotic, the accuracy of the final result for $\epsilon = 1, 2$ is limited.

It has been proposed[8] to calculate directly the renormalization-group functions in the dimensions of interest $(d = 2, 3)$, as powers in the coupling constant g. As the series are divergent and the zero g^* is now of order one, nonperturbative methods to sum the series are required.

Recently many terms have been calculated.[9] A realistic calculation then became possible. Guessing correctly that the Kth term in the series would grow for large orders as $K!$, Baker et al.[10] decided to use a Borel transformation, and to sum the series of the Borel transform by the Padé method.

Since then Lipatov[11] has proposed a method to estimate the behavior of the perturbation series at large orders. Application of this method to the $(\vec{\phi}^2)_d^2$ field theory[12] gave the large-order behavior of the various quantities of interest for $d = 1, 2, 3$. The correlation functions are given in terms of functional integrals. Their large-order behavior can be obtained by a steepest-descent calculation of the path integrals, in which the relevant saddle point is a finite action solution (instanton) of the classical Euclidean field equations for g negative

$$(-\Delta + m^2)\vec{\phi}(x) + \frac{1}{3!}g\vec{\phi}(x)[\vec{\phi}(x)]^2 = 0 \ . \tag{6}$$

For any physical quantity $A(g)$ with an expansion

$$A(g) = \sum_K A_K g^K \ , \tag{7}$$

the large-order behavior has the form

$$A_K = cK^{b_0}(-a)^K K![1 + O(1/K)], \quad K \to +\infty \ . \tag{8}$$

The quantities a and b_0 have been calculated in Ref. 12, the different values of the constant c in Ref. 13. This additional information has been used by Baker et al.[9] to improve their summation procedure based on Padé approximants.

On the other hand, new summation methods then became available. It is natural to use a Borel transformation

$$B(g) = \sum_K g^K \frac{A_K}{K!} \ , \tag{9}$$

$$A(g) = \int_0^\infty e^{-t}B(gt)dt \ .$$

The large-order-behavior calculation gives us the

location, the nature, and the residue of singularity of the Borel transform $B(g)$ closest to the origin. Making some additional assumptions on the analyticity properties of $B(g)$, it is possible to sum the series giving $B(g)$. Using the Borel summability[14] of $A(g)$ allows us then to calculate $A(g)$, as will be explained in more detail in Sec. II.

This is the method we have used in Ref. 1 and in the present article. Very recently, it has also been applied to the summation of the ϵ expansion.[7]

The contents of this article are the following: In Sec. II we discuss the problem of the resummation of asymptotic series using the Borel transformation. In Sec. III we explain in more detail how we have used these summation methods in practice. In Sec. IV we have summed the perturbative expansion in two cases in which the exact result is known, a simple integral and the ground-state energy of the anharmonic oscillator. In Sec. V the critical exponents of ϕ_2^4 are calculated. In Sec. VI we present an extensive discussion of the most interesting example, the critical exponents of the n-vector model, in three dimensions. In Sec. VII we discuss results obtained from the ϵ expansion. Section VIII contains our conclusions.

II. RESUMMATION OF ASYMPTOTIC SERIES

A. The problem

Let $A(g)$ be some function analytic in a neighborhood of the origin in an angle $\pi\alpha$, which for convenience we shall assume to be centered on the positive real axis. In this angle it admits an asymptotic expansion

$$\left| A(g) - \sum_{n=0}^K A_n g^n \right| \le C(K+1)|g|^{K+1} \tag{10}$$

for

$$|\arg g| \le \frac{1}{2}(\pi\alpha) \text{ and } C(K) = MC^K(K!)^\rho \ .$$

The asymptotic expansion does not define in general the function $A(g)$ completely. At $|g|$ fixed the best estimate of $A(g)$ obtainable from the series corresponds to the minimum in K of $C(K)|g|^K$. The final ambiguity will correspond to functions analytic in the angle and bounded by $\epsilon(g)$

$$\epsilon(g) = \min_{\{K\}} C(K)|g|^K \sim \exp[-(c|g|)^{-1/\rho}] \ . \tag{11}$$

But there are cases in which no such function exists.

When α is larger than ρ, according to a classical theorem (Phragmen-Lindelöf), there exists no analytic function bounded by $\epsilon(g)$ in the angle. The function $A(g)$ is therefore uniquely defined,[15] and the

only problem is to reconstruct it from the series. This is the case we shall consider from now on.

B. Borel transform (Ref. 15)

When the function $A(g)$ satisfies the condition $\alpha > \rho$, it has a Borel representation:

$$A(g) = \int_0^\infty e^{-t} B(gt^\beta) dt \ , \tag{12}$$

with $\beta < \alpha$.

Expanding in powers of g

$$B(g) = \sum_K B_K g^K, \quad B_K = \frac{A_K}{\Gamma(K\beta + 1)} \ . \tag{13}$$

When $\beta = \rho$, $B(g)$ is analytic in the union of a circle and an angle

$$|\arg g| < \frac{\pi}{2}(\alpha - \beta) \ . \tag{14}$$

If $\beta > \rho$, it is an entire function. From the asymptotic expansion of $A(g)$, one obtains the Taylor series of $B(g)$ at the origin, which defines $B(g)$ uniquely, and therefore also $A(g)$ through the Borel representation.

C. Calculation of the Borel transform

If $\beta = \rho$, the Taylor series defines $B(g)$ only in a circle while we need it on the whole positive axis. We have to make an analytic continuation of the Taylor series.

It would seem, therefore, that the situation $\beta > \rho$ is better because $B(g)$ is now an entire function. But that advantage is fictitious. Only if we knew all the terms of the series and could sum them with infinite accuracy could we calculate $A(g)$. If we can use only a finite number of terms, then after integration we get back the series we started from. Therefore even in this case a resummation procedure is needed.

From now on we shall limit ourselves to the case $\beta = \rho$.

In the absence of any knowledge about the singularities of $B(g)$, one method which can be used to perform the analytic continuation is the Pade approximation.[16]

If we know more about the analytic properties of $B(g)$, it is in general preferable to use a conformal transformation, which maps a part of the domain of analyticity containing the real positive axis onto a circle centered at the origin, whose radius has been normalized to 1 (Ref. 17)

$$g = g(u) \ , \tag{15}$$

with

$$g \sim u, \quad u \to 0 \tag{16}$$

and

$$g \sim [1/(1 - u)^\gamma], \quad u \to 1 \ . \tag{17}$$

From the assumed analyticity properties of $A(g)$, we obtain the condition

$$\gamma \le \alpha - \rho \ . \tag{18}$$

We have transformed the asymptotic expansion in a new series

$$A(g) = \sum_{K=0}^\infty U_K \int_0^\infty e^{-t} [u(gt^\rho)]^K dt \ . \tag{19}$$

The coefficients U_K are the Taylor-series coefficients of a function analytic in a circle of radius 1.

For K large, the integrals are dominated by values of u close to 1. It is possible to evaluate them by steepest descent: The saddle-point equation is

$$-1 + Kgt^{\rho-1} \frac{u'(gt^\rho)}{u(gt^\rho)} = 0 \ . \tag{20}$$

From the behavior of $u(g)$ near $u = 1$, we then get

$$t \sim \frac{K^{\gamma/(\rho+\gamma)}}{g^{1/(\rho+\gamma)}} \tag{21}$$

and therefore

$$\int_0^\infty e^{-t} [u(gt^\rho)]^K dt \sim \exp\left[-C_1 \frac{K^{\gamma/(\rho+\gamma)}}{g^{1/(\rho+\gamma)}} \right] \ , \tag{22}$$

$$K \to \infty \ .$$

A typical situation is one in which the U_K have the same behavior in K so that the natural domain of convergence of such an expansion is

$$\text{Re } g^{-1/(\rho+\gamma)} > C_2 \ . \tag{23}$$

For g small this yields

$$|g| \to 0 \ ,$$
$$|\arg g| \le \frac{1}{2}\pi(\rho + \gamma) \le \frac{1}{2}\pi\alpha \ , \tag{24}$$

which is in agreement with the original assumptions.

We have given here a very qualitative but rather general discussion. We shall concentrate now on the specific examples we are interested in.

D. ϕ_d^4 field theories

For the field theories ϕ_d^4, $d = 0, 1, 2, 3$, it has been shown that the-series are Borel summable.[14]

In addition the large-order behavior of the series has been obtained explicitly and has the form[12,13]

$$A(g) = \sum_K A_K g^K \ ,$$

$$A_K = cK^{b_0}(-a)^K K![1 + O(1/K)], \quad K \to \infty \ , \tag{25}$$

with

$$a > 0 \; .$$

We shall therefore consider the Borel (actually Borel-Leroy) transform $B_b(g)$ of $A(g)$ defined by

$$B_b(g) = \sum_{K=0}^{\infty} \frac{A_K}{\Gamma(K+b+1)} g^K \; , \tag{26}$$

where b is a useful parameter, but which will play no role in the first part of the discussion. Then

$$A(g) = \int_0^{\infty} t^b e^{-t} B_b(gt) \, dt \; . \tag{27}$$

From the large-order behavior we know that $B_b(g)$ is analytic in a circle of radius $1/a$. Furthermore we know that the singularity of $B_b(g)$ closest to the origin is located at the point $-1/a$.

In addition, if, as it is extremely likely, all other finite-distance singularities of $B_b(g)$ come from subleading instanton contributions (at least as long as the theory is super-renormalizable, $d < 4$), then all singularities of $B_b(g)$ lie on the negative real axis.

This result is proven for ϕ_0^4 and ϕ_1^4. We shall assume it for ϕ_2^4 and ϕ_3^4.

Then we can map the cut g plane onto a circle with a mapping leaving the origin invariant

$$u = \frac{(1+ag)^{1/2}-1}{(1+ag)^{1/2}+1} \leftrightarrow g = \frac{4}{a}\frac{u}{(1-u)^2} \; . \tag{28}$$

We obtain now an expansion of $B_b(g)$ convergent in the whole cut plane

$$B_b(g) = \sum_K U_K [u(g)]^K \; . \tag{29}$$

From this we get an expansion for $A(g)$

$$A(g) = \sum_K U_K I_K(g) \; , \tag{30}$$

with

$$I_K(g) = \int_0^{\infty} t^b e^{-t} [u(gt)]^K dt \; . \tag{31}$$

E. Natural domain of convergence

We shall now study the domain of convergence of this expansion. We shall therefore evaluate the behavior of $I_K(g)$, for K large, by the steepest-descent method. We have

$$I_K(g) = \int_0^{\infty} t^b e^{-t} \left[\frac{(1+agt)^{1/2}-1}{(1+agt)^{1/2}+1} \right]^K dt \; . \tag{32}$$

The saddle point is given by

$$\frac{b}{t} - 1 + \frac{K}{t(1+agt)^{1/2}} = 0 \; . \tag{33}$$

For $Kag \gg 1$

$$t \sim (K^2/ag)^{1/3} \tag{34}$$

and therefore

$$I_K(g) \sim K^{2b/3} \exp[-3(K^2/ag)^{1/3}] \; . \tag{35}$$

Note that for K not too large, so that agK is of order 1, $I_K(g)$ decreases exponentially in K.

The convergence of the expansion depends now on the behavior of the coefficients U_K.

If the coefficients U_K are bounded by

$$|U_K| < M_\epsilon \exp(\epsilon K^{2/3}), \quad \forall \epsilon > 0 \; , \tag{36}$$

then the new expansion converges for all g such that

$$\mathrm{Re}\, g^{-1/3} > 0 \; , \tag{37}$$

or

$$-\frac{3}{2}\pi < \arg g < \frac{3}{2}\pi \; . \tag{38}$$

The function $A(g)$ is thus analytic not only in the cut plane but also in a second sheet up to $\frac{3}{2}\pi$.

Here only the simple integral ϕ_0^4 will have this property.

If the coefficients U_K behave like

$$U_K \sim \exp(CK^{2/3}), \quad K \to \infty \; , \tag{39}$$

then the expansion converges in a region

$$\mathrm{Re}\, \frac{1}{(ag)^{1/3}} > \frac{1}{3}C \; . \tag{40}$$

It is not known whether the anharmonic oscillator has this property. When $|g|$ goes to zero, the singularities must become tangent to the axis $\arg g = \frac{1}{2}(3\pi)$. This condition is satisfied by the anharmonic oscillator. But in addition the domain of the singularities must be included in

$$|g| > C'(\frac{3}{2}\pi - \theta)^3 \; .$$

Even less of course is known for ϕ_2^4 and ϕ_3^4.

If g is real positive then two situations can occur

$$g < \frac{1}{a}(\frac{1}{3}C)^{-3} \; , \tag{41}$$

and the expansion converges;

$$g > \frac{1}{a}(\frac{1}{3}C)^{-3} \; , \tag{42}$$

and the expansion does not converge but is a new kind of "asymptotic expansion." Only the numerical analysis of the series can tell us in which situation we are.

As the series in the case of ϕ_2^4 and ϕ_3^4 are rather short, we shall not be able to answer definitely to this question. On the other hand, we shall be helped by the fact that the values of g will be such that

$$agK < 2.5 \; , \tag{43}$$

so that the integrals $I_K(g)$ will decrease faster than in

the asymptotic regime. In addition we shall use the parameter b as a variational parameter to improve the "convergence," as will be explained in Sec. III.

III. APPLICATION

We shall now examine in more detail how we can apply the method we have explained in Sec. II.

A. Choice of the function

The first problem which arises is to decide to which function we should apply the summation method, for instance, $A(g)$, $A^{-1}(g) \cdots$. Although the results should be the same, at finite order differences will appear. As a general rule, it appears that the function for which the series most rapidly reaches its asymptotic regime in K gives the best convergence.

Another question is the following. Let us define a *shift of order K* of the original series by

$$A(g) = A_0 + A_1 g + \cdots + g^K A_{(K)}(g) \ , \qquad (44)$$

where we apply our resummation method to the function $A_{(K)}(g)$.

Of course, if we have only a finite number of coefficients of the series we cannot take K too large. On the other hand, $A(g)$ itself is not necessarily the best choice, as we shall see later. A reason can be found to explain this fact:

The Borel transform of $A_{(K)}(g)$ behaves for g large as $B_b(g)/g^K$. After mapping, the singularity of $B_b(g)$ at $g = \infty$ is transformed in a singularity in u at $u = 1$. It is certainly useful to adjust K so that the singularity of $B_b(g(u))/g^K$ at $u = 1$ becomes as weak as possible.

An alternative way of producing the same effect[18] is to expand, in powers of $u, B_b(g(u))(1-u)^\alpha$. It is easy to verify that for $\alpha = 2$ K, the two methods are identical when the order in the expansion in powers of u is larger than, or equal to 2 K. For orders smaller than 2 K the method with shifting is in general better. On the other hand, the second method allows one to compensate more precisely the possible singularity of $B_b(g(u))$ and to go up to higher values of α for which in the shifting method there would be no terms left.

B. Choice of the parameter b

To explain the role of the parameter b we shall use a more precise form of the large-order behavior of the A_K's

$$A_K = c(-a)^K \Gamma(K + b_0 + 1)[1 + O(1/K)] \ , \quad (45)$$

where a, b_0, c are numbers which have been explicitly calculated in all cases of interest here.

Therefore the coefficients of the Borel transform behave like

$$B_b(g) = \sum_K B_K g^K \ ,$$
$$B_K = c(-a)^K K^{b_0 - b}[1 + O(1/K)] \ . \qquad (46)$$

This shows that the leading singularity of $B_b(g)$ at $g = -1/a$ is of the form

$$B_b(g) = c\Gamma(1 + b_0 - b)(1 + ag)^{b - b_0 - 1}[1 + O(1 + ag)] \ . \qquad (47)$$

Therefore, by varying b, we can modify the singularity of $B_b(g)$ at $g = -1/a$, or of $B_b(g(u))$ at $u = -1$. To weaken the singularity it seems reasonable to take $b \geq b_0 + 1.5$. This qualitative criterion does not fix b very precisely. We have therefore taken it as a variational parameter derived from the following variational principle:

The function $B_b(g(u))$ has an expansion in terms of coefficients $U_K(b)$ which are *smooth* functions of K and b. If we truncate the expansion of B_b at order L, it is reasonable to choose for the "best" b the zero b^* (which can be complex) of smallest module of $U_L(b)$. In this way the coefficients $U_{L+1}(b)$, $U_{L+2}(b) \cdots$ which we do not know will be minimized in general in terms of the information we have, i.e., the coefficients $U_K(b)$ up to order L.

C. Subtraction of the asymptotic behavior

Up to now we have not used the constant c which normalizes the large-order behavior of A_K. One can imagine various ways of doing it. One procedure which we shall use is the following: We shall subtract from the function $A(g)$ a known function $C(g)$ which has the same large-order behavior as $A(g)$. We shall write

$$C(g) = \sum_K c_K g^K \ , \qquad (48)$$

$$A(g) = C(g) + \sum_K (A_K - c_K) g^K \ , \qquad (49)$$

and apply our summation method to the series $(A_K - c_K)$. If we want our method to be reasonably efficient, we must choose a function which already at low orders has coefficients c_K close to A_K. An empirical choice which seems to give good results is

$$C(g) = \int_0^\infty e^{-t} t^b D(gt) dt \ , \qquad (50)$$

with D given (for $b' \geq 1$) by

$$D(g) = c(-)^{b'} \Gamma(b_0 - b + 1)(ag)^{-b'}(1 + ag)^{b - b_0 - 1}$$
$$- \sum_{K = -b'}^{-1} c \frac{\Gamma(b_0 - b + 1 + b' + K)}{\Gamma(K + b' + 1)} (-ag)^K \ , \qquad (51)$$

where b' is an integer equal to b if b is itself an integer, or to $(b+0.5)$ if b is a half integer. Note that if $b'=b$, the function $C(g)$ does not depend on b.

The function $D(g)$

$$D(g) = \sum_{K=0}^{\infty} D_K g^K \ ,$$

$$D_K = c(-a)^K \frac{\Gamma(K+b_0+b'-b+1)}{\Gamma(K+b'+1)} \ , \tag{52}$$

is such that the D_K give the large order of B_K $= A_K/\Gamma(K+b+1)$, as can be seen from the behavior [Eq. (45)] of A_K.

Then one has

$$A(g) = C(g) + \int_0^{\infty} e^{-t} t^b \left[\sum_{k=0}^{\infty} (B_K - D_K)(gt)^K \right] dt \ , \tag{53}$$

where we apply all our methods to the last term up to the known order of B_K.

D. Fit of the coefficients

As a final result we shall give numbers with error bars which show in what range we believe, after these calculations, the result should lie. As we have no mathematical bound on the errors, the error bars are somewhat subjective. As an ultimate check, we have fitted the known series coefficients in various ways using explicitly the large-order behavior. In this way we have obtained predictions for the coefficients we do not know yet. We have incorporated them in our analysis and checked that the limit of this longer series was lying in the range we predicted. The additional information we have made use here is the smoothness of the coefficients A_K in K.

E. Remarks

The systematic use of all these variations of the same method has had two opposite effects. On one hand, it has helped us find the most efficient methods to sum our series and so decrease the uncertainty over the final results. On the other hand, we have found in this way a large number of plausible results, and this has increased the uncertainty over the results, but at the same time it has, we hope, delivered us from obvious biases due to the overly special structure of one of these methods.

Practically the parameter α has been varied up to values as large as 6 or 7, and the range of variation of b was always larger than 10.

F. Pseudo-ϵ expansion (Ref. 19)

When we calculate the critical exponents from ϕ_2^4 and ϕ_3^4, we shall have to solve first one equation

$$W(g^*) = 0 \tag{54}$$

and calculate the others series $\gamma(g), \nu(g) \cdots$ for $g = g^*$. As a result the final errors on the exponents will be the sum of the error of the series of the exponent and of the error coming from g^*. To avoid this one can proceed in the following way.

Let us define

$$W(g, \epsilon') = -\epsilon' g + \sum_{2}^{\infty} W_K g^K \ . \tag{55}$$

Then

$$W(g, 1) = W(g) \ . \tag{56}$$

It is possible now to calculate g^* as a power series in ϵ', and to substitute this series in the exponents. Cumulation of error is therefore avoided, at the price of having series with more complicated structures.

We have then applied systematically all the methods explained above to these ϵ' series for ϕ_2^4 and ϕ_3^4.

IV. NUMERICAL APPLICATIONS TO CASES WHERE THE RESULT IS EXACTLY KNOWN

A. The integral

$$Z(g) = \int_{-\infty}^{+\infty} dx \exp[-(x^2 + gx^4)] \ . \tag{57}$$

The integral $Z(g)$ corresponds to the vacuum-vacuum amplitude of field theory ϕ_0^4, which is a complicated way of saying that the coefficients Z_K of the expansion of $Z(g)$ in powers of g count the number of Feynman diagrams of a ϕ^4 field theory with the proper weighting factors.

The large-order-behavior calculation shows here that

$$Z(g) = \sum_K Z_K g^K \ ,$$

$$Z_K \underset{K \to \infty}{\sim} \frac{1}{(2\pi)^{1/2}} (-4)^K \frac{K!}{K} \ . \tag{58}$$

For half-integer values of the parameter b, the Borel transform $B_b(g)$ of $Z(g)$ has simple expressions in terms of the variable u

$$g = \frac{u}{(1-u)^2} \ , \tag{59}$$

$$B_b(g(u)) = \begin{cases} \dfrac{(1-u)^{1/2}}{1+u}, & b = -\frac{1}{2} \ , \\ 2(1-u)^{1/2}, & b = \frac{1}{2} \ . \end{cases} \tag{60}$$

For $b = -\frac{1}{2}$ the dominant singularity is at $u = -1$, so that for K large the U_K behave like

$$U_K \sim \frac{(-1)^K}{\sqrt{2}} [1 + O(1/K)] \ . \tag{61}$$

Thus our method converges for all values of g satisfying

$$|\arg g| < \frac{3}{2}\pi . \tag{62}$$

For $b = \frac{1}{2}$ the dominant singularity is at $u = 1$, and the U_K behave like

$$U_K \sim \frac{c'}{K^{3/2}} . \tag{63}$$

As b has increased by one unit, the large-K behavior of $I_K(g)$ has increased by a factor $K^{2/3}$, so the expansion converges more rapidly for $b = \frac{1}{2}$ by a factor $K^{-5/6}$.

In general, increasing b by one unit decreases the contribution of the singularity at $u = -1$ by a factor $K^{-1/3}$.

Now to decrease the strength of the singularity at $u = 1$, we can shift $A(g)$

$$A(g) = A(0) + gA_{(1)}(g) . \tag{64}$$

For $b = \frac{3}{2}$, the corresponding Borel transform $B_{(1)}(g)$ is

$$B_{(1)}(g(u)) = 2 \left[\frac{(1-u)^{1/2} - 1}{u} \right] (1-u)^2 . \tag{65}$$

The singularity at $u = 1$ is now weaker, and therefore the U_K decrease faster. The convergence is improved by a factor $1/K^2$.

Now if we take $b = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$, then the first term of the expansion gives the exact result.

The case of the integral is of course ideal, because

the only singularities of the Borel transform $B(g(u))$ are power-law singularities located at $u = \pm 1$. In general other singularities will be present on the circle $|u| = 1$, which we shall not be able to modify, so that the efficiency of this method will be more limited.

The numerical investigations confirm this theoretical analysis.

B. Anharmonic oscillator

We have applied our methods to the ground-state energy $E(g)$ of the anharmonic oscillator (ϕ_1^4 field theory). For this example the method of Borel transformation and mapping has been proposed by Loeffel.[17] As we have explained already, it is not clear if the particular mapping we have chosen will generate a convergent expansion. On the other hand, considering that $B(g)$ is analytic in a cut plane, if the method converges, it is optimal.

We have first calculated with 20 terms of the series and various values of the coupling constant ranging from 0.05 to 50 and compared the results with values obtained from a numerical solution of the Schrödinger equation.

From the large-order-behavior calculations we know that

$$E(g) = \frac{1}{2} + \sum_1^\infty E_K g^K , \tag{66}$$

$$E_K \underset{K \to \infty}{\sim} (-3)^K K^{-1/2} K! [1 + O(1/K)] . \tag{67}$$

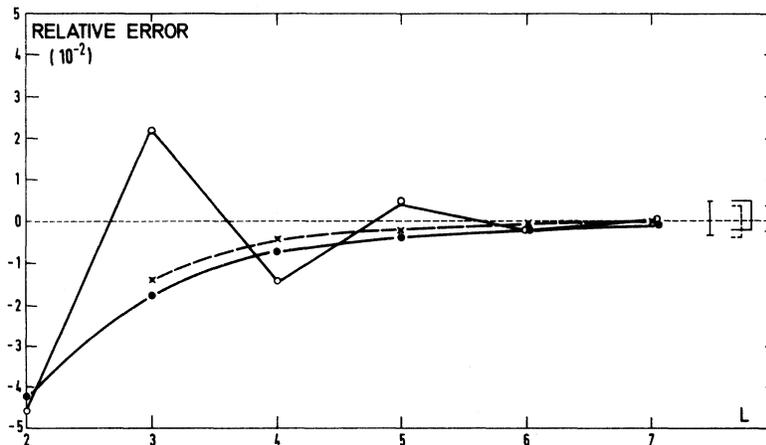In addition the behavior of $E(g)$ for g large is

FIG. 1. Anharmonic oscillator ($g = 0.5$): Relative error ($E/E_{\text{exact}} - 1$) (times 10^{-2}) as a function of the order L of the perturbation series, with no shift. Points ○ represent typical oscillatory behavior ($b = -0.5$), points ● represent typical monotonic behavior ($b = 2.5$), and points × represent values for the "best" b found at each order L by (complex) zero b^* of $U_L(b)$ (see Sec. III B). On the right are shown corresponding typical error bars, respectively, with vertical lines ——, — — —, and — - — - -, for the result at order $L = 6$. The vertical bar |———|, represents the final error bar (for $L = 6$), taking into account all shifts of the series.
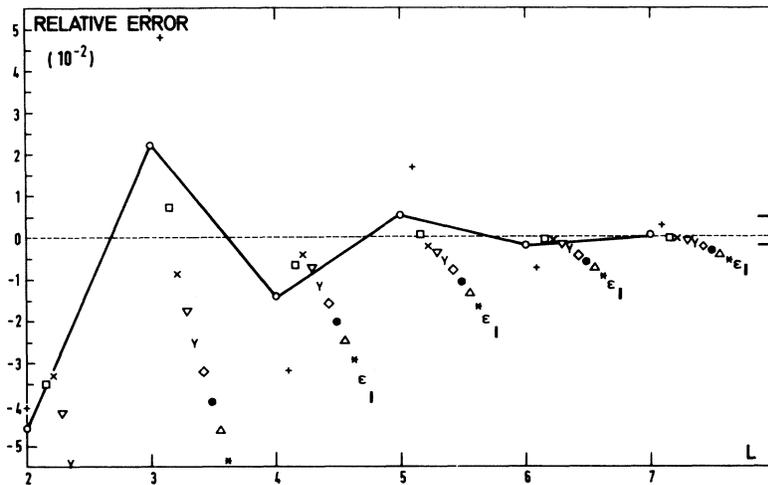
FIG. 2. Anharmonic oscillator ($g = 0.5$): Relative error ($E/E_{\text{exact}} - 1$) (times 10^{-2}) as a function of the order L of the perturbation series, with no shift. At each order L, points from left to right correspond to increasing values of b, starting from $b = -1$ (for point $+$) and increasing by steps of 1. Points O again represent typical oscillatory behavior, with the corresponding error bar for the result at order $L = 6$ on the right.

known from scaling arguments

$$E(g) \underset{g \to +\infty}{\sim} g^{1/3} . \tag{68}$$

As a result we expect the relevant domain in b to be

$$b \geq 1 \tag{69}$$

and the best value of α to be

$$\alpha = \frac{2}{3} . \tag{70}$$

Considering the sensitivity of the results in α, we have calculated only for integer values.

The first obvious result is that, up to order 20 at least, the new series converges towards the exact result.

The second remark is that the rate of convergence

decreases with increasing g as expected.

For $g = 50$, the relative accuracy with 20 terms, and for "good" values of α and b, is of the order of a few percent.

It has been argued in Ref. 20, using 60 terms of the series and a more accurate calculation, that there is some evidence that a convergence of the form

$$\exp - (K/g^2)^{1/3} \tag{71}$$

is impossible. Series like the new series obtained here, would only be also asymptotic but with a very small residual error of the order of $\exp(-200/g)$, instead of $\exp(-1/(3g))$ for the original perturbative expansion. Certainly such a result cannot be checked
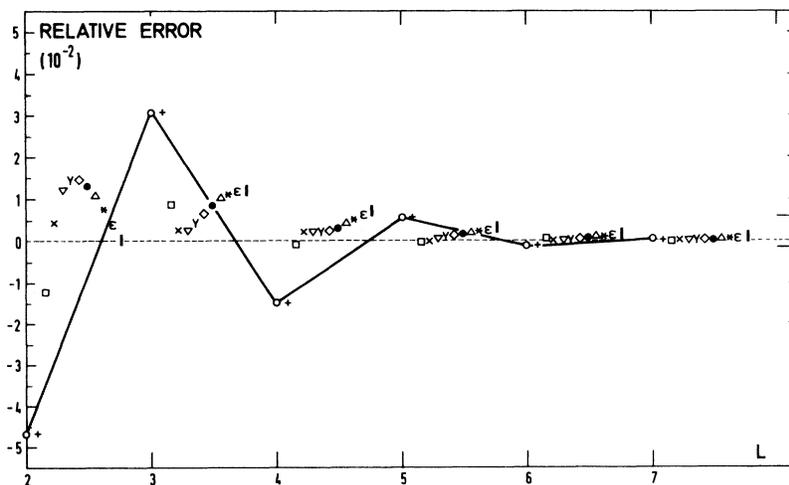

FIG. 3. Same as Fig. 2, but with the parameter α (defined in Sec. III A) equal to 1 (b starts from 0).
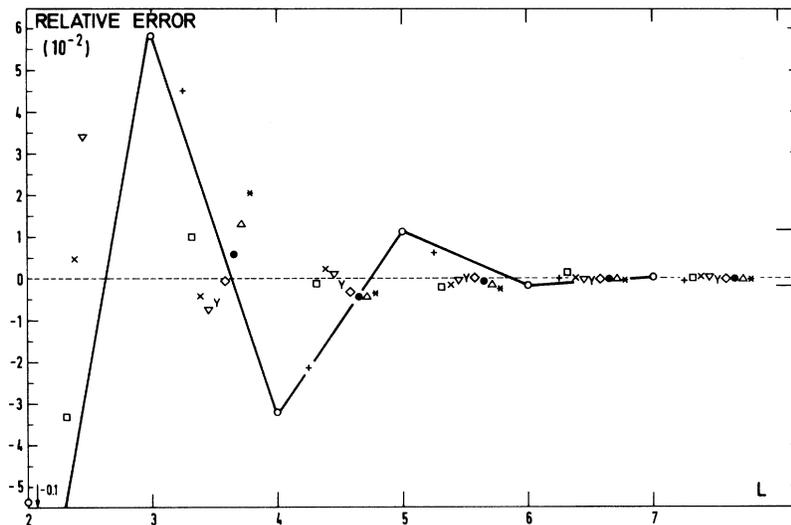
FIG. 4. Same as Fig. 2, but with a shift of order 1 of the series (b starts from 1.5).

here, and for any practical purpose our new series can be considered as convergent.

In a second step we have chosen one value of g for which the rate of convergence seems to be comparable to the rate observed in ϕ_2^4 and ϕ_3^4

$$g = 0.5 \qquad (72)$$

and we have systematically varied b, shifted the series, and varied α.

As noted already in Ref. 18, the best convergence is obtained for α close to the theoretical value $\frac{2}{3}$, in our case $\alpha = 1$. One clearly sees this in Figs. 2 and 5, where results of explorations in b at each order of the series are shown for different shifts and α's. As the exact result does not depend on the parameter b, one expects the best convergence to be associated with

the weakest variation in b; this happens for the value $\alpha = 1$.

Now for α fixed the best values of b are generally found between 1 and 2.5 although occasionally they can be as high as 3 or 4.

In a third step we have used these series to find empirical criteria to estimate the error made in truncating the series. We have looked for criteria such that the error would be overestimated and this for a large range of α and b. In general, as b increases the obtained sequences, as function of the corresponding order L of the perturbation series, show first oscillations and then, after a transient set of b's, a monotonic behavior for higher b (see Fig. 1).

It appears that typical reasonable and conservative criteria for the error on the result S_L at order L are
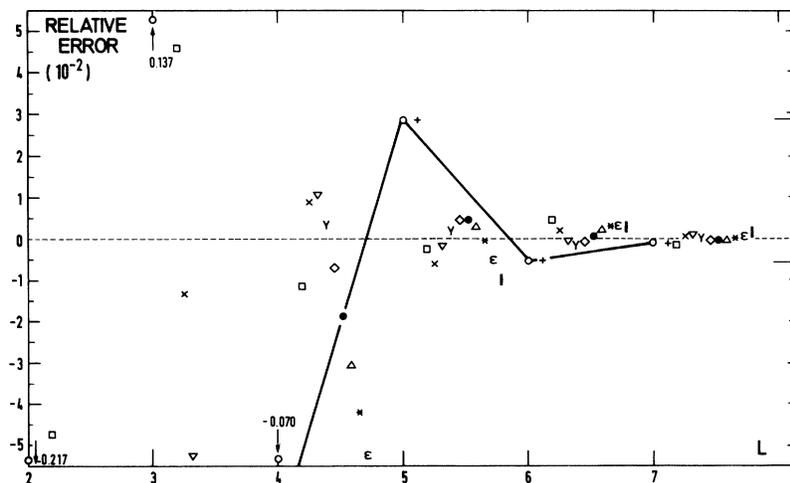

FIG. 5. Same as Fig. 2, but with the parameter α equal to 3 (b starts from 0.5).

from the study of anharmonic oscillator:

(i) Take the last truly (as b increases) oscillatory curve, and then take as a possible error bar the separation between S_L and S_{L-1}.

(ii) Take, after the transient set of curves as b increases, the first truly monotonic curve, S_{L-1} then giving an estimation of the upper (or lower) limit of the error bar.

Padé approximants made on the S_l (l up to L) give results consistent with such criteria. Furthermore, these criteria are applied for each α (or shift) and for both methods with or without use of the constant c of the asymptotic behavior. Then a reasonable union of such error bars around the optimal α is finally taken as the error bar on the result.

This is illustrated in Figs. 1 to 5 for $L = 6$ in view of the application to $(\vec{\phi}^2)^3_1$. We have also plotted in Fig. 1 the sequence obtained from the "best" b found at each order by (complex) zero b^* of $U_L(b)$ (see Sec. III B).

The criteria given above finally yield for the anharmonic oscillator with $g = 0.5$, at order 6 of the perturbation series:

$$\frac{E}{E_{\text{exact}}} - 1 = (1 \pm 4) \times 10^{-3} \ . \tag{73}$$

V. CRITICAL EXPONENTS IN THE ϕ^4_2 FIELD THEORY

We shall now study our first real field theoretical example. It is commonly believed[2,3] that the ϕ^4_2 field theory from the point of view of critical properties belongs to the same universality class as the two-dimensional Ising model, so that universal quantities like the critical exponents $\gamma, \nu \cdots$ should be the same in both models. We can therefore compare our results with the values exactly known of the Ising model

$$\gamma = 1.75 \ , \tag{74}$$

$$\nu = 1.0 \ , \tag{75}$$

and therefore

$$\eta = 2 - \gamma/\nu = 0.25 \ . \tag{76}$$

On the other hand, the critical coupling constant is

TABLE I. Estimates of critical exponents in the ϕ^4 field theory in $d = 2$.

$g^* =$	1.85 ± 0.10
$\omega =$	1.3 ± 0.2
$\gamma =$	1.79 ± 0.09
$\nu =$	0.97 ± 0.08
$\eta =$	0.13 ± 0.07
$\eta^{(2)} =$	-0.85 ± 0.07

only known from high-temperature series expansions.[21]

$$g^* = 1.751 \pm 0.005 \ . \tag{77}$$

The exponent ω which characterizes the leading correction to the scaling laws has not yet been identified. A possible candidate is 1.

Unfortunately, for technical reasons, the series in g are rather short: $\eta(g)$, $\nu(g)$, $\gamma(g)$, $\omega(g)$ are only known up to order g^4 and $W(g)$ up to order g^5. In addition, as one can expect from the ϵ expansion, the critical coupling is larger, so that the convergence is slow compared for instance to $d = 3$. As a result, this very interesting comparison is not completely conclusive. Our estimates are given in Table I. The constants appearing here in the large-order behavior (8) are given in Table II. The critical exponents γ

TABLE II. Constants (Refs. 12 and 13) appearing in the large-order behavior Eq. (8).

(i) For the n-vector model $(\vec{\phi}^2)^2$ in three dimensions

$$a = 0.147\,774\,22\,\frac{9}{(n+8)}$$

$$b_0 = \begin{cases} 5 + \frac{1}{2}n & \text{for } \omega(g) \\ 2 + \frac{1}{2}n & \text{for } \eta(g) \\ 3 + \frac{1}{2}n & \text{for } W(g) \text{ and the other exponents } . \end{cases}$$

c given by

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
for $W(g)$:	0.085 489	0.039 962	0.016 302	0.005 961
for $\omega(g)$:	$-0.014\,212$	$-0.005\,905$	$-0.002\,168$	$-0.007\,207$
for $\eta(g)$:	0.002 884	0.001 797	0.000 880	0.000 366
for $\eta^{(2)}(g)$:	0.010 107	0.006 299	0.003 084	0.001 281

and with $\quad c_{\nu-1} = c_{\eta^{(2)}}; \ c_\nu = -\frac{1}{4}c_{\eta^{(2)}}$

$$c_{\gamma-1} = \frac{1}{2}c_{\eta^{(2)}}; \ c_{\beta-1} = 2c_{\eta^{(2)}}$$

(ii) For ϕ^4_2 field theory

$$a = 0.238\,659\,217$$

$$b_0 = \begin{cases} 5 & \text{for } \omega(g) \\ 2 & \text{for } \eta(g) \\ 3 & \text{for } W(g) \text{ and the other exponents } . \end{cases}$$

$$c = \begin{cases} 0.048\,86 & \text{for } W(g) \\ -0.023\,32 & \text{for } \omega(g) \\ 0.003\,468 & \text{for } \eta(g) \\ 0.010\,49 & \text{for } \eta^{(2)}(g) \text{ and } \nu^{-1}(g) \\ 0.005\,245 & \text{for } \gamma^{-1}(g) \end{cases}$$

and ν seem to be compatible with Ising-like exponents. The exponent η is much too small but the series for $\eta(g)$ has only three terms, so that it is extremely difficult to assign an error to the obtained result. The most serious problem comes from the quantity $\eta^{(2)}$ which is given by

$$\eta^{(2)} = \frac{1}{\nu} + \eta - 2 \ . \tag{78}$$

For the Ising model

$$\eta^{(2)} = -0.75 \ . \tag{79}$$

The series, on the other hand, suggests rather

$$\eta^{(2)} = -0.85 \pm 0.07 \ , \tag{80}$$

which is somewhat far from the Ising value.

The estimates for the critical coupling constant g^* are also higher than the value given by the high-temperature series, but the errors are so large that this difference is probably not significant.

From this short analysis, two facts emerge: (i) The results are in qualitative agreement with Ising-like values. (ii) On the other hand, we observe small deviations from the Ising-model values which are somewhat high compared to the expected uncertainties over the series estimate.

In such a confusing situation, it would be extreme-ly interesting if one could calculate one or two terms more of the series even with an accuracy not better than 10^{-3}.

VI. THREE-DIMENSIONAL n-VECTOR MODEL $(\vec{\phi}^2)^2_3$

We shall discuss in the first part of this section the case $n = 1$, which corresponds to Ising-like systems, because in this case the comparison with high-temperature (HT) series is the most meaningful.

With the normalization used in Ref. 9, the constants appearing here in the large-order behavior (8) are given in Table II.

A. Detailed discussion of the case $n = 1$

We have again calculated first the zero g^* of $W(g)$, and then the various critical exponents separately γ, ν, β, $\eta^{(2)}$, η, ω, the relations between these exponents being a check of the method. We have also calculated these exponents directly through the pseudo-ϵ expansion.

We give Figs. 6 to 18 as examples of only some of the numerous results obtained from the various methods discussed previously.

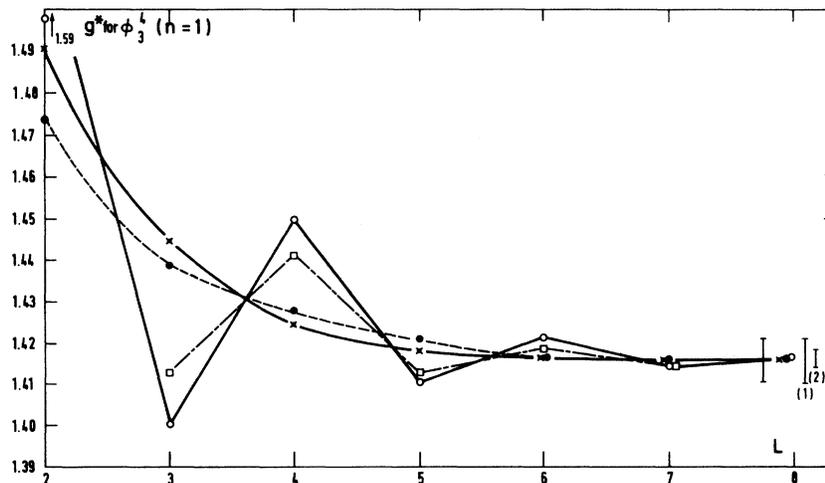FIG. 6. Some typical convergences for the value of g^*, zero of $W(g)$, for ϕ^4 theory with $d = 3$ and $n = 1$, as a function of the order L of the perturbation series of $W(g)/g$. The two last orders ($L = 7$ and 8) correspond to a fit of the known terms (up to $L = 6$) of the series (see Sec. III D). Oscillatory behavior is labeled by O, normal method, shift of order 1 ($b = 2.5$) and □, from pseudo-ϵ expansion, for $\alpha = 9$, "best" b at each order by (complex) b^*. Monotonic behavior is labeled by ●, normal method, shift of order 2 ($b = 8.5$) and ×, method using the asymptotic constant c, shift of order 0 ($b = -1$). Vertical bar $\vdash\!\!\!\dashv$ represents the final estimate (81). Vertical bars (1) and (2) represent domains of variation of Padé approximants applied, for $L = 6$ and fixed b, on the series of the results at each order, for a domain of variation of b equal to 5 around the "best" b at order 6: (i) for the normal method and (ii) for the method using the asymptotic constant c.
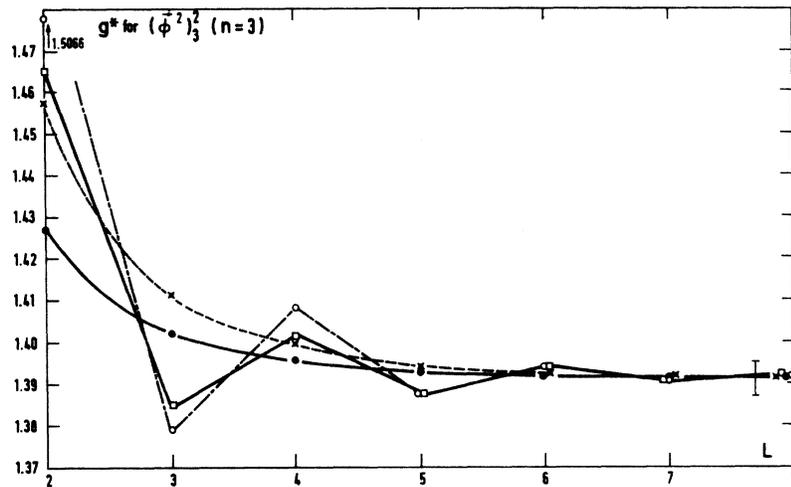
FIG. 7. Some typical convergences for the value of g^*, zero of $|W(g)$, for $(\vec{\phi}^2)^2$ theory with $d = 3$ and $n = 3$, as a function of the order L of the perturbation series of $W(g)/g$. The two last orders ($L = 7$ and 8) correspond to a fit of the known terms (up to $L = 6$) of the series. Oscillatory behavior is labeled by: O, normal method, shift of order 1 ($b = 3.5$) and □, method using the asymptotic constant c, shift of order 1 ($b = 1.5$). Monotic behavior is labeled by: ●, normal method, shift of order 2 ($b = 9.5$) and ×, method using the asymptotic constant c, shift of order 1 ($b = 6$). Vertical bar |———| represents the final estimate.

1. Coupling constant g^*

Let us first discuss the determination of g^*, some typical behavior being given in Fig. 6. As for the anharmonic oscillator (and for ϕ_2^4 field theory), we have here both oscillatory and monotonic sequences as function of the order L of the perturbation series.

The criteria used to estimate the result have been explained in Sec. IV B.

Notice that the structure of the sequences of the "best" b's (see Sec. III B) is not always smooth and regular. In fact, in determining these best b's, we have used all the information on the series. We have then to order 6 of $W(g)/g$ an optimal estimate of g^*;
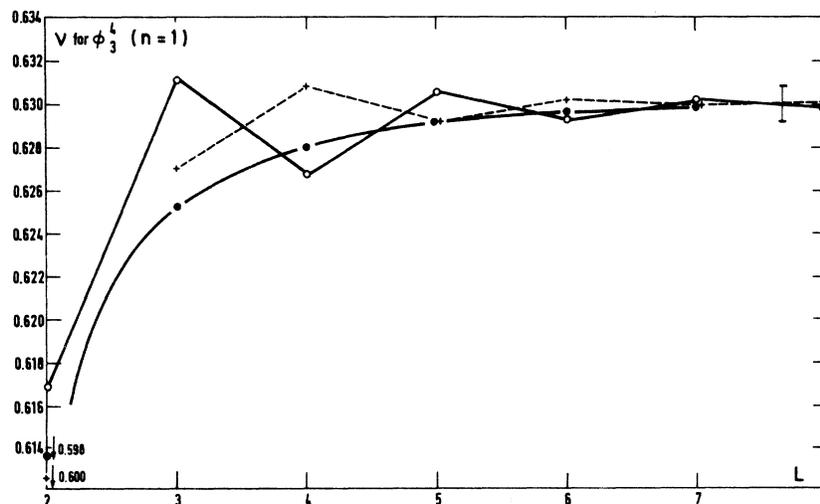

FIG. 8. Some typical convergences for the critical exponent ν, for ϕ_3^4 theory with $n = 1$, as a function of the order L of the perturbation series of $\nu(g^*)$, here with $g^* = 1.416$. The two last orders ($L = 7$ and 8) correspond to a fit of the known terms (up to $L = 6$) of the series. Points O are from normal method, shift of order 1, $b = 2$. Points ● and + are from method using the asymptotic constant c, respectively, with $\alpha = 3$ and $\alpha = 4$, $b = 7$ and $b = 2$. Vertical bar |———| represents the final estimate (83) for $\nu(1.416)$.
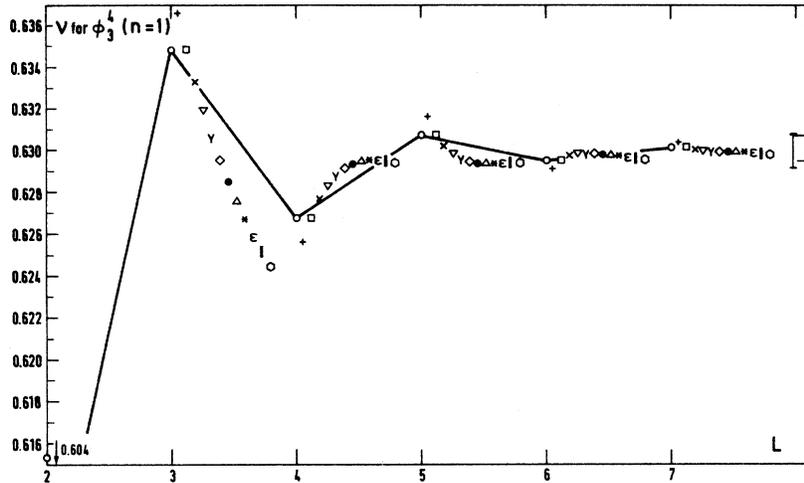
FIG. 9. Critical exponent $\nu = \nu(g^*)$ with $g^* = 1.416$ for ϕ_3^4 theory with $n = 1$, as function of the order L of the perturbation series: Normal method with shift of order 2. At each order L, points from left to right correspond to increasing values of b, starting from $b = 4$ (for point +) and increasing by steps of 1. Points O represent typical oscillatory behavior, with the corresponding error bar for the result at order $L = 6$ on the right. The last order ($L = 7$) corresponds to a fit of the known terms (up to $L = 6$) of the series. Vertical bar $\vdash$——$\dashv$ represents the final estimate (83) for $\nu(1.416)$.

however, such sequences do not help very much in general to estimate the error bar, which is better given by the criteria of oscillatory behavior obtained from other b's.

Use of the constant c multiplying the large-order term gives here better monotonic behavior than for the normal method without c. However, this is not general and depends on the specific series.

Variation of the parameter α (and shifts of the series) shows an optimal convergence around $\alpha = 2$

(shift of order 1) for g^*.

Figure 6 also shows values of g^* obtained if we use a fit of the known coefficients to predict orders 7 and 8 of the series for $W(g)/g$, as explained in Sec. III D. We then obtain two results which confirm the criteria we use to estimate the error: (a) The oscillatory (or monotonic) behavior selected up to order 6 remains the same at higher orders, and (b) there is a rapid convergence inside the error bar estimated at order 6.
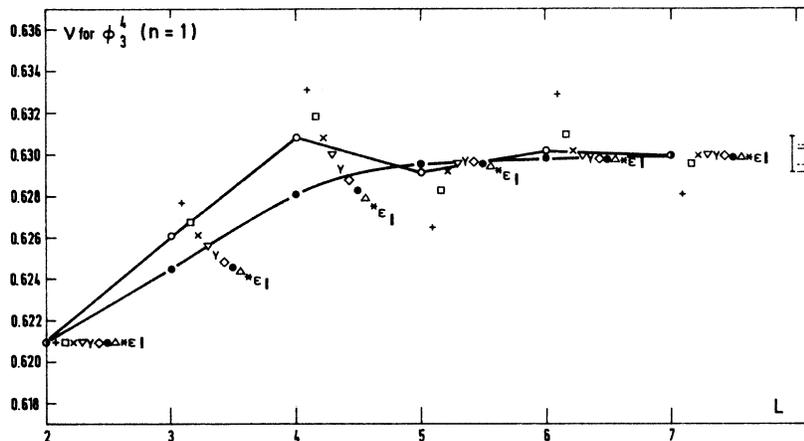

FIG. 10. Same as Fig. 9, but by the method of using the asymptotic constant c, with shift of order 2. Parameter b increases from 0 by steps of 2. Points O and ● represent typical oscillatory and monotonic behavior, with the corresponding error bars for the result at order $L = 6$ on the right.
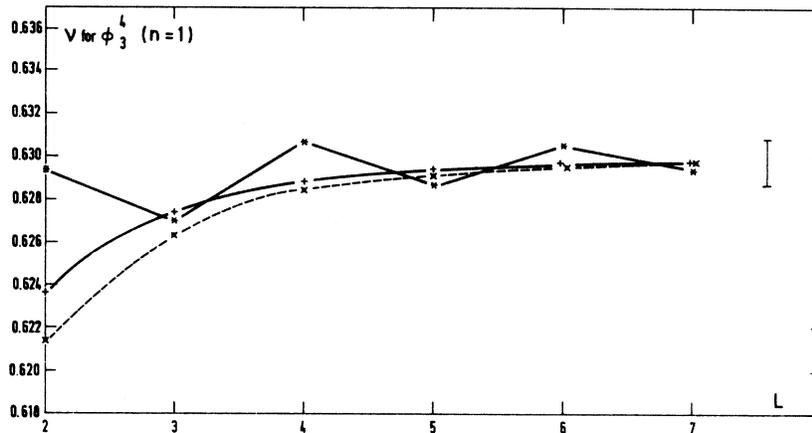
FIG. 11. Same as Fig. 8, but for pseudo-ϵ- expansion. Points $\times$ and $*$ are from normal method, with shift of order, respectively, 1 and 2, $b = 3.5$ and -1.5. Points $+$ are from the method using the asymptotic constant c, with shift of order 1, $b = 1$. Vertical bar $\vdash\!\!\!\dashv$ represents the final estimate (87).

In this way, we have found that even a calculation which would not be very accurate of the eighth order of the series for $W(g)$ would improve the determination of g^*.

Another check of our results is obtained, as shown in Fig. 6, by extrapolation of the successive values of g^* at each order with use of Padé approximants, varying the parameter α and b in a large range around their optimal values.

We have also used as a check the pseudo-ϵ expansion of Sec. III F. For larger α (of order 9) there are here oscillations for the sequences of best b's found at each order by (complex) zero b^* of $U_L(b)$, an example being given in Fig. 6.

Only a few examples are given in Fig. 6. After a global study of the results obtained by all the methods we used, we finally obtained the estimate for $g^*(n = 1)$

$$g^* = 1.416 \pm 0.005 \ . \tag{81}$$

2. Critical exponents γ and ν

We give in Figs. 8–10 for ν and 13–15 for γ, some examples obtained by calculating ν and γ as $\nu(g^*)$ and $\gamma(g^*)$ for $g^* = 1.416$. We see essentially
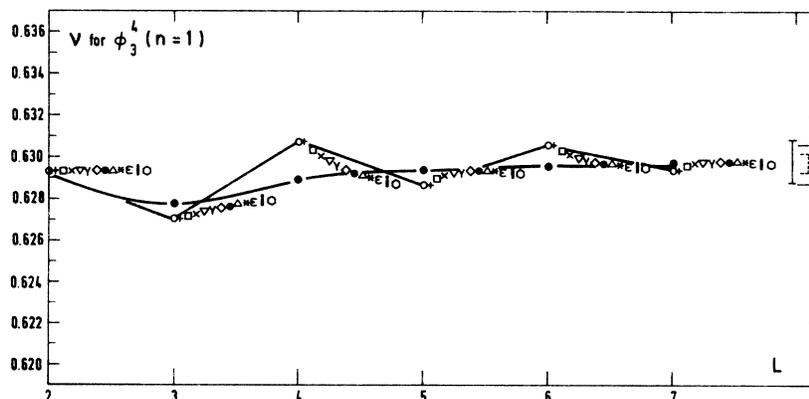

FIG. 12. Same as Figs. 9 and 10, but for pseudo-ϵ expansion: normal method with shift of order 2. Parameter b increases from -1.5 by steps of 1.
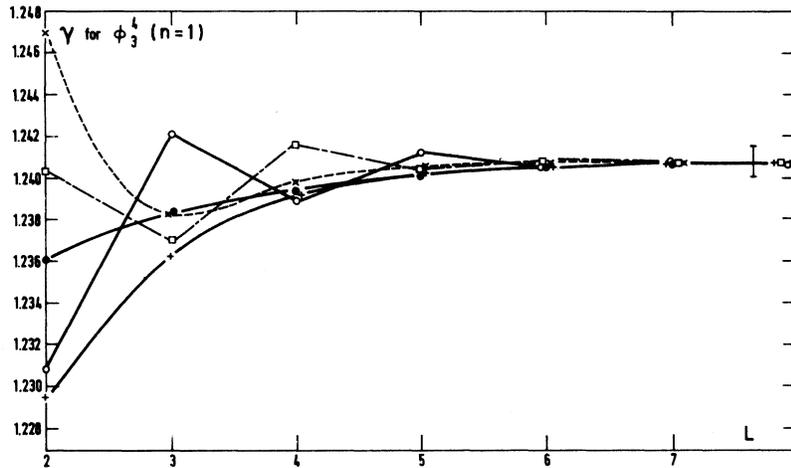
FIG. 13. Some typical convergences for the critical exponent γ, for ϕ_3^4 theory with $n = 1$, as function of the order L of the perturbation series of $\gamma(g^*)$, here with $g^* = 1.416$. The two last orders ($L = 7$ and 8) correspond to a fit of the known terms (up to $L = 6$) of the series. Points O and ● are from normal method, respectively, for a shift of order 1 ($b = 4$) and for $\alpha = 1$ ($b = 6.5$). Points □, ×, and + are from the method using the asymptotic constant c, respectively, for a shift of order 1 ($b = 2.5$ and 6.0) and with no shift ($b = 3$). Vertical bar |———| represents the final estimate (82) for $\gamma(1.416)$.

again the same structures as discussed above.

As mentioned for the anharmonic oscillator, we also use in general as a criterion the variation of the results as a function of b at each order of the series. As the exact result does not depend on the parameter b, the best convergence is in general associated with the weakest variation in b. Figures 9, 10, 14, and 15

give examples of typical variations in b (note that b varies by steps of either 1 or 2). We obtain an optimal convergence with a shift of order 1 for γ and 2 for ν.

Let us remark, as it is clear for instance from Figs. 9 and 10, that we often have, for the obtained sequences as function of the order, oscillations of op-
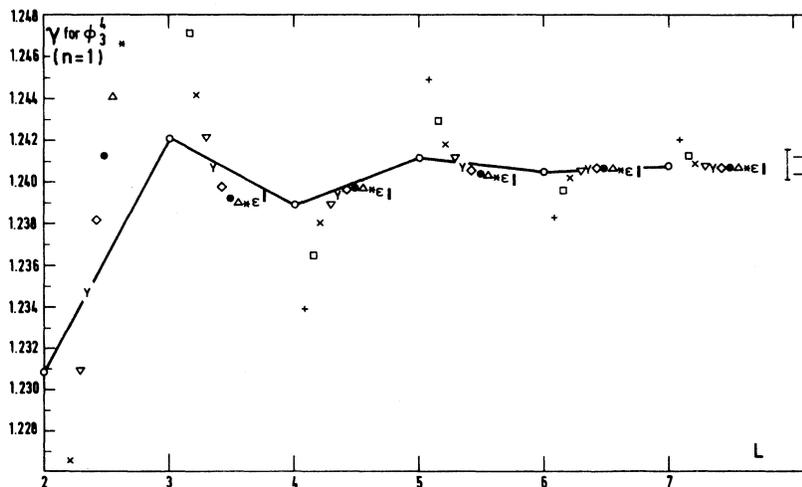

FIG. 14. Critical exponent $\gamma = \gamma(g^*)$ with $g^* = 1.416$ for ϕ_3^4 theory with $n = 1$, as function of the order L of the perturbation series: normal method with shift of order 1. At each order L, points from left to right correspond to increasing values of b, starting from $b = 1$ (for point +) and increasing by steps of 1. Points O represent typical oscillatory behavior, with the corresponding error bar for the result at order $L = 6$ on the right. The last order ($L = 7$) again corresponds to a fit of the known terms (up to $L = 6$) of the series. Vertical bar |———| represents the final estimate (82) for $\gamma(1.416)$.
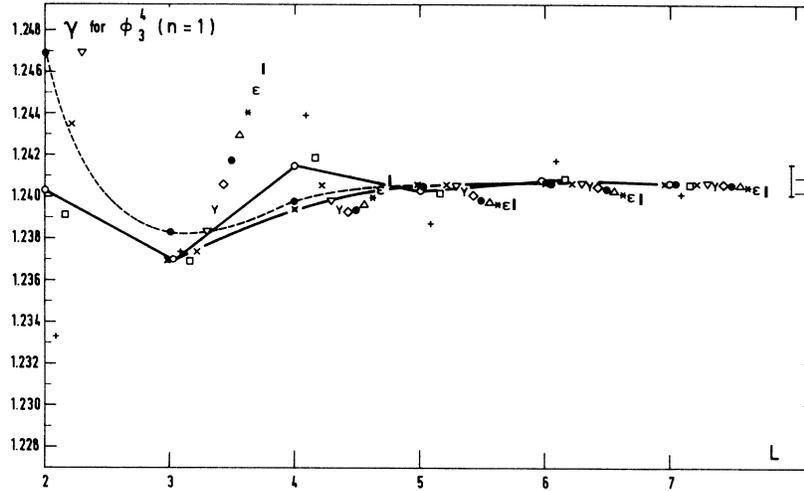
FIG. 15. Same as Fig. 14, but by the method using the asymptotic constant c, with shift of order 1. Parameter b increases from 0 by steps of 2. Points ✖ correspond to the "best" b obtained at each order by (complex) zero b^*.

posite phases for the normal method and for the method using the constant c of the large order. This is useful to estimate the higher and lower limit of the final error bar on the result.

Let us also mention that we have obtained here better results for the series of $\nu^{-1}(g)$ and $\gamma^{-1}(g)$ than for the direct series. As expected, the series for which all terms oscillate yields the best convergence. Finally it is clear in the figures shown that what we have said above for the use of "predicted" orders

from the fit of the known coefficients is also true here.

After a global study of the results obtained by all our methods, we finally have the following estimates:

$$\gamma(1.416) = 1.2408 \pm 0.0008 \ , \tag{82}$$

$$\nu(1.416) = 0.6300 \pm 0.0008 \ . \tag{83}$$

Taking into account the uncertainty on the value of g^* around 1.416 given in Eq. (81), we then obtain for
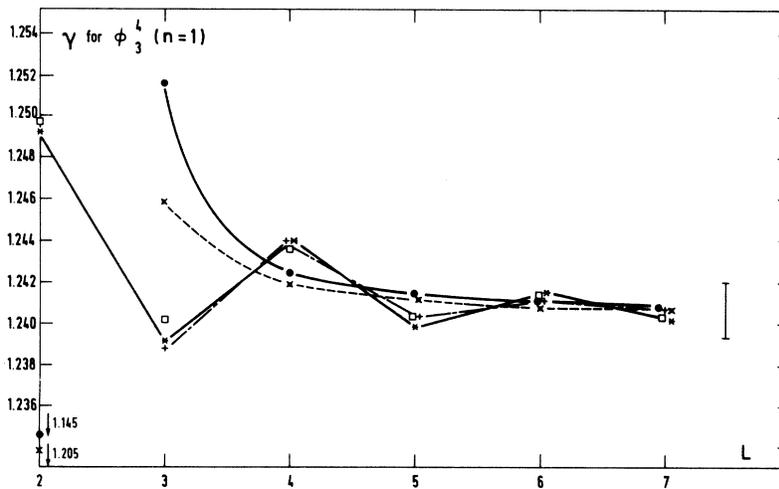
FIG. 16. Same as Fig. 13, but for pseudo-ϵ expansion. Points ✕ are from the method using the asymptotic constant c with $\alpha = 6$ ($b = 1$). Other points are from the normal method: □ and ● for $\alpha = 5$ ($b = -0.5$ and 3.5), * for $\alpha = 6$ ($b = -0.5$), and + for $\alpha = 7$ corresponding to the "best" b obtained at each order by (complex) zero b^*. Vertical bar |———| represents the final estimate (86).
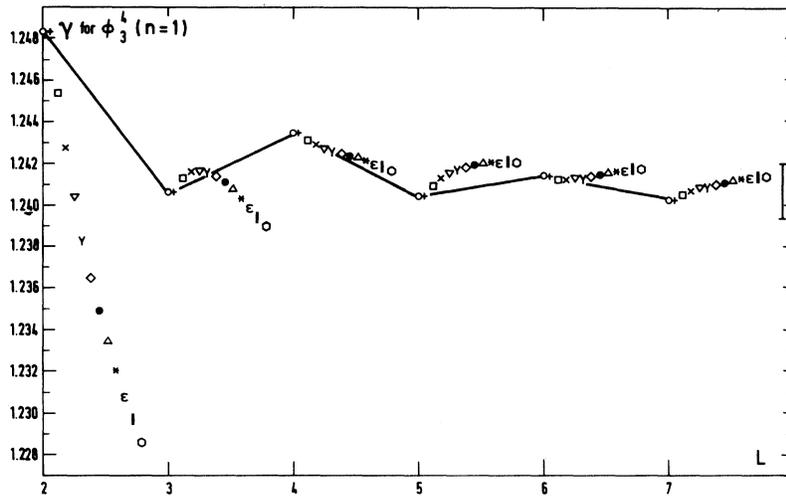
FIG. 17. Same as Figs. 14 and 15, but for pseudo-ϵ expansion: normal method with shift of order 1. Parameter b increases from -1.5 by steps of 1.

the critical exponents

$$\gamma = 1.241 \pm 0.0020 \ , \tag{84}$$

$$\nu = 0.630 \pm 0.0015 \ . \tag{85}$$

As explained above, we have checked these results using the pseudo-ϵ expansion of Sec. III F. Some examples are given in Figs. 11, 12 for ν and 16–18 for γ. We obtained sequences of slightly different structure. The oscillations are somewhat larger, and for γ the monotonic curves converge from above. Also,

the optimal convergence is for higher values of α (around 5). The behavior of the "best" b's obtained by (complex) zero b^* is monotonically decreasing for $\alpha = 5$ (Fig. 18; points $\times$) and becomes oscillatory for $\alpha = 7$ (Fig. 16; points $+$).

The results obtained from this pseudo-ϵ expansion

$$\gamma = 1.2407 \pm 0.0013 \ , \tag{86}$$

$$\nu = 0.6298 \pm 0.0010 \ , \tag{87}$$

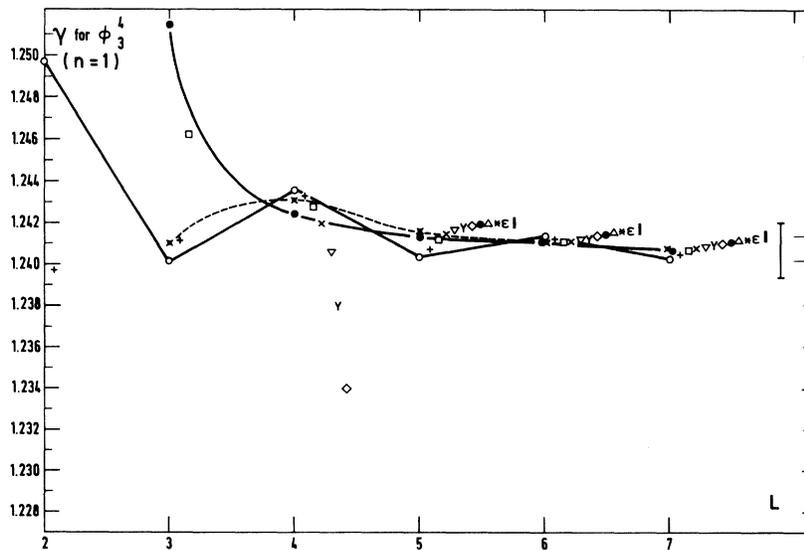are therefore in good agreement with the method us-

FIG. 18. Same as Fig. 17, pseudo-ϵ expansion, but for the normal method with $\alpha = 5$. Parameter b increases from 0.5 by steps of 2. Points $\times$ correspond to the "best" b obtained at each order by (complex) zero b^*.

TABLE III. Estimates of critical exponents in the three-dimensional n-vector model $(\vec{\phi}^2)_3^n$.

	$n=0$	$n=1$	$n=2$	$n=3$
g^*	1.421 ±0.008	1.416 ±0.005	1.406 ±0.004	1.391 ±0.004
ω	0.80 ±0.04	0.79 ±0.03	0.78 ±0.025	0.78 ±0.02
γ	1.1615 ±0.0020	1.241 ±0.0020	1.316 ±0.0025	1.386 ±0.0040
ν	0.588 ±0.0015	0.630 ±0.0015	0.669 ±0.0020	0.705 ±0.0030
η	0.027 ±0.004	0.031 ±0.004	0.033 ±0.004	0.033 ±0.004
β	0.302 ±0.0015	0.325 ±0.0015	0.3455 ±0.0020	0.3645 ±0.0025
$\eta^{(2)}$	−0.2745 ±0.0035	−0.3825 ±0.0030	−0.474 ±0.0025	−0.549 ±0.0035
α	0.236 ±0.0045	0.110 ±0.0045	−0.007 ±0.006	−0.115 ±0.009
$\Delta_1, \omega\nu$	0.470 ±0.025	0.498 ±0.020	0.522 ±0.018	0.550 ±0.016

ing g^*. In the pseudo-ϵ expansion, we avoid the uncertainty on g^*, but the structure of the series is more complicated.

B. General results

Everything we have said for g^*, γ, and ν for $n=1$ applies also for other critical exponents and other values of n for the n-vector model, and we only give here the final results. The best estimates are shown in Table III. Note that, as we mention in Sec. III E,

TABLE IV. Verification of scaling relations between critical exponents for the three-dimensional n-vector model $(\vec{\phi}^2)_3^n$: First column (A) gives the directly computed η, β, and $\eta^{(2)}$; second column (B) gives the same exponents calculated by the relations $\eta=2-\gamma/\nu$, $\beta=\frac{1}{2}(3\nu-\gamma)$ and $\eta^{(2)}=(1-\gamma)/\nu$, using the estimates of Table III for the directly computed γ and ν.

	A	B
$n=0$		
η	0.027 ±0.004	0.025 ±0.008
β	0.302 ±0.0015	0.3015 ±0.0035
$\eta^{(2)}$	−0.2745 ±0.0035	−0.2745 ±0.004
$n=1$		
η	0.031 ±0.004	0.030 ±0.008
β	0.325 ±0.0015	0.3245 ±0.0035
$\eta^{(2)}$	−0.3825 ±0.003	−0.3825 ±0.004
$n=2$		
η	0.033 ±0.004	0.033 ±0.010
β	0.3455 ±0.002	0.3455 ±0.0045
$\eta^{(2)}$	−0.474 ±0.0025	−0.4725 ±0.005
$n=3$		
η	0.033 ±0.004	0.034 ±0.014
β	0.3645 ±0.0025	0.3645 ±0.0065
$\eta^{(2)}$	−0.549 ±0.0035	−0.5475 ±0.008

our more complete analysis has somewhat widened, despite more information, the uncertainty in the results, as compared to those obtained in Ref. 1.

One sees that, when n increases, the accuracy in the determination of g^* improves surprisingly (see Fig. 7), while in the case of the critical exponents it deteriorates, as expected from the study of large orders.

The critical exponents given in Table III have all been calculated independently. Table IV presents verification of scaling relations between them. Indeed our theory is scale invariant order by order in perturbation theory, and scaling laws between exponents must be satisfied within the error bars. Taking for instance γ and ν as the two independent critical exponents, we give in Table IV the estimates obtained for η, $\eta^{(2)}$, and β, either by direct calculations (Table III), or though the relations

$$\eta = 2 - \gamma/\nu \ , \tag{88}$$

$$\eta^{(2)} = (1-\gamma)/\nu \ , \tag{89}$$

$$\beta = \tfrac{1}{2}(3\nu-\gamma) \ , \tag{90}$$

using the estimates of the directly computed γ and ν.

The various results obtained in this way are remarkably consistent.

VII. SUMMATION OF THE ϵ EXPANSION

Kazakov et al.[7] have recently calculated the fourth-order contribution to the critical exponents ν and ω for the ϵ expansion,[5] the fourth-order contribution to η being already known.[6]

Using the Lipatov estimates for the ϵ expansion,[11,12] they have applied the same summation methods as ours. Their results are in good agreement with those obtained in our calculation directly in 2 and 3 dimensions. Note that the series are at

TABLE V. Values of critical exponents obtained in the framework of the ϵ expansion, using the homographic transformation (91) of Sec. VII.

$d = 3$; $n = 0$				
$\omega = 0.80$	$\gamma = 1.163$	$\nu = 0.589$	$\eta = 0.034$	$\beta = 0.305$
$d = 3$; $n = 1$				
$\omega = 0.80$	$\gamma = 1.242$	$\nu = 0.632$	$\eta = 0.04$	$\beta = 0.330$
$d = 3$; $n = 2$				
$\omega = 0.79$	$\gamma = 1.324$	$\nu = 0.676$	$\eta = 0.05$	$\beta = 0.357$
$d = 3$; $n = 3$				
$\omega = 0.78$	$\gamma = 1.395$	$\nu = 0.713$	$\eta = 0.05$	$\beta = 0.379$
$d = 2$; $n = 0$				
$\omega = 1.53$	$\gamma = 1.43$	$\nu = 0.77$	$\eta = 0.30$	$\beta = 0.58$
$d = 2$; $n = 1$				
$\omega = 1.40$	$\gamma = 1.82$	$\nu = 1.03$		
$d = 1$; $n = 0$				
$\omega = 2.44$	$\gamma = 2$	$\nu = 1.3$		

the same length as the series in two dimensions but shorter than those in three dimensions.

We have redone this calculation but with the following slight modification: we know that, as a function of ϵ, the exponents for $n = 2$ and 3 have a singularity at $\epsilon = 2$ ($d = 2$). For $n = 1$ a singularity probably lies at $\epsilon = 3$ ($d = 1$), and for $n = 0$ presumably at $\epsilon = 4$ ($d = 0$). Therefore the Borel transform has an exponential behavior for large values of the argument. To avoid this problem, we have made a homographic transformation

$$\tilde{\epsilon} = \frac{\epsilon}{(1 - \epsilon/\rho(n))} \ , \tag{91}$$

with

$$\rho(0) = 4, \quad \rho(1) = 3 \ ,$$

$$\rho(n) = 2, \quad \text{for } n \geq 2 \ .$$

The new Borel transform probably now has a power-law behavior, which can be handled with the parameter α defined in Sec. III A.

We give in Table V the results for $d = 3$ ($n = 0, 1, 2, 3$), for $d = 2$ ($n = 0, 1$), and for $d = 1$ ($n = 0$). Because these results are less accurate, we do not give the apparent errors, which are always such that the results are compatible with the direct calculations in two and three dimensions.

Finally, for $n = 1$ we have verified that, if we choose the values of b and α such that the two-dimensional results are exactly those of the Ising model, then the three-dimensional results calculated with the same values of b and α are very close to the results obtained by direct calculation in fixed dimen-

sion $d = 3$. For example we have

$$\gamma_{(2)} = 1.75 \rightarrow \gamma_{(3)} = 1.240 \ ,$$

$$\nu_{(2)} = 1.0 \rightarrow \nu_{(3)} = 0.631 \ , \tag{92}$$

$$\eta_{(2)} = 0.25 \rightarrow \eta_{(3)} = 0.033 \ .$$

VIII. CONCLUSIONS

We shall now make a few final remarks about the calculations of the critical exponents for the n-vector model in three dimensions.

First, it appears that the results are very much independent of the summation method. Essentially same type of estimates are obtained using Padé approximants[9] or other resummation techniques.[20] Also, results from the ϵ expansion are consistent with those obtained directly at a fixed dimension.

Second, the estimate of the apparent errors is as usual more delicate and somewhat subjective, and therefore more method dependent. Nevertheless, there seems to be reasonable agreement between all types of calculations made in the framework of field theory.

Then, if we compare such results with those given by experiments or by high-temperature series, as shown in Table VI, we see the following situation: (i) At first sight, the results appear to be quite similar, (ii) nevertheless, significant discrepancies, mainly with high-temperature results, appear, apparently[22] incompatible with quoted errors.

The situation is therefore at present somewhat confusing,[77] and it does not seem possible yet to draw definitive conclusions from numerical calculations about the identity of lattice models and continuous field theory. May be more terms both in high-temperature series expansions and in ϕ^4 field theory would lead to more accurate results so that we could see if the differences become more or less significant.

TABLE VI(a). Three-dimensional critical exponents obtained by high-temperature series analysis.

$n = 1$

$\gamma =$ $1.250\begin{cases}+0.003\\-0.007\end{cases}$ (Ref. 23)	1.250 ± 0.003 (Refs. 21, 78, and 24–27)
1.246 ± 0.005 (Refs. 28 and 29)	1.250 ± 0.001 (Ref. 30)
1.245 ± 0.003 (Ref. 22)	$\simeq 1.241$ (Ref. 79)
$\nu =$ $0.638\begin{cases}+0.002\\-0.008\end{cases}$ (Ref. 23)	$0.638\begin{cases}+0.002\\-0.001\end{cases}$ (Refs. 31, 21, and 25)
0.638 ± 0.002 (Ref. 22)	
$\alpha =$ 0.125 ± 0.020 (Ref. 23)	$\simeq 0.125$ (Refs. 32 and 33)
$\simeq 0.110$ (Ref. 22)	0.13 ± 0.01 (Ref. 29)
$\beta =$ 0.312 ± 0.005 (Refs. 34 and 25)	$0.303–0.318$ (Refs. 30 and 25)
0.311 ± 0.008 (Ref. 35)	0.312 ± 0.004 (Ref. 30)
$\Delta_1, \omega\nu =$ 0.6 ± 0.1 (Ref. 23), $\simeq 0.496$ (Ref. 79)	0.50 ± 0.08 (Ref. 36)
0.50 ± 0.05 (Ref. 37)	0.83 ± 0.11 (Ref. 27)

$n = 2$

$\gamma =$ 1.318 ± 0.010 (Ref. 38)	1.333 ± 0.010 (Ref. 39)
1.315 ± 0.015 (Ref. 40)	1.33 ± 0.02 (Ref. 41)
$\nu =$ 0.670 ± 0.006 (Ref. 38)	0.678 ± 0.005 (Ref. 39)
0.673 ± 0.006 (Ref. 40)	
$\alpha =$ -0.02 ± 0.03 (Ref. 38)	0.02 ± 0.05 (Ref. 41)
0.02 ± 0.02 (Ref. 40)	
$\Delta_1, \omega\nu =$ 0.60 ± 0.08 (Ref. 39)	$\eta = 0.04 \pm 0.01$ (Ref. 38)

$n = 3$

$\gamma =$ $1.42\begin{cases}+0.02\\-0.01\end{cases}$ (Ref. 42)	$1.375\begin{cases}+0.02\\-0.01\end{cases}$ (Refs. 43 and 40)
1.405 ± 0.02 (Refs. 44 and 46)	1.36 ± 0.04 (Refs. 45 and 46)
$\nu =$ 0.725 ± 0.015 (Ref. 42)	$0.7025\begin{cases}+0.010\\-0.005\end{cases}$ (Ref. 43)
0.717 ± 0.007 (Refs. 44 and 46)	0.70 ± 0.02 (Ref. 46)
$\alpha =$ -0.14 ± 0.06 (Refs. 44 and 46)	-0.07 ± 0.04 (Refs. 55, 33, and 43)
-0.09 ± 0.03 (Ref. 56)	
$\beta =$ 0.38 ± 0.03 (Refs. 47 and 46)	0.35 ± 0.05 (Ref. 46)
$\eta =$ 0.043 ± 0.014 (Ref. 43)	0.040 ± 0.008 (Refs. 44 and 48)
$\Delta_1, \omega\nu =$ 0.54 ± 0.10 (Ref. 42)	

$n = 0$

$\gamma =$ 1.165 ± 0.003 (Ref. 9), 1.1615 ± 0.0005 (Ref. 80)	$\simeq 1.167$ (Refs. 49 and 50)	$1.162–1.167$ (Ref. 50)
$\nu =$ 0.600 ± 0.005 (Ref. 9)	$\simeq 0.6$ (Refs. 49, 51, and 52)	$\simeq 0.59$ (Ref. 52)
$\alpha \simeq$ 0.25 (Refs. 49, 53, and 54)		

$$\Delta_1, \omega\nu \simeq 0.465 \text{ (Ref. 80)}$$

TABLE VI.(b) Experimental results for three-dimensional critical exponents.

$n = 1$

$\gamma =$	1.240 ±0.007 (Ref. 57)		1.23−1.28 (Ref. 58)
	1.23 ±0.02 (Ref. 59)		1.24 ±0.015 (Ref. 60)
	1.20 ±0.05, 1.21 ±0.03, 1.14 ±0.1,		1.18 ±0.08,
	1.25 ±0.05, 1.22 ±0.06 (Ref. 61)		
$\nu =$	0.625 ±0.003, 0.625 ±0.005 (Ref. 57)		0.63 ±0.02 (Ref. 59)
	0.65 ±0.02 (Ref. 60)		
$\alpha =$	0.113 ±0.005 (Ref. 62)		
	0.13 ±0.02, 0.08 ±0.02, 0.125 ±0.01,		0.11 ±0.03,
	0.12 ±0.04 (Ref. 61)		
$\beta =$	0.328 ±0.004, 0.316 ±0.008 (Ref. 63)		0.328 ±0.007 (Ref. 64)
	0.321−0.329 (Ref. 58)		0.305 ±0.005 (Ref. 60)
	0.34 ±0.02, 0.347 ±0.01, 0.354 ±0.004, 0.33 ±0.05,		
	0.34 ±0.01 (Ref. 61)		
$\eta =$	0.016 ±0.007, 0.016 ±0.014 (Ref. 57)		
	0.03−0.06 (Ref. 60)		
$\Delta_1, \omega\nu =$	0.50 ±0.03 (Ref. 62)		

$n = 2$

$\nu =$	0.675 ±0.001 (Ref. 65)	0.66−0.68 (Ref. 66)
$\alpha =$	−0.026 ±0.004 (Ref. 65)	

$n = 3$

$\gamma =$	1.40 ±0.03 (Ref. 60)	1.30−1.35 (Ref. 67)
	1.397 ±0.034 (Ref. 68)	
$\nu =$	0.70 ±0.02 (Ref. 60)	0.724 ±0.008 (Ref. 68)
$\alpha =$	−0.09−−0.12 (Ref. 67)	
$\beta =$	0.357 ±0.012, 0.358 ±0.003 (Ref. 67)	
	0.346 ±0.007 (Ref. 69)	0.336 ±0.002 (Ref. 70)
	0.378 ±0.010 (Ref. 71)	0.385 ±0.005 (Ref. 72)
	0.37 ±0.015 (Ref. 60)	

$n = 0$

$\nu =$	0.595 ±0.020 (Ref. 73)	0.55 ±0.02 (Refs. 74 and 76)
	0.575 ±0.01 (Refs. 75 and 76)	0.57−0.60 (Ref. 76)
	≃0.6 (Ref. 49)	

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