Approximate critical surface of the bond-mixed square-lattice Ising model

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The critical surface of the quenched bond-mixed square-lattice spin-$\frac{1}{2}$ first-neighbor-interaction ferromagnetic Ising model (with exchange interactions J_1 and J_2) has been investigated. Through renormalization group and heuristical procedures, a very accurate [error inferior to 3×10^{-4} in the variables $t_i \equiv \tanh(J_i/k_B T)$] approximate *numerical* proposal for all points of this surface is presented. This proposal simultaneously satisfies *all* the available exact results concerning the surface, namely $p_c = \frac{1}{2}$, $t_c = \sqrt{2} - 1$, both limiting slopes in these points, and $t_2 = (1 - t_1)/(1 + t_1)$ for $p = \frac{1}{2}$. Furthermore an *analytic* approximation [namely, $(1 - p)\ln(1 + t_1) + p \ln(1 + t_2) = \frac{1}{2}\ln 2$] is also proposed. In what concerns the available exact results, it only fails in reproducing one of the two limiting slopes, where there is an error of 1% in the derivative: These facts result in an estimated error less than 10^{-3} (in the t variables) for any point in the surface.

I. INTRODUCTION

During the last few years, considerable attention has been focused on various interesting experimental and theoretical aspects offered by disordered magnetic systems. In particular, quite an amount of theoretical work has been devoted to quenched bond- and site-diluted and bond- and site-mixed ferromagnetic Heisenberg and Ising models.[1-13] Because of its relative simplicity, the quenched bond-disordered square-lattice spin-$\frac{1}{2}$ first-neighbor-interaction ferromagnetic Ising model has deserved special attention, mainly in what concerns its phase diagram. Consequently a certain amount of exact results[14-21] are now available, as well as several approximate treatments.[20, 22-27]

In order to be more specific, let us introduce some notation. The Hamiltonian of our system will be given by

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j \quad (\sigma_i, \sigma_j = \pm 1) , \qquad (1)$$

where $\langle i,j \rangle$ runs over all the first-neighboring couples of sites of a (cyclic) square lattice, and J_{ij} is a random variable which takes the value J_1 with probability $1 - p$ and the value J_2 with probability p, the distribution being independent for each bond. For future convenience we shall use the variable[24, 25, 27, 28] $t \equiv \tanh(J_{ij}/k_B T)$ (hereafter referred to as the *thermal transmittivity* of the bond). Hence the J_{ij} probability distribution may be characterized by

$$P(t) = (1 - p)\delta(t - t_1) + p\delta(t - t_2) , \qquad (2)$$

where $t_i \equiv \tanh(J_i/k_B T)$ for $i = 1, 2$, and $t_i \geq 0$ (i.e., $J_i \geq 0$) by hypothesis (since we are dealing with the

ferromagnetic case). In this way, Eqs. (1) and (2) completely define the quenched isotropic homogeneous bond-mixed Ising model whose equilibrium statistics (at temperature T) we want to study. After remarking that the transformation $(1 - p, t_1, t_2) \rightarrow (p, t_2, t_1)$ alters nothing but our arbitrary conventions, let us stress that this model contains some interesting particular cases: (a) The pure Ising model corresponds to $p = 1$ ($\forall t_1$), or $p = 0$ ($\forall t_2$), or $t_1 = t_2$ ($\forall p$); (b) the bond-percolation limit corresponds[3, 4, 29, 30] to $t_1 = 0$ and $t_2 = 1$, or $t_1 = 1$ and $t_2 = 0$; (c) the bond-diluted Ising model corresponds to $t_1 = 0$ or $t_2 = 0$; (d) the equal-concentration mixed Ising model corresponds to $p = \frac{1}{2}$.

The present model is generally expected to have (though there is no rigorous proof), in the (p, t_1, t_2) space [or equivalently in the (p, T, α) space, where $\alpha \equiv J_1/J_2$], a critical surface which separates the paramagnetic and ferromagnetic regions (which are connected through a second-order phase transition). The accurate quantitative determination of this surface constitutes the central aim of the present paper. Some essential exact values are already known, namely, Refs. 14-16, 18-20, respectively,

$$t_c \equiv t_2(p = 1) = \sqrt{2} - 1 ,$$

$$p_c \equiv p(t_1 = 0, t_2 = 1) = \frac{1}{2} ,$$

$$\left.\frac{\partial t_2}{\partial p}\right|_{t_1 = 0, t_2 = t_c, p = 1} = -(6\sqrt{2} - 8) \cong -0.485 ,$$

$$\left.\frac{\partial t_2}{\partial p}\right|_{t_1 = 0, t_2 = 1, p = p_c} = -4\ln 2 \cong -2.773 ,$$

$$t_2(p = \frac{1}{2}) = \frac{1 - t_1}{1 + t_1}, \quad \forall t_1 \in [0, 1] .$$

To the best of our knowledge no other independent quantitative exact results (concerning the critical surface) are available.

In the present paper we develop three different *approximative* calculations of the critical surface: through a renormalization group (the t-variable renormalization group or t-RG hereafter) in Sec. II, through heuristic arguments in Sec. III, and through another renormalization group (the s-variable renormalization group or s-RG hereafter, where the s variable will be introduced later) in Sec. IV. All three are remarkably close among them, and satisfy (either exactly or quite closely) the above exact results. Furthermore everything leads to the conclusion that the whole unknown *exact* surface runs *between* the heuristic and the s-RG ones. Through a convenient interpolation, we take advantage of this fact and calculate (still in Sec. IV) a fourth critical surface which has *no pretension of analytical exactitude,* but which *exactly* reproduces *all* the above results, and whose ac-

curacy we strongly suspect (although we cannot prove it) to be greater than 3×10^{-4} in the t_2 variable for any given (p, t_1) pair.

II. t RENORMALIZATION GROUP

The present treatment is a natural generalization of one (namely, RG2) of the six renormalization-group treatments (denoted by RG1 to RG6) developed by two of us (C.T. and S.V.F.L.) in Ref. 27; also, it closely follows the lines of Ref. 24. It improves both, as we will see later. It consists in constructing a real-space renormalization group (RG) which renormalizes the probability distribution given by Eq. (2) into

$$P'(t) = (1 - p')\delta(t - t_1') + p'\delta(t - t_2') \tag{3}$$

through use of a self-dual H-shaped cluster[24,25,27,29-31] (very suitable for the square lattice), whose probability distribution is given by

$$
\begin{aligned}
P_H(t) = {}&(1-p)^5\delta\!\left[t - \frac{2t_1^2 + 2t_1^3}{1 + 2t_1^3 + t_1^4}\right] + p(1-p)^4\delta\!\left[t - \frac{2t_1^2 + 2t_1^2 t_2}{1 + 2t_1^2 t_2 + t_1^4}\right] \\[2mm]
&+ 4p(1-p)^4\delta\!\left[t - \frac{t_1 t_2 + t_1^2 + t_1^2 t_2 + t_1^3}{1 + t_1^2 t_2 + t_1^3 + t_1^3 t_2}\right] + 2p^2(1-p)^3\delta\!\left[t - \frac{2t_1 t_2 + 2t_1^2 t_2}{1 + t_1 t_2^2 + t_1^2 t_2^2 + t_1^3}\right] \\[2mm]
&+ 4p^2(1-p)^3\delta\!\left[t - \frac{t_1^2 + t_1 t_2 + t_1 t_2^2 + t_1^2 t_2}{1 + t_1 t_2^2 + t_1^2 t_2 + t_1^3 t_2}\right] + 2p^2(1-p)\delta\!\left[t - \frac{t_1^2 + t_2^2 + 2t_1^2 t_2}{1 + 2t_1^2 t_2 + t_1^2 t_2^2}\right] \\[2mm]
&+ 2p^2(1-p)^3\delta\!\left[t - \frac{2t_1 t_2 + t_1 t_2^2 + t_1^3}{1 + 2t_1 t_2^2 + t_1^2 t_2^2}\right] + 2p^3(1-p)^2\delta\!\left[t - \frac{t_2^3 + 2t_1 t_2 + t_1^2 t_2}{1 + 2t_1^2 t_2 + t_1^2 t_2^2}\right] \\[2mm]
&+ 2p^3(1-p)^2\delta\!\left[t - \frac{t_1^2 + t_2^2 + 2t_1 t_2^2}{1 + 2t_1 t_2^2 + t_1^2 t_2^2}\right] + 4p^3(1-p)^2\delta\!\left[t - \frac{t_2^2 + t_1 t_2 + t_1 t_2^2 + t_1^2 t_2}{1 + t_1 t_2^2 + t_1 t_2^3 + t_1^2 t_2}\right] \\[2mm]
&+ 2p^3(1-p)^2\delta\!\left[t - \frac{2t_1 t_2 + 2t_1 t_2^2}{1 + t_2^3 + t_1^2 t_2^2 + t_1^2 t_2}\right] + 4p^4(1-p)\delta\!\left[t - \frac{t_2^2 + t_2^3 + t_1 t_2 + t_1 t_2^2}{1 + t_2^3 + t_1 t_2^2 + t_1 t_2^3}\right] \\[2mm]
&+ p^4(1-p)\delta\!\left[t - \frac{2t_2^2 + 2t_1 t_2^2}{1 + t_2^4 + 2t_1 t_2^2}\right] + p^5\delta\!\left[t - \frac{2t_2^2 + 2t_2^3}{1 + 2t_2^3 + t_2^4}\right] \; .
\end{aligned}
\tag{4}
$$

We now require that the equations

$$\langle t \rangle_{P'} = \langle t \rangle_{P_H} \; , \tag{5a}$$

$$\langle t^2 \rangle_{P'} = \langle t^2 \rangle_{P_H} \; , \tag{5b}$$

$$\langle t^3 \rangle_{P'} = \langle t^3 \rangle_{P_H} \; , \tag{5c}$$

be satisfied [$\langle \cdots \rangle$ denotes the standard mean value; for example, $\langle t^n \rangle_{P'} = (1 - p')\,(t_1')^n + p'\,(t_2')^n$]. In this way we obtain p', t_1', and t_2' as functions of p, t_1, and t_2, or, in brief,

$$(p', t_1', t_2') = R(p, t_1, t_2) \; . \tag{6}$$

The solutions of the equation $(p, t_1, t_2) = R(p, t_1, t_2)$ give us the fixed points. Through recurrent application of transformation (6), we get flow lines in the (p, t_1, t_2) space (see Fig. 1). Their structure enables us to numerically establish the approximate critical surface we are looking for. Furthermore, the linearization of Eq. (6) in the neighborhood of every interesting fixed point normally leads to a nonvanishing Jacobian matrix whose largest eigenvalue allows for calculation of the approximate critical exponent ν (associated to the correlation length ξ), and whose eigenvectors reveal the slopes of the approximate critical surface.
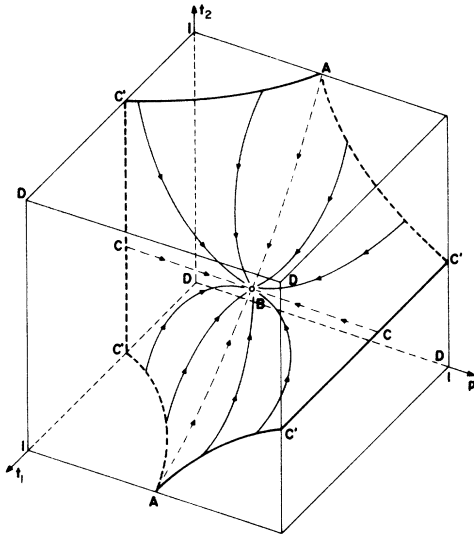
FIG. 1. t-RG, s-RG and heuristic critical surfaces (they are indistinguishable within the present scale); the flow lines correspond to both t-RG and s-RG; the line ABA lies in the plane $p = \frac{1}{2}$; the line CBC corresponds to $t_1 = t_2 = \sqrt{2} - 1$; the surface is invariant under the transformation (p, t_1, t_2) $\rightarrow (1 - p, t_2, t_1)$.

The particular RG we are dealing with provides two interesting physically distinct fixed points (besides the trivial ones $t_1 = t_2 = 0$ and $t_1 = t_2 = 1$, marked D in Fig. 1). The first of those points is the percolation fixed point $(p = \frac{1}{2}, t_1 = 0, t_2 = 1)$, or equivalently $(p = \frac{1}{2}, t_1 = 1, t_2 = 0)$, which is a source (marked A in Fig. 1). The other one is the Ising fixed point $(p = \frac{1}{2}, t_1 = t_2 = \sqrt{2} - 1)$, marked B in Fig. 1, but

which physically is the same as the points C and C', as well as the line segments joining them. This is a saddle point, stable within the critical surface, and unstable out of it. In other words, this situation clearly defines a critical surface, namely, the set of all the points whose flow lines lead to the Ising fixed point B. It is worth remarking that the flowing sense of the flow lines seems to suggest that the critical exponents of the two-dimensional mixed Ising model are those of the pure Ising model for the whole critical surface, with the unique exception of the pure percolation points. However, this is a controversial point because the two-dimensional pure-Ising model is a marginal one in what concerns the Harris criterion,[16] as its specific-heat critical exponent α_t vanishes. The main results of the present critical surface are given in Table I, and we notice that they agree fairly well with the known exact ones. In Table II we present a few points of the three-dimensional curve obtained by intersecting the critical surface with the cylindrical surface $\alpha = 0.2 = J_1/J_2 = \text{arg tanh } t_1/\text{arg tanh } t_2$ (see also Fig. 2).

Let us now compare the present results with those obtained in Refs. 24 and 27. All three treatments lead to the exact results $p_c = \frac{1}{2}$ and $t_c = \sqrt{2} - 1$, but the present one leads to better agreement with exact values for the diluted model (more precisely for $(\partial t_2/\partial p)|_{p=1, t_1=0, t_2=t_c}$ and $(\partial t_2/\partial p)|_{p=p_c, t_1=0, t_2=1}$ as well as for the equal concentration mixed model (where comparison is possible only with Ref. 24). The reason for the better performance of the present RG is that all three parameters (p, t_1, t_2) are allowed to renormalize, whereas only two are allowed to do so in RG2 of Ref. 27 (where $t_1' = t_1 = 0$) and in Ref. 24, (where $\text{argtanh} t_1'/\text{argtanh} t_2' = \text{argtanh} t_1/\text{argtanh} t_2 = \alpha$).

TABLE I. Main results associated with the present three approximate, as well as to the exact critical surfaces [the heuristic and s-RG surfaces preserve the exact relation $t_2 = (1 - t_1)/(1 + t_1)$ for $p = \frac{1}{2}$].

	t_c	$-\dfrac{\partial t_2}{\partial p}\bigg\|_{\substack{p=1 \\ t_1=0}}$	p_c	$-\dfrac{\partial t_2}{\partial p}\bigg\|_{\substack{p=1/2 \\ t_1=0}}$	$-\dfrac{\partial t_2}{\partial t_1}\bigg\|_{\substack{p=1/2 \\ t_1=0}}$
t-RG	$\sqrt{2} - 1$	0.485 28	$\frac{1}{2}$	2.811	2.020
Heuristic	$\sqrt{2} - 1$	$(\ln 2)/\sqrt{2}$ $\simeq 0.490$	$\frac{1}{2}$	$4 \ln 2$	2
s-RG	$\sqrt{2} - 1$	0.485 281 4	$\frac{1}{2}$	2.781	2
Exact	$\sqrt{2} - 1$[a]	$6\sqrt{2} - 8$[b] $\simeq 0.485\ 281\ 4$	$\frac{1}{2}$[c]	$4 \ln 2$[d] $\simeq 2.773$	2[e]

[a]Reference 14.   [d]Reference 18.
[b]Reference 16.   [e]References 19 and 20.
[c]Reference 15.

TABLE II. Comparison of the present three approximate critical lines associated with $\alpha \equiv J_1/J_2 = 0.2$ (see also Fig. 2).

t_2	$k_B T_c/J_2 =$ 1/arg tanht_2	$t_1 =$ tanh (α arg tanh t_2)	p (t-RG)	p (heuristic)	p (s-RG)
$\sqrt{2}-1$	2.269	0.08791	1	1	1
0.45	2.063	0.09664	0.91013	0.91053	0.91014
0.5	1.820	0.10942	0.80412	0.80475	0.80417
0.55	1.617	0.12305	0.71482	0.71546	0.71491
0.6	1.443	0.13775	0.63745	0.63798	0.63758
0.65	1.290	0.15383	0.56852	0.56890	0.56867
0.7	1.153	0.17174	0.50523	0.50541	0.50539
0.75	1.028	0.19217	0.44499	0.44497	0.44516
0.8	0.910	0.21625	0.38491	0.38468	0.38508
0.85	0.796	0.24608	0.32072	0.32029	0.32090
0.9	0.679	0.28622	0.24378	0.24316	0.24394
0.95	0.546	0.35080	0.12563	0.12496	0.12570
0.976	0.454	$\sqrt{2}-1$	0	0	0

Furthermore, universality is automatically preserved if all three parameters renormalize, whereas t_1 (or α) dependence of ν (or any other exponent) is to be expected in the other cases.

The approximate critical exponents obtained within the present RG are $\nu_t \cong 1.149$ [to be compared with the exact value 1], $\nu_p \cong 1.423$ [to be compared with

the possibly exact value $\ln 3/(2 \ln \frac{3}{2}) \cong 1.355$],[32] and finally $\phi = 1$ (which is the exact value). A crossover exponent ϕ equal to unity comes from the fact that the Jacobian matrix of R in the percolation point has its three eigenvalues degenerate. The values we have obtained for ν_p, ν_t, and ϕ have already appeared in previous works,[24,25,27,29,30] as they essentially depend on the cluster chosen to perform decimation and not on the degree of freedom allowed for renormalization in the parameter space.

III. HEURISTIC APPROACH

Let us now present a heuristic reasoning which led nevertheless to a quite accurate approximation for the critical surface we are looking for. It is well known[33] that the critical temperature of the pure rectangular Ising model is given by

$$\sinh(2J_1/k_B T_c) \sinh(2J_2/k_B T_c) = 1 \quad . \tag{7}$$

If we define[27] the dual thermal transmittivity t^D of a given one t by

$$t^D \equiv \frac{1-t}{1+t} \tag{8}$$

then Eq. (7) may be rewritten as

$$t_1 = t_2^D \quad (\text{or } t_1^D = t_2) \quad . \tag{7'}$$

On the other hand the natural definition[31] for the dual occupancy probability p^D of a given one p is

$$p^D \equiv 1 - p \quad . \tag{9}$$

Hence the well-known[15] critical line $p_1 + p_2 = 1$ for anisotropic bond percolation in square lattice may be
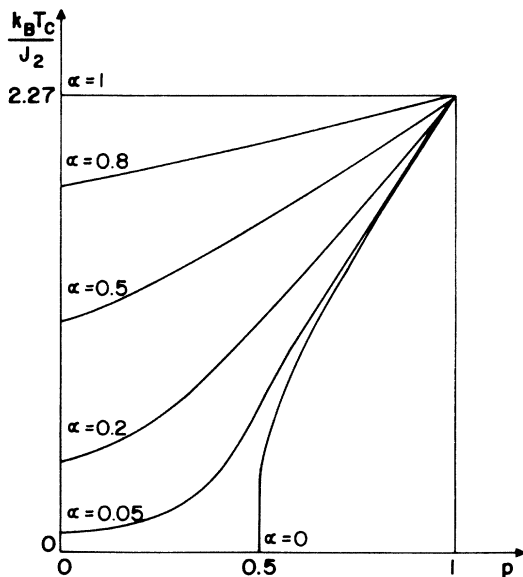

FIG. 2. The critical temperature as a function of J_2-bond concentration for different values of $\alpha \equiv J_1/J_2$ (the t-RG, heuristic and s-RG curves are indistinguishable within the present scale). The ferromagnetic (paramagnetic) phase is stable on the low (high) temperature and high (low) concentration side of the corresponding critical line.

rewritten as

$$p_1 = p_2^D \quad (\text{or} \, p_1^D = p_2) \quad , \tag{10}$$

which puts it onto the same grounds as the above *thermal* critical phenomenon, as described by Eq. (7'). Finally let us recall that the critical line of the equal concentration mixed model is also given[19,20] by Eq. (7'), and that the self-duality of the isotropic square lattice imposes $t_c = t_c^D$ and $p_c = p_c^D$, which immediately lead [through Eqs. (8) and (9)] to the well-known critical values $t_c = \sqrt{2} - 1$ and $p_c = \frac{1}{2}$.

After these preliminary considerations, let us now proceed to rewrite Eq. (7') in the more symmetric form

$$(1 + t_1)(1 + t_2) = 2 \quad ,$$

hence

$$\ln(1 + t_1) + \ln(1 + t_2) = \ln 2 \quad (p = \tfrac{1}{2}) \quad . \tag{7''}$$

Furthermore $t_c = \sqrt{2} - 1$, which leads to

$$\ln(1 + t_1) = (\tfrac{1}{2})\ln 2 \quad (p = 0)$$

and

$$\ln(1 + t_2) = \tfrac{1}{2}\ln 2 \quad (p = 1) \quad .$$

A natural extension of these last two expressions and (7''), applicable to $p \in [0,1]$, is given by

$$(1 - p)\ln(1 + t_1) + p\ln(1 + t_2) = \tfrac{1}{2}\ln 2 \quad . \tag{11}$$

Let us now test this expression in a particular case *completely unrelated* to the cases used to build Eq. (11), namely, the diluted model (that is, $t_1 = 0$). We have

$$p\ln(1 + t_2) = \tfrac{1}{2}\ln 2 \quad ; \tag{11'}$$

hence

$$(\partial t_2/\partial p)_{p=p_c, t_2=1} = -4\ln 2$$

(which is the exact result) and

$$(\partial t_2/\partial p)_{p=1, t_2=t_c} = -(\ln 2)/\sqrt{2}$$

(less than 1% error — see Table I). This performance (which certainly surprised us) is superior to that obtained by an excellent approximate algorithm proposed by Oguchi and Ueno [Eq. (3.5) of Ref. 20], which may be rewritten as

$$(1 - p)t_1 + pt_2 = (1 - p)\frac{1 - t_1}{1 + t_1} + p\frac{1 - t_2}{1 + t_2} \tag{12}$$

which also exactly reproduces the pure and equal concentration models, but leads, in the diluted limit, to

$$\frac{\partial t_2}{\partial p}\bigg|_{p=p_c, t_2=1} = -\frac{8}{3} \quad (3.8\% \; error) \quad ,$$

$$\frac{\partial t_2}{\partial p}\bigg|_{p=1, t_2=t_c} = -\frac{1}{2} \quad (3.0\% \; error) \quad .$$

To develop the reasoning further, we define a new variable s by

$$s \equiv \frac{\ln(1 + t)}{\ln 2} \quad , \tag{13}$$

whence we are led to put [according to Eq. (8)] as dual variable s^D

$$s^D(t) = s(t^D) = 1 - s(t) \quad . \tag{14}$$

Now this is a very interesting property: If we consider the s-probability distribution

$$P(s) = (1 - p)\delta(s - s_1) + p\delta(s - s_2)$$

corresponding to Eq. (2), and its *dual probability distribution*

$$P_D(s) = (1 - p)\delta(s - s_1^P) + p\delta(s - s_2^P) \quad ,$$

Eq. (14) implies that $\langle s \rangle_p + \langle s \rangle_{P_D} = 1$ (see also Ref. 27). In view of all this, Eq. (11) may be written in a natural way as

$$\langle s \rangle_P = \langle s \rangle_{P_D} = \tfrac{1}{2} \quad , \tag{11''}$$

the first of these equalities being justified by the fact that the square lattice is self-dual. Incidentally, we notice that Eq. (12) may also be written as

$$\langle t \rangle_P = \langle t \rangle_{P_D}$$

which is justified on similar grounds, and of which Eq. (9) of Ref. 27 is but a particular case (more precisely the RG3 approximation presented in Ref. 27 corresponds to the diluted-model limit of Oguchi and Ueno's approximation).
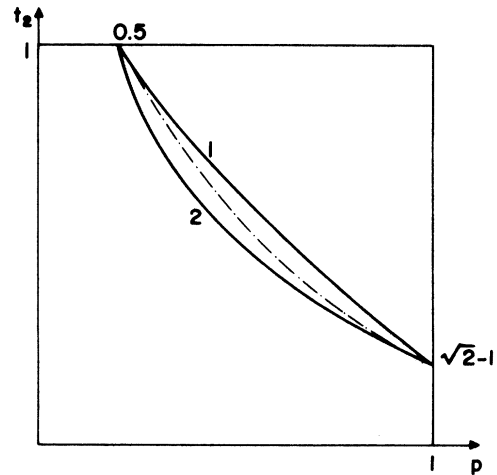
Let us finally remark that Eq. (11'') has the same

FIG. 3. Relative position (out of scale) of the (supposed) exact critical frontier (broken line) and the heuristic (line 1) and s-RG (line 2) frontiers, in the plane $t_1 = 0$ (diluted model).

form as the square-lattice critical line of a certain generalized bond-percolation problem[31] where the aleatory variable (named *fidelity* and noted *a* has a *probabilistic* nature; in particular the dual transformation is $a^D = 1 - a$, in clear analogy with both Eqs. (9) and (14). We have not succeeded in understanding the physical concepts underlying Eq. (11) and its peculiar properties. Nevertheless, there is no doubt that the numerical results describe very closely the exact surface (see Figs. 1 and 2 and Tables I and II). In Fig. 3 we have represented (out of scale) the relative position of the present approximation with respect to the possibly exact result, for the case $t_1 = 0$.

IV. s RENORMALIZATION GROUP AND COMPARISON

We took advantage of the encouraging properties associated to the s variable (and, in particular, the fact that it leads to the *exact* critical line for $p = \frac{1}{2}$, which the t-RG did not), and constructed another RG (noted s-RG) to approximate the critical surface. Equations (5) will be replaced by

$$\langle s(t) \rangle_{P'} = \langle s(t) \rangle_{P_H} \ , \tag{15a}$$

$$\langle [s(t)]^2 \rangle_{P'} = \langle [s(t)]^2 \rangle_{P_H} \ , \tag{15b}$$

$$\langle [s(t)]^3 \rangle_{P'} = \langle [s(t)]^3 \rangle_{P_H} \ , \tag{15c}$$

where P' and P_H are still those, respectively, given by Eqs. (3) and (4). If we take into account the one-to-one analytic relation between s and t, it is clear that, if the distributions had enough parameters to allow the imposition of infinitely many equations (instead of only three), the t-RG and the s-RG would have led to one and the same transformation. But as only a finite number of relations are demanded, then there is certainly no reason for t-RG and s-RG to be the same, and indeed they are not. In fact, although s-RG leads to a structure of fixed points and flow lines much the same as that of t-RG (see Fig. 1), the present surface is *closer* to the supposed exact result (see Table I).

It has already been remarked that, if a point (p,t_1,t_2) belongs to the exact critical surface, the complementary point $(1 - p,t_2,t_1)$ also belongs to it. This is an immediate consequence of the definitions. Now it is also to be expected that the point (p,t_1^p,t_2^p) [see Eq. (8)] will also be on the exact surface, since the lattice is self-dual. In particular, this is the case for $p = \frac{1}{2}$, whence Eq. (7'). In this way, only one region of the (p,t_1) plane is relevant in the determination of t_2 for the critical surface; for instance, the region $(t_1 \leq \sqrt{2} - 1, \ p \geq \frac{1}{2})$. The other regions can be immediately analyzed, once we have the results for

the first one; it suffices to apply the transformations

$$(p,t_1,t_2) \rightarrow (p,t_1^p,t_2^p)$$

$$\rightarrow (1 - p,t_2,t_1)$$

$$\rightarrow (1 - p,t_2^p,t_1^p) \ .$$

For this reason the comparison will be made only for the region $(t_1 \leq \sqrt{2} - 1, \ p \geq \frac{1}{2})$.

Our three approximate critical surfaces and the exact one share the pure percolation points (A, in Fig. 1), and the pure Ising lines (lines $C'C$ and CB in Fig. 1). The s-RG and heuristical approximations satisfy the transformation $(p,t_1,t_2) \rightarrow (p,t_1^p,t_2^p)$, and in particular the equal concentration mixed line (AB in Fig. 1), but the t-RG does not. The exact surface shares, with the heuristic one, the slopes near the pure percolation points, and (within an error inferior to 10^{-7}) with the s-RG one, the slopes in the pure Ising points (C' in Fig. 1). See in Fig. 3 (out of scale) how the heuristic, the s-RG and the supposed exact surfaces cut the plane $t_1 = 0$.

In the region considered, we may order the critical surfaces in a simple manner: Starting in Fig. 1 from the plane $t_2 = 1$ and going towards the plane $t_2 = 0$, we first cross the heuristic critical surfce, then the supposed exact one, then comes the s-RG and finally the t-RG ones.

Let us now discuss a point which no doubt deserves analysis: We have assumed all the time that the *whole* unknown exact critical surface is being bounded (by above or below) by our successive approximations, and more precisely that it lies between the heuristic and the s-RG ones. What are the facts on which we have built our conviction, in spite of the absence of a rigorous proof? To answer this question let us first introduce the definitions

$$\tau_1 \equiv \frac{\partial t_2}{\partial p} \bigg|_{p=1, t_1=0, t_2=t_c}$$

and

$$\tau_2 \equiv \frac{\partial t_2}{\partial p} \bigg|_{p=p_c, t_1=0, t_2=1} \ .$$

A greek-letter superscript indicates the various approaches we have been considering: RG1 to RG6 of Ref. 27, the present t-RG, heuristical and s-RG approximations, Oguchi and Ueno's approximation [see Eq. (12)], and finally the exact solution. Our reasons are as follows: (a) Whenever $\tau_1^\gamma < \tau_1^\delta$ ($\tau_2^\gamma < \tau_2^\delta$) we have $\tau_2^\gamma \geq \tau_2^\delta$ ($\tau_1^\gamma \geq \tau_1^\delta$); this fact allows us to order *all* the approaches (the exact as well as the approximate ones); (b) whenever the limiting slopes τ_1 and τ_2 suggested that a given critical line (or surface) was, let us say, below another one [in the region $(t \leq \sqrt{2} - 1, \ p \geq \frac{1}{2})$], this was indeed verified (in *all* the approximative approaches) for the *whole* line (or surface) in that region; (c) last but not least,

TABLE III: Typical values of the critical surface $t_2(p,t_1)$ [top number: $t_2(s$-RG); bottom number: $\bar{t}_2$, where $\bar{t}_2$ is our best proposal].

$p\backslash t_1$	0	0.1	0.2	0.3	0.4	$\sqrt{2}-1$	0.5	0.6	0.7	0.8	0.9	1
0.5	1	0.8182	0.6667	0.5385	0.4286	$\sqrt{2}-1$	0.3333	0.2500	0.1765	0.1111	0.0505	0
	1	0.8182	0.6667	0.5385	0.4286	$\sqrt{2}-1$	0.3333	0.2500	0.1765	0.1111	0.0505	0
0.6	0.7811	0.6717	0.5777	0.4959	0.4238	$\sqrt{2}-1$	0.3598	0.3026	0.2511	0.2044	0.1619	0.1229
	0.7817	0.6720	0.5778	0.4959	0.4238	$\sqrt{2}-1$	0.3598	0.3025	0.2509	0.2042	0.1616	0.1225
0.7	0.6397	0.5745	0.5172	0.4662	0.4203	$\sqrt{2}-1$	0.3790	0.3414	0.3071	0.2757	0.2461	0.2197
	0.6403	0.5748	0.5173	0.4662	0.4203	$\sqrt{2}-1$	0.3790	0.3414	0.3070	0.2755	0.2463	0.2193
0.8	0.5414	0.5055	0.4734	0.4443	0.4178	$\sqrt{2}-1$	0.3935	0.3713	0.3508	0.3317	0.3141	0.2975
	0.5417	0.5057	0.4734	0.4443	0.4178	$\sqrt{2}-1$	0.3935	0.3713	0.3507	0.3316	0.3139	0.2972
0.9	0.4693	0.4541	0.4402	0.4275	0.4158	$\sqrt{2}-1$	0.4050	0.3950	0.3857	0.3770	0.3688	0.3612
	0.4694	0.4541	0.4402	0.4275	0.4158	$\sqrt{2}-1$	0.4050	0.3950	0.3856	0.3769	0.3688	0.3611
1	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$
	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$

Thorpe and McGurn[26] also state (certainly for quite independent reasons), without general proof, that, in this kind of problem, the limiting slopes so strongly tie the critical line (or surface) that nothing happens but a smooth variation linking the extremities.

For these reasons, we can state, with a certain degree of confidence, that the exact surface lies between our heuristic and s-RG critical surfaces. Furthermore, let us recall that the heuristic one approaches very closely the exact surface in the neighborhood of $p = \frac{1}{2}$ (as suggested by the derivatives), and that the same thing happens with the s-RG one in the neighborhood of $p = 1$. So, we adopt, as a final *numerical* proposal, with no pretension of *analytic* exactitude, the following linear interpolation for all values of p and t_1 for which there is a phase transition:

$$\bar{t}_2 = 2[(1-p)t_2(\text{heuristic}) + (p - \tfrac{1}{2})t_2(s-\text{RG})] \ .$$

$$(16)$$

Typical numerical results for this last proposal are presented in Table III (below those associated to s-RG). We estimate the error of $\bar{t}_2$ to be inferior to 3×10^{-4} for all pairs (p,t_1).

V. CONCLUSION

We have developed a *numerical* proposal for the exact critical surface of the isotropic homogeneous quenched bond-mixed square-lattice spin-$\frac{1}{2}$ first-neighbor-interaction ferromagnetic Ising model. The high accuracy that has been obtained (see typical values in Table III), with an estimated error less than 3×10^{-4} in the t variable, confirms that adequate use

of renormalization-group techniques provides a powerful tool for constructing numerical approximations of critical quantities, and its performance may compete with the best standard approximation procedures (low- and high-temperature series, Monte Carlo methods, etc.). Let us, however, recall that, in what concerns the critical exponents ν_p and ν_t, the precision obtained in the present work was very poor (6% error in ν_p and 15% error in ν_t); to improve in this sense we must consider larger cells (see for instance Ref. 30).

Our best proposal $[\bar{t}_2(p,t_1)]$ simultaneously satisfies all the available exact results [namely, $p_c = \frac{1}{2}$,[15] $t_c = \sqrt{2}$ -1,[14] $(\partial t_2/\partial p)|_{p=p_c, t_1=0} = -4$ ln2,[18] $(\partial t_2/\partial p)|_{p=1, t_1=0}$ $= 8 - 6\sqrt{2}$,[16] and[19,20] $t_1 = (1 - t_2)/(1 + t_2)$ for $p = \frac{1}{2}$]. However, if a very high accuracy is not required, the simple *analytic* relation [Eq. (11)]

$$(1-p)\ln(1+t_1) + p\ln(1+t_2) = \tfrac{1}{2}\ln2$$

provides an excellent approximation to the critical surface we are interested in, as it satisfies all the above quoted exact results with the unique exception of $(\partial t_2/\partial p)|_{p=1, t_1=0}$, where it has an error inferior to 1%: these facts result in an error for t_2 (or t_1) inferior to 10^{-3} for any point in the surface.

As a last comment let us say that the present results exhibit the importance, within the renormalization-group framework, of allowing the parameters of the problem to have the maximum possible freedom for renormalization, in order to avoid undesirable artificial effects either on the critical lines or surfaces, or on the critical exponents (apparent loss of universality).

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