PHYSICAL REVIEW B VOLUME 21, NUMBER 8 15 APRIL 1980

Contribution of electron-electron normal scattering processes to the electrical resistivity of thin wires

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(Received 29 October 1979)

We have examined, using Monte Carlo techniques, the mean free path of electrons in a cylindrical wire. The electrons were allowed to undergo isotropic scattering from impurities, diffuse scattering at the walls of the wire, and both isotropic and anisotropic electron-electron normal scattering. A wide range of impurity and electron-electron free paths were used so that the Knudsen minimum and the transition to Poiseuille flow could be examined. An analytic expression for the electronic free path at very large impurity and electron-electron free paths has been obtained from theoretical considerations. The data are consistent with recent low-temperature measurement on potassium. They can be extended to situations involving other normal scattering processes such as that of phonons in a wire.

I. INTRODUCTION

Rowlands, Duvvury, and Woods[1] measured the electrical resistivity of potassium at temperatures below 2 K. They found their data could be fitted with the formula

$$\rho = \rho_0(1 + AT^{1.5}), \tag{1}$$

where ρ is the resistivity at temperature T, ρ_0 is the resistivity at $T = 0$ K, and $A = (86 \pm 10) \times 10^{-6}$ $K^{-1.5}$. They proposed that the temperature-dependent term could arise due to the presence of electron-electron normal processes in a thin wire.

It is well known (cf. Ziman[2]) that electron-electron normal scattering will not contribute to the bulk resistivity of metals with a spherical Fermi surface. This is due to the fact that such processes conserve momentum and hence current. Freeman, Blatt, and Bass[3] suggested that the processes could contribute to the resistivity in a thin wire by knocking electrons into the walls. These electrons could then scatter diffusely from the walls and reduce the current.

Rowlands *et al.*, by analogy with the Knudsen[4] formula for the resistance to the flow of rarified gases in a cylindrical tube due to molecular collisions, proposed that an appropriate formula for the resistance to the electron flow in thin wires would be

$$\rho = \rho_0(1 + ka/\lambda_e), \tag{2}$$

where a is the wire radius, λ_e a mean free path for electron-electron processes, and k is a constant, which in the gas case was found empirically to be of order unity by Knudson. They suggested that the usual T^2 dependence of $1/\lambda_e$, and the existence of dislocation nets might combine to produce a $T^{1.5}$ dependence. They also suggested that ρ_0 could be something other than the size-limited resistivity which analogy with the Knudsen formula would suggest. In the Knudsen case ρ_0 is proportional to $1/a$, and hence the dependence of ρ on the residual resistivity would vanish from the temperature-dependent part.

The primary objective of this study is to determine k for the case of electron-electron normal processes in a cylindrical wire. In view of a large residual resistivity in the experiments of Rowland *et al.*, and for that matter in all low-temperature experiments, effects of impurity scattering on k are also examined.

In the course of setting up this study, two interesting and related problems were encountered. Whitworth[5] studied the contribution of phonon-phonon normal processes to the thermal resistivity of liquid helium. He found a dip in the resistivity which he called the "Knudsen minimum." He noted that the minimum had never been explained theoretically, and that it was not obvious, without detailed theoretical study, why phonon-phonon processes in which phonons were knocked into the walls of the container should predominate over the processes in which phonons were knocked away from the walls. Clearly such a predominance is needed if the phonon free path is first to decrease as the phonon-phonon free path is decreased. Moreover, a reversal in the predominance is needed to explain the subsequent increase in the phonon free path as the phonon-phonon free path is reduced still further.

Ziman noted that the Knudsen minimum had not been observed in solids; he also raised the question as to whether the free path of such phonons would increase or decrease as the phonon-phonon collisions of the normal type were increased. In view of the questions raised by Ziman and Whitworth, it therefore is an objective of the present study to see if, in addition to obtaining the con-

stant k, the geometrical origin of the increased or reduced free path can be understood in terms of the reduction of the free path for electron-electron normal processes, and more generally, for any momentum-conserving processes.

The above remarks apply to situations in which the free path for the normal processes is larger or of the same order as the size of the container. When the free path is much smaller than the dimension of the container a region of Poiseuille flow is encountered in which the free path associated with the resistance to flow becomes many times larger than the free path for the normal processes.

Gurzhi[6] dealt with the latter region for the case of electron-electron normal processes. He argued that the actual path of the electron would, when the path for the electron-electron normal processes became much less than the dimensions of the wire, resemble a random walk. Then the free path to be used in determining electrical resistivity should be proportional to a^2/λ_e where a is the wire radius and λ_e is the free path for electron-electron collisions. In a subsequent paper Gurzhi[7] developed a theory of the region using a magnetohydrodynamic approach.

There is no data in the literature as to the values of λ_e for which the transition to Poiseuille flow will take place and the Gurzhi formulas would be appropriate. It is therefore an objective of this study to determine this transition for various impurity concentrations. Accordingly, we have explored the behavior of resistivity for the region of the free path well outside the Knudsen minimum.

II. THEORY

The original intention in this paper was to solve the linearized Boltzmann equation subject to the boundary condition imposed by diffuse scattering at the walls of a cylindrical wire. This had been solved exactly by Dingle[8] for the case in which only impurity scattering was present, and at first sight it seemed the work he did could be extended to include electron-electron normal processes. In fact it was not found possible to extend his theory to include the normal processes. This is in part because of the complicated integrals which arise in electron-electron scattering even when a spherical Fermi surface is assumed, and in part because there is no simple analytic result over much of the range of impurity free paths λ_i even when no electron-electron scattering is present.

The approach finally adopted here was to treat the electron as a particle which traveled at the Fermi velocity, and to determine, with Monte Carlo techniques on a computer, the average distance (or mean free path) λ_m traveled by the electron before it collided with either a wall or an impurity. Collisions with electrons were taken only to alter the direction of the electron's travel and not to contribute directly to the mean free path. The mean free path between collisions with impurities was taken as λ_i, and the mean free path between collision with electrons was taken to be λ_e. In what follows these free paths, and any others, will be understood to be in units of the wire radius. The results then are quite general, and independent of the absolute value of λ_e or λ_i.

A mean free path $\lambda_m(\rho)$ at radius ρ was first determined by launching particles in random directions from location ρ and establishing the average distance traveled by the electron before a wall or impurity collision took place. This supposes the wall scattering to be totally diffuse rather than specular. Collisions with electrons were taken only to alter the direction as mentioned in the previous paragraph. The average mean free path λ_m was then obtained by averaging $\lambda_m(\rho)$ over the cylindrical cross section of the wire once $\lambda_m(\rho)$ had been obtained at selected radii.

Two types of electron-electron scattering were considered. In the first type the scattering was taken to be isotropic. This isotropic scattering was not taken to destroy the momentum gained from the electric field by the electron, only to alter the electron direction isotropically. The electron following such a collision would emerge on average with the local drift velocity. In the second type of scattering a mechanism which favored forward scattering was used. This was based on a screened Coulomb interaction between the electrons.

Ziman[9] determined the contribution of screened Coulomb electron-electron umklapp processes to electrical resistivity when a spherical Fermi surface was assumed. The calculation is quite involved, but the point to note is that the integrand is proportional to the rate of scattering of a pair of electrons from a state $\vec{k}_1, \vec{k}_2$ to a new state $\vec{k}_3, \vec{k}_4$. Once the complicated integrations over $\vec{k}_2$, and $\vec{k}_4$ have been carried out, the remaining integrand I is a measure of the scattering rate of electrons from state $\vec{k}_1$ to state $\vec{k}_3$ per unit solid angle. The integrand Ziman obtains is

$$I = \frac{1}{(|\vec{k}_3 - \vec{k}_1|^2 + q^2)^2} \frac{1}{|\vec{k}_3 - \vec{k}_1|}, \tag{3}$$

when the reciprocal lattice vector is set equal to zero for normal processes. Here $1/q$ is the Thomas-Fermi screening length. If we now let $\vec{k}_3$ and $\vec{k}_1$ lie on a spherical Fermi surface and let θ be the angle between them we obtain

$$I \propto \frac{1}{(1-\cos\theta)^{1/2}} \frac{1}{(2.612-\cos\theta)^2}, \quad (4)$$

where a Thomas-Fermi screening length appropriate for potassium has been introduced. This formula was then used to scatter electrons anisotropically.

In the introduction to the theory, it was noted that no theoretical results were obtained starting with the Boltzmann equation. It is, however, possible to develop an approximate result for λ_m which is suitable for large values of λ_e and λ_i. The theory is based on the case of isotropic electron-electron normal processes.

If N particles are launched from a location ρ and $N_e(\rho)$ of these collide with electrons and the remainder undergo their first collision with a wall or an impurity we may write

$$\lambda_m(\rho) = \lambda(\rho) + 1/N \sum_{i=1}^{N_e(\rho)} d(\rho_i). \quad (5)$$

Here $\lambda(\rho)$ is the average distance traveled before the first collision of any type. It can be evaluated as follows.

Consider the average distance traveled by electrons launched from location ρ, with a polar angle θ with respect to the axis of the wire and azimuthal angle ϕ. Let the mean free path for collisions[10] with electrons or impurities be λ_T. The average distance traveled before electrons collide with an electron, impurity, or wall may be written

$$\lambda_T\, 1 - \exp\{-d(\rho,\theta,\varphi)/\lambda_T\}, \quad (6)$$

where d is the distance from the wall to the launch site. It follows that the average distance traveled in all directions is

$$\lambda(\rho) = \frac{\lambda_T}{4\pi} \int\int [1 - \exp(-d/\lambda_T)] \sin\theta\, d\theta\, d\varphi. \quad (7)$$

Evaluation of the double integral is not possible analytically for all λ_T. It is of importance to note that the integral is almost identical to that evaluated by Dingle in determining the free path needed to determine electrical resistivity from the Boltzmann equation. His integral has a factor of $3\cos^2\theta$ in the integrand.

Dingle was able, by a variety of mathematical methods, to obtain the series expansion solution of the integral for $\lambda_T \gg 1$ and $\lambda_T \ll 1$. It is a straightforward procedure to repeat his work without the factor of $3\cos^2\theta$ in the integrand. We find, for the region of interest $\lambda_T \gg 1$,

$$\lambda = \tfrac{4}{3} - (1/2\lambda_T)[\ln(\lambda_T/2) + 2.449] + O(1/\lambda^2) \quad (8)$$

which may be compared with Dingle's result

$$\lambda = 2 - (3/2\lambda_T)[\ln(\lambda_T/2) + 1.059] + O(1/\lambda^2). \quad (9)$$

Note then that the two methods give somewhat different λ when only impurity scattering is present and $\lambda_T = \lambda_i$. On the other hand both give similar temperature dependence of λ if the temperature dependence is contained in λ_T.

In Eq. (5) the $d(\rho_i)$ represent the distance traveled by the ith electron before striking a wall or impurity following its collision with an electron at location ρ_i. Since the electron-electron scattering is isotropic, this distance is just $\lambda_m(\rho_i)$ on the average, and we may write

$$\lambda_m(\rho) = \lambda(\rho) + 1/N \sum_{i=1}^{N_e(\rho)} \lambda_m(\rho_i). \quad (10)$$

In the actual Monte Carlo calculations to be discussed later,[11] we find that $\lambda_m(\rho)$ varies from 0.85 to 1.33 times its average value of λ_m. Accordingly we may write

$$\lambda_m(\rho) = \lambda(\rho) + [N_e(\rho)/N] k(\rho) \lambda_m, \quad (11)$$

where $k(\rho)$ lies between 0.85 and 1.33. It can be shown that $N_e(\rho)/N$ is $\lambda(\rho)/\lambda_e$, and hence we may write

$$\lambda_m(\rho) = \lambda(\rho)[1 + k(\rho)\lambda_m/\lambda_e]. \quad (12)$$

If we now average over the cross section of the cylinder and solve for λ_m we obtain

$$\lambda_m = \lambda/(1 - k''\lambda/\lambda_e), \quad (13)$$

where k'' is again between 0.85 and 1.33. Substitution of λ from Eq. (8) into Eq. (13) then gives an estimate of the dependence of λ_m on λ_T. It must be noted that the accuracy to which k'' is known makes the introduction of the terms of order $1/\lambda_T^2$ from Eq. (8) of little value and so they have not been determined.

III. PROCEDURE

The problem here is to obtain an estimate of

$$\lambda_m = \int_0^1 \lambda_m(\rho)\rho\, d\rho \Big/ \int_0^1 \rho\, d\rho. \quad (14)$$

Here $\lambda_m(\rho)$ is the average distance traveled by an electron launched from radius ρ in a random direction before it collides with an impurity or a wall of the wire. On route to such a collision the electron may undergo one or more electron-electron collisions.

It is of interest to compare the method used here with that of Luthi and Wyder.[12] They launched a particle down the wire and then allowed it to move finite steps along its trajectory. After each step the computer was asked to decide if the particle had undergone a collision with an impurity, a wall, or a small-angle phonon collision which was what they were studying.

TABLE I. λ_m for isotropic scattering.

λ_i	λ_e									
	10^5	10	5	2	1	0.5	0.2	0.1	0.05	0.025
10^5	1.33	1.30	1.31	1.37	1.53	1.86	2.97	4.7[a]	8.5[a]	16.1[a]
10	1.15	1.15	1.17	1.23	1.34	1.58	2.22	3.1[a]	4.2[a]	
5	1.03	1.04	1.06	1.11	1.19	1.37	1.80	2.28[a]	2.8[a]	3.3[a]
2	0.80	0.82	0.82	0.85	0.89	0.97	1.14	1.30	1.43[a]	
1	0.59	0.59	0.60	0.61	0.63	0.66	0.71	0.76	0.80	
0.5	0.38	0.38	0.39	0.39	0.39	0.39	0.41	0.42	0.43	0.44[a]
0.2	0.178	0.180	0.178	0.179	0.180	0.179	0.180	0.182	0.183	
0.1	0.094	0.094	0.094	0.093	0.094	0.094	0.094	0.094	0.094	
0.05	0.047	0.048	0.048	0.048	0.048	0.048	0.047	0.047	0.048	0.047[a]

[a]Values are accurate to ±2%, all other values are accurate to ±1%.

Our method consisted of using a random number generator to launch the particle in a random direction, and then using the generator to establish how far the particle traveled before its first collision with an impurity or electron. If this distance were outside the wall of the wire, then the distance traveled was taken as the distance to the wire wall. If the distance were inside, then the random number generator was used to determine if the collision was with an electron or impurity. In the latter case the distance traveled to the collision was taken as the distance traveled by the electron, but in the former case the random number generator was used to launch the particle from its new radial location, and the process was continued until it reached an impurity or a wall. In that case the total distance traveled by the electron from its launch until it hit the wall or impurity was used as the distance traveled by the electron.

Data on $\lambda_m(\rho)$ and λ_m were obtained by averaging over 9000 launches the distance traveled by the electrons. Values of ρ selected were $0, 0.1, \ldots,$ 1.0 and the value of λ_m was then obtained from a Simpson's rule integration over these 11 points.

In order to obtain the more accurate values of λ_m and $\lambda_m(\rho)$ needed to examine the Knudsen minimum 230 000 launches were used at eight radial locations and an eight-point Gaussian integration was then performed to obtain λ_m.

Even with only 9000 launches it took about 5 min of computer time to obtain λ_m on the Burroughs 6700. A total time of about 2 h was needed for the cases where 230 000 launches were used.

As an indication of the accuracy of the Monte Carlo values the standard deviation from the mean of 13 samples, each sample averaged over 1000 launches, was determined for various combinations of λ_i and λ_e. In this way it was found that the runs on λ_m using 9000 shots typically had a standard deviation of 1%, whereas using 230 000 shots the standard deviation was 0.002. These standard deviations are given as guides to the accuracy in the tables of the next section.

IV. RESULTS

The results of the isotropic scattering study are shown in Table I for a wide range of λ_e and λ_i. Consider first the situation in which there is no electron-electron scattering ($\lambda_e = 10^5$) and the impurity scattering ranges from $\lambda_i = 10^5$ down to

TABLE II. λ_m for anisotropic scattering.

λ_i	λ_e									
	10^5	10	5	2	1	0.5	0.2	0.1	0.05	0.025
10^5	1.33	1.28	1.29	1.30	1.34	1.46	1.88	2.6[a]	4.1[a]	7.3[a]
10	1.15	1.15	1.15	1.16	1.20	1.30	1.59	2.08[a]	2.8[a]	
5	1.03	1.04	1.04	1.05	1.09	1.15	1.38	1.66	2.12[a]	2.7[a]
2	0.80	0.81	0.81	0.82	0.84	0.88	0.98	1.10[a]	1.24[a]	
1	0.59	0.59	0.59	0.60	0.61	0.62	0.66	0.70	0.75[a]	
0.5	0.38	0.38	0.38	0.38	0.38	0.39	0.40	0.41	0.42[a]	0.44[a]
0.2	0.178	0.180	0.178	0.179	0.179	0.179	0.179	0.182	0.182	
0.1	0.093	0.094	0.093	0.094	0.094	0.094	0.094	0.093	0.094	
0.05	0.047	0.047	0.047	0.047	0.047	0.047	0.048	0.047	0.047	0.047[a]

[a]Values are accurate to ±2%, all other values are accurate to ±1%.

TABLE III. Details of λ_m for isotropic scattering.

λ_i	λ_e								
	10^7	200	100	50	20	10	5	2	1
10^7	1.329	1.325	1.319	1.315	1.306	1.302	1.312	1.37[a]	1.53[a]
100	1.306	1.300	1.297	1.294	1.285	1.285	1.296	1.363	
10	1.148	1.150	1.150	1.150	1.151			1.23[a]	1.34[a]

[a]Values are accurate to ±1%, all other values are accurate to ±0.002.

0.05. Note that the introduction of impurities monotonically lowers the free path, as one expects. The walls of the wire totally determine the free path for $\lambda_i = 10^5$, but reduce the free path only 5% below its impurity value when $\lambda_i = 0.05$. In fact when λ_i is about 0.2 (one-tenth the diameter of the wire) the walls reduce the free path only about 10%.

Now consider the situation in which there is no impurity scattering ($\lambda_i = 10^5$) and the value of λ_e varies from $\lambda_e = 10^5$ to 0.025. Here a dip in the free path is first observed, the Knudsen minimum, and then the free path rises above its value with walls alone. Note that the free path almost doubles as λ_e is reduced from 0.05 to 0.025; that is the region described by Gurzhi in which $\lambda_m \propto 1/\lambda_e$ is being entered.

It is quite interesting to obtain data for $\lambda_e = 0.025$. Here, with a free path of 16.1, each electron undergoes roughly 650 collisions before the walls of the wire are reached and a great deal of computer time was required to follow thousands of these from their launch to the edge of the wire.

Finally consider the column in which $\lambda_e = 0.05$ and λ_i varies from 10^5 down to 0.05. The free path is seen to decrease monotonically, and below $\lambda_i = 0.2$, the electron-electron contribution to the free path is not detectable. Incidentally, the smallest value of λ_e suggested by Rowlands *et al.*[1] is about 1 cm at $T = 1$ K. It is evident that a very large wire would be required before the region of Poiseuille flow was encountered. In their experiment $\lambda_e \gtrsim 5$, and this is the region of the tables appropriate to study their data.

The results of the anisotropic-scattering study are shown in Table II. Here we see behavior similar to that of Table I, but the increase in λ_m as λ_e goes from $\lambda_e = 10^5$ to 0.025 is much reduced in this anisotropic case. Note also that as λ_e varies from $\lambda_e = 10^5$ down to 1 there is very little change in λ_m. The anisotropic scattering is not as effective in altering the free path from its wall and impurity value as is the isotropic scattering.

In Table III are the results of a detailed study of the Knudsen minimum in the isotropic-scattering cases. Here 230 000 launches of electrons from each of eight radii in the wire (the points of an 8-point Gaussian) gave an accuracy of ±0.002. The dip can be seen clearly for both $\lambda_i = 10^7$ and 100, but has vanished (to an accuracy of ±0.002) for $\lambda_i = 10$. Note that for $\lambda_i = 10^7$ the dip is a minimum in the vicinity of $\lambda_e = 10$ and is over at $\lambda_e = 2$.

Similar data are presented for the anisotropic case in Table IV. The dip is clearly visible for $\lambda_i = 10^7$ and 100, and is not evident at $\lambda_i = 10$ in this case. An interesting feature of the data is the fact that the dip is both deeper and broader when anisotropic scattering is present than in the isotropic case.

In Table V are shown the values of λ_m calculated with Eq. (13) developed in the theory section. For λ_i and $\lambda_e \geq 100$ the agreement is good, but for λ_i and $\lambda_e \simeq 10$ the agreement is only to a few percent. This is not surprising since the terms omitted in the calculation of λ are of order $1/\lambda_i^2$ and hence of order 1%. The value of k'' used was unity for the table. Better agreement can be obtained with a larger value. The agreement suggests we use the formula to determine the temperature dependence of λ_m in the region before the minimum is reached.

At first sight it is puzzling that the anisotropic scattering should give rise to a deeper and broader

TABLE IV. Details of λ_m for anisotropic scattering.

λ_i	λ_e								
	10^7	200	100	50	20	10	5	2	1
10^7	1.329	1.327	1.319	1.315	1.304	1.291	1.288	1.298	1.34[a]
100	1.306	1.301	1.298	1.294	1.283	1.277	1.272	1.283	
10	1.148	1.149	1.148	1.148	1.147	1.148			1.20[a]

[a]Values are accurate to ±1%, all other values are accurate to ±0.002.

TABLE V. Prediction of λ_m for isotropic scattering.

λ_i	λ_e					
	10^7	200	100	50	20	10
10^7	1.333	1.324	1.318	1.310	1.290	1.274
100	1.302	1.297	1.293	1.287	1.272	1.255
10	1.130	1.129	1.128	1.125	1.119	1.107

Knudsen minimum. The explanation for this curious effect can be seen in Tables VI and VII. There the value of $\lambda_m(\rho)$ are presented at selected radii for $\lambda_i = \infty$ and selected values of λ_e.

Consider the isotropic case. Recall that the average value of λ_m reached a minimum at $\lambda_e = 10$ and then began to rise. The data of Table VI show this to be understandable in terms of two competing mechanisms. Near the center of the wire ($\rho = 0.02$) the free path reaches a minimum at $\lambda_e = 20$ and then begins to rise. On the other hand, near the edge of the wire, the free path is continuing to decrease even at $\lambda_e = 5$. The scattering leading to an increase in the free path then has begun to dominate once $\lambda_e = 10$ is reached. [Preliminary studies show that even at very small values of λ_e the value of λ_m ($\rho = 1.0$) remains at about 1, while the value of λ_m ($\rho = 0.0$) has grown to many times the radius of the wire.]

Now if we examine Table VII we see that the path lengthening which occurs in the central region is just starting to be effective at $\lambda_e = 5$, while the path shortening at and near the walls is about as effective as that which is found in the isotropic case. Since the anisotropic scattering is less effective in deviating electrons from their initial trajectory, it is perhaps not surprising that in central regions the free path is not enhanced as rapidly as in the isotropic case. On the other hand, electrons launched near the walls can be deflected into the wall even if only scattered through a small angle.

The above observations also provide some insight into the Knudsen minimum. As Whitworth noted, the problem is to understand why it is that path-lengthening processes begin to dominate over path-shortening processes. We see from Table VI that at long enough λ_e the path is reduced everywhere over the cross section of the wire. Then as λ_e is decreased the path is gradually increased, starting at the center and gradually spreading toward the walls. At some λ_e the lengthening begins to dominate and we have passed through the minimum.

If we try to see why it is that the path is shortened at all, for large λ_e, it is evident that no simple explanation exists. In examining 10 000 trajectories at $\rho = 1.0$ for $\lambda_i = 10^7$ and $\lambda_e = 10$ there was no sign of any systematic reduction in the free path even though the average value was reduced 12% below the value with no electron-electron scattering. That the effect is an average one was seen in the fact that for 10 000 launches from the center of the wire at $\rho = 0.0$ for $\lambda_i = 10^7$ and $\lambda_e = 10$ the value of the average was 2% below the average with no electron-electron scattering; yet the fully converged value in this case gave the same value as if there were no electron-electron scattering. The study showed that the alteration of path length by a few percent in the vicinity of the minimum cannot be understood in terms of some simple explanation.

V. COMPARISON OF EXPERIMENT AND THEORY

By writing the resistivity proportional to $1/\lambda$, the experimental results of Rowlands *et al.*[1] may be put in the form

$$\Delta\lambda = \lambda_0 - \lambda = \lambda_0 A T^{1.5},$$

where $\Delta\lambda$ is the dip in the free path from its value at $T = 0$. The largest value of $\Delta\lambda$ obtained in their experiment is about 10^{-4} at a temperature of 2 K, and is for their cleanest sample in which $\lambda_0 \approx 0.33$.

We may compare this result with the data of Tables III and IV by using the values of λ_e that Rowlands *et al.* used. They proposed values be-

TABLE VI. $\lambda_m(\rho)$ for the isotropic case and no impurity scattering.

λ_e	ρ								
	0.020	0.102	0.237	0.408	0.592	0.763	0.898	0.980	Average [a]
10^7	1.57	1.57	1.55	1.50	1.42	1.31	1.18	1.05	1.333
200	1.57	1.56	1.53	1.50	1.42	1.30	1.16	1.04	1.325
100	1.57	1.55	1.54	1.49	1.41	1.30	1.15	1.03	1.319
50	1.56	1.55	1.53	1.49	1.41	1.29	1.15	1.03	1.315
20	1.55	1.55	1.53	1.49	1.40	1.28	1.14	1.00	1.306
10	1.57	1.56	1.54	1.49	1.40	1.27	1.13	0.99	1.302
5	1.61	1.60	1.58	1.52	1.42	1.28	1.11	0.97	1.312

[a] Values are accurate to ±0.002, all other values are accurate to ±0.01.

TABLE VII. $\lambda_m(\rho)$ for the anisotropic case and no impurity scattering.

λ_e	ρ								
	0.020	0.102	0.237	0.408	0.592	0.763	0.898	0.980	Average [a]
10^7	1.57	1.57	1.55	1.50	1.42	1.31	1.18	1.05	1.333
200	1.57	1.56	1.54	1.50	1.42	1.31	1.16	1.04	1.327
100	1.56	1.55	1.54	1.49	1.41	1.30	1.15	1.03	1.319
50	1.56	1.55	1.53	1.49	1.41	1.29	1.15	1.03	1.315
20	1.54	1.54	1.52	1.48	1.40	1.28	1.14	1.00	1.304
10	1.54	1.54	1.51	1.47	1.39	1.26	1.12	0.99	1.291
5	1.55	1.54	1.52	1.47	1.39	1.26	1.11	0.98	1.288

[a]Values are accurate to ±0.002, all other values are accurate to ±0.01.

tween 5 and 50 at 2 K. The Knudsen minimum is seen to lie in just this range. Unfortunately our computation was not capable of detecting $\Delta\lambda$ of order 10^{-4}, and such computer calculations would have involved too much time. Note that at $\lambda_i = 100$ ($\lambda_0 = 1.306$) the dip in λ_m is substantially greater than 10^{-4}, but at $\lambda_i = 10$ ($\lambda_0 = 1.150$) there is no dip detected in the isotropic case, and the anisotropic dip has decreased to about one-sixth of its original value. This decrease, were it to continue down to $\lambda_0 = 0.33$ of the experiment, could well give rise to the dip of 10^{-4} observed. Note that in Table I $\lambda_0 = 0.38$ at $\lambda_i = 0.5$, so that if isotropic point scattering were involved in the experiment a very small value of λ_i would be involved. Note also that the calculation predicts the increase in the temperature-dependent part of the resistivity with increased ρ_0 observed in the experiment.

It is also of interest to note that $\Delta\lambda$ goes as roughly $T^{1.4}$, as λ_e goes from 200 to 50, when $\lambda_i = 100$ in both isotropic and anisotropic cases. Moreover, for $\lambda_i = 10^7$, there is a range of λ_e values predicted by theory for which $\Delta\lambda$ goes as $(\ln\lambda_e)/\lambda_e$ when isotropic scattering is involved. In this range, if we take $\lambda_e \propto T^{-2}$ as is usually done, then again a dependence of roughly $T^{1.5}$ is encountered. This temperature dependence, while it may not persist to small values of λ_i, is strongly suggestive that Rowlands *et al.* are observing electron-electron normal processes.

VI. DISCUSSION

We have examined, using Monte Carlo techniques, the mean free path of electrons in a cylindrical wire. The electrons were allowed to undergo isotropic scattering from impurities, diffuse scattering from the walls of the wire, and both isotropic and anisotropic scattering from electron-electron normal processes. A wide range of impurity and electron-electron free paths were used so that the Knudsen minimum and the transition to Poiseuille flow could be mapped out.

It was found that the Knudsen minimum can be understood as due to competition between normal scattering lengthening the free path at the center of the wire and reducing it near the boundaries of the wire. The minimum occurs when the former processes begin to dominate the latter. We have found an analytic expression for the free path in the region where λ_e and λ_i are very large, and evidence of a temperature dependence which is roughly $T^{1.5}$ in that region.

The results are consistent with the main features of the experiment of Rowlands *et al.*[1] Thus the magnitude, temperature dependence, and proportionality to ρ_0, all match our findings in a qualitative way. Unfortunately it was not possible to extend either the theory, or Monte Carlo calculations, into a region of impurity concentration which is consistent with their experiments. It would be of considerable interest to see if the experimental work could be carried out in a region where the Monte Carlo calculations could be used to determine the magnitude and temperature dependence of the effect.

ACKNOWLEDGMENTS

It is a pleasure to thank Paul Tulonen, who set up the computer programs and carried out the computations needed for the Monte Carlo calculations. I would also like to thank the staff of the Computer Center at Brock University for their patience and assistance with the various phases of the work, in particular, Bill Bell. This work was supported in part by a grant from the National Science and Engineering Research Council of Canada, and in part by the Air Force Office of Scientific Research under Contract No. F 49620-78-C-0019.

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