Critical amplitude ratio of the confluent singular term for the specific heat: Calculations to order ϵ^2 for systems with continuous symmetry

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The critical amplitude ratio of the confluent singular term for the specific heat has been shown to be universal, and the value has been calculated to order ϵ^2 for systems with continuous symmetry using renormalized ϕ^4 theory. The theoretical results are in good agreement with Ahler's experimental values at the superfluid transition in ^{4}He.

I. INTRODUCTION

One of the major advances in condensed matter physics in recent years has been the development of the renormalization-group theory of critical phenomena.[1] The theory provides both a qualitative and quantitative description of the singularities which appear at critical points. For example, the theory predicts that on the coexistence curve, in the asymptotic critical region, a typical thermodynamic property of the system can be written

$$f(t)=f_0|t|^{-\lambda}(1+g_0|t|^{x}) \quad , \tag{1.1}$$

where t is the reduced temperature measured from T_c, λ and x are critical exponents, and f_0 and g_0 are the amplitudes of the leading and confluent singular terms, respectively. A major accomplishment of the theory was the confirmation of the hypothesis of universality,[2] which greatly reduces the complexity of critical phenomena by dividing systems into a small number of equivalence classes. Within each class the critical exponents are the same, the systems even share a common equation of state once two thermodynamic scales "temperature" and "magnetization" have been chosen. Later the hypothesis of two-scale-factor universality,[3] which postulates that the length scale is universally related to the two thermodynamic scales was established,[4] and it was shown that just as there are 12 critical exponents and 10 relations among them there are 12 fundamental leading critical amplitudes and 10 universal relations among them.[5]

Now, when critical exponents are known with high accuracy, via ϵ expansion and knowledge of the asymptotic behavior of perturbation series,[6] when universal relations between critical amplitudes are known and have been calculated at least to second order in $\epsilon=4-d$[5], we are almost in a position to make a detailed comparison between theory and experiment. There is one major stumbling block: due to the finite resolution of experiment the confluent singular term shown in Eq. (1.1) must be taken into account in the data analysis.[7] The presence and origin of this term was first understood from renormalization-group analysis.[8] Although the "correction to scaling" exponent x, which was first calculated in a ϵ expansion by Wegner,[8] is now known with high accuracy, little work has been done on the correction to scaling amplitudes.[9]

In this paper we use renormalized perturbation theory[10,11] to prove universality of the ratio of the correction to scaling amplitudes above and below T_c and calculate the ratio to order ϵ^2.

The paper is organized as follows. In Sec. II we review the basic ideas of renormalized perturbation theory and discuss the origin of the confluent singular term. In Sec. III the relation between renormalized perturbation theory and thermodynamics is explained, and it is shown that the specific heat on the coexistence curve can be written in the form Eq. (1.1) with an additional additive constant which is the same above and below T_c. The universality of the ratio of the correction to scaling amplitudes, D^+/D^- is established. The detailed calculation of D^+/D^- is given in Sec. IV.

II. RENORMALIZED PERTURBATION THEORY

In this section we review, briefly, renormalized perturbation theory and its relation to the theory of critical phenomena.[10,11] In the critical region the true Hamiltonian may be replaced by a Ginzburg-Landau effective Hamiltonian,[11]

$$\beta H=\int d^dx\, H(\vec{x}) \quad ,$$
$$H(\vec{x})=\tfrac{1}{2}[\vec{\nabla}\vec{\phi}(\vec{x})]^2+\tfrac{1}{2}\mu^2\phi^2(\vec{x})+\frac{\lambda}{4!}[\phi^2(\vec{x})]^2 \quad , \tag{2.1}$$

where $\vec{\phi}(\vec{x})$ is an n-component local vector field whose statistical average is the order parameter of our problem.

$$\phi^2(\vec{x}) = \sum_{i=1}^{n} \phi_i^2(\vec{x}) \ ,$$

$$[\vec{\nabla}\phi(\vec{x})]^2 = \sum_{k=1}^{d} \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_k} \phi_i(\vec{x}) \right)^2 \ .$$

The "bare mass" μ^2 is related to the temperature by $\mu^2 \sim T - T_0$, where T_0 is the mean-field transition temperature; λ is the bare-coupling constant. In momentum space there is an ultraviolet cutoff $\Lambda \simeq 1/a$ where a is the lattice spacing in the original problem.

In renormalized perturbation theory the one-particle irreducible (1PI) Green's functions $\Gamma^{(L,N)}(p_1 \cdots p_L, q_1 \cdots q_N; \mu^2, \Lambda, \lambda, \overline{\phi})$, which contain N ϕ fields and L ϕ^2 fields are renormalized in such a way that the corresponding renormalized functions $\Gamma_R^{(L,N)}$ are finite in the infinite cutoff limit when the space dimensionality $d \lesssim 4$. The magnetization $\overline{\phi}$ is zero when $T > T_c$ (the critical temperature), and the magnetic field H is set equal to zero. All $\Gamma^{(L,N)}$ can be renormalized multiplicatively except $\Gamma^{(2,0)}$, which requires additional additive renormalization. The relations between $\Gamma^{(L,N)}$ and $\Gamma_R^{(L,N)}$ are

$$\Gamma_R^{(L,N)}(p_1 \cdots p_L, q_1 \cdots q_N; t, \Lambda, g, \kappa, M) = Z_{\phi^2}^L Z_\phi^{N/2} \Gamma^{(L,N)}(p_1 \cdots p_L, q_1 \cdots q_N; \mu^2, \Lambda, \lambda, \overline{\phi}) \ , \quad (2.2)$$

$$\Gamma_R^{(2,0)}(p, -p; t, \Lambda, g, \kappa, M) = Z_{\phi^2}^2 [\Gamma^{(2.0)}(p, -p; \mu^2, \Lambda, \lambda, \overline{\phi}) - \Gamma^{(2.0)}(p, -p; \lambda, \Lambda)|_{p^2=\kappa^2}] \ . \quad (2.3)$$

Here t, g, and M are renormalized temperature, coupling constant, and magnetization, respectively, κ is an arbitrary momentum scale, and Z_ϕ and Z_{ϕ^2} are renormalization constants. The subtraction term $\Gamma^{(2,0)}(p, -p; \lambda, \Lambda)|_{p^2=\kappa^2}$ in Eq. (2.3) is evaluated at $T = T_c$. The renormalization constants and bare-coupling constant are computed in the massless theory ($t = 0$) as power series in g from the renormalization conditions

$$\Gamma_R^{(0,2)}(p, -p; t=0, \Lambda, g, \kappa, M=0)|_{p^2=0} = 0 \ , \quad (2.4)$$

$$\frac{\partial}{\partial p^2} \Gamma_R^{(0,2)}(p, -p; 0, \Lambda, g, \kappa, 0)|_{p^2=e^l\kappa^2} = 1 \ , \quad (2.5)$$

$$\Gamma_R^{(0,4)}(p_i; 0, \Lambda, g, \kappa, 0)|_{s,p} = g \ , \quad (2.6)$$

$$\Gamma_R^{(2,1)}(k_1, k_2, p; 0, \Lambda, g, \kappa, 0)|_{\overline{s,p}} = 1 \ , \quad (2.7)$$

$$\Gamma_R^{(2,0)}(p, -p; 0, \Lambda, g, \kappa, 0)|_{p^2=\kappa^2} = 0 \ , \quad (2.8)$$

where s,p means $p_i \cdot p_j = \frac{1}{4}\kappa^2(4\delta_{ij} - 1)$; $\overline{s,p}$ means $k_i^2 = \frac{3}{4}\kappa^2$, $\vec{k}_1 \cdot \vec{k}_2 = -\frac{1}{4}\kappa^2$, and $p^2 = (\vec{k}_1 + \vec{k}_2)^2 = \kappa^2$. The arbitrary dimensionless parameter l characterizes the renormalization scheme.

When the temperature is not exactly at T_c we may write

$$\mu^2 = \mu_c^2 + (\mu^2 - \mu_c^2) = \mu_c^2 + \delta\mu^2 \ , \qquad \mu_c^2 = T_c - T_0 \ , \quad (2.9)$$

and expand $\Gamma^{(L,N)}$ around μ_c^2 and $\overline{\phi} = 0$

$$\Gamma^{(L,N)}(k_i, p_i; \mu^2, \phi, \lambda, \Lambda) = \sum_{I,J} \frac{(\delta\mu^2)^I (\overline{\phi})^J}{I!J!} \Gamma^{(L+I,N+J)}(k_i, l_i = 0, p_i, q_i = 0; \mu_c^2, \overline{\phi} = 0, \lambda, \Lambda) \ . \quad (2.10)$$

Multiplying $\Gamma^{(L+I,N+J)}$ by $Z_{\phi^2}^{L+I} Z_\phi^{(N+J)/2}$ will make it finite as $\Lambda \to \infty$. If we define

$$\delta\mu^2 = Z_{\phi^2} t \ , \qquad \overline{\phi} = Z_\phi^{1/2} M \ , \quad (2.11)$$

it follows that:

$$\Gamma_R^{(L,N)}(k_i, p_i; t, M, g, \Lambda) = Z_{\phi^2}^L Z_\phi^{N/2} \Gamma^{(L,N)}(k_i, p_i; \mu^2, \overline{\phi}, \lambda, \Lambda) \ , \quad (2.12)$$

which is finite order by order in g as $\Lambda \to \infty$. The shift in transition temperature which may be calculated directly from Eq. (2.4) as a power series in λ, is shown in Fig. 1. In the actual calculation all Feynman diagrams are calculated in dimensionally regularized form.[12] If we note that the dimensional regularization of the integral of power is zero,[11] it follows that $\mu_c^2 = 0$, and could therefore be set equal to zero initially, as we have anticipated in Eq. (2.12).

The renormalized 1PI functions therefore are calculated as follows: first we expand $\Gamma^{(L,N)}(\vec{p}, \vec{q}; \lambda, \overline{\phi}, \mu^2)$, $(L,N) \neq (2,0)$, $(1,0)$ and $(0,1)$, as a power series in λ then transform from λ to g and μ^2 to t and $\overline{\phi}$ to M by Eq. (2.11); finally we multiply $\Gamma^{(L,N)}$ by $Z_{\phi^2}^L$, Z_ϕ^N calculated from the massless theory ($t = 0$).

The critical behavior of the theory may be studied via the renormalization-group equation which is obtained by differentiating Eqs. (2.2) and (2.3)

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}-\tfrac{1}{2}\eta(u)\left(N+M\frac{\partial}{\partial M}\right)-\left(\frac{1}{\nu(u)}-2\right)\left(L+t\frac{\partial}{\partial t}\right)\right]\Gamma_R^{(L,N)}(q,p;t,M)=\delta_{N,0}\delta_{L,2}\kappa^{-\epsilon}B(u) \ , \quad (2.13)$$

where $u=\kappa^{-\epsilon}g$ is the dimensionless coupling constant

$$\beta(u)=-\epsilon\left(\frac{\partial \ln u_0}{\partial u}\right)^{-1}, \quad u_0=\kappa^{-\epsilon}\lambda \ , \quad (2.14)$$

$$\frac{1}{\nu(u)}-2=\beta(u)\frac{\partial \ln Z_{\phi^2}}{\partial u} \ , \quad (2.15)$$

$$\eta(u)=\beta(u)\frac{\partial \ln Z_\phi}{\partial u} \ . \quad (2.16)$$

The right-hand side of Eq. (2.13), which is due to additive renormalization, will be calculated explicitly in Sec. IV.

The homogeneous solution of Eq. (2.13) may be derived in terms of an arbitrary parameter ρ.

$$[\rho d/d\rho-\tfrac{1}{2}N\eta(u(\rho))-[\nu^{-1}(u(\rho))-2]L]\times\Gamma_R^{(L,N)}(\rho_i;t(\rho),u(\rho),\kappa(\rho))=0 \ , \quad (2.17)$$

where

$$\kappa(\rho)=\kappa\rho \ , \quad (2.18)$$

$$\ln\rho=\int_u^{u(\rho)}\frac{du'}{\beta(u')} \ , \quad (2.19)$$

$$t(\rho)=t\exp\left[-\int_u^{u(\rho)}\left(\frac{1}{\nu(u')}-2\right)\frac{du'}{\beta(u')}\right] \ , \quad (2.20)$$

$$M(\rho)=M\exp\left(-\tfrac{1}{2}\int_u^{u(\rho)}\frac{\eta(u')}{\beta(u')}du'\right) \ , \quad (2.21)$$

$$u(\rho=1)=u \ , \quad (2.22)$$

then from dimensional analysis it follows that:

$$\Gamma_R^{(L,N)}(p_i;t,u,\kappa,M)=(\rho\kappa)^{d-2L-N(d-2)/2}\left(\frac{M(\rho)}{M}\right)^N\left(\frac{t(\rho)}{t}\right)^L\Gamma_R^{(L,N)}\left(\frac{p}{\rho\kappa};\frac{t(\rho)}{\rho^2\kappa^2},\frac{M(\rho)}{(\rho\kappa)^{d/2-1}},u(\rho),1\right) \ . \quad (2.23)$$

We fix ρ by requiring

$$\frac{t(\rho)}{\rho^2\kappa^2}=1 \ . \quad (2.24)$$

When $t\to0$, from Eq. (2.20) $t(\rho)\to0$ also; hence, from Eq. (2.24) the critical domain corresponds to $\rho\to0$; further, from Eq. (2.19), if $\beta(u)$ has a nontrivial zero (fixed point) u^* with positive derivative at u^*, the $\rho\to0$ implies $u(\rho)\to u^*$. Introducing the scales[5,11]

$$X(\rho,u)=\exp\left(-\int_u^{u(\rho)}[\nu^{-1}(u)-\nu^{-1}]/\beta(u')\,du'\right) \ , \quad (2.25)$$

$$Y(\rho,u)=\exp\left(-\tfrac{1}{2}\int_u^{u(\rho)}[\eta(u')-\eta]/\beta(u')\,du'\right) \ ,$$

where $\nu(u^*)=\nu$ and $\eta(u^*)=\eta$ are the usual critical exponents, it follows that:

$$\rho=[X(\rho,u)t]^\nu=\tilde{t}^{\,\nu} \quad (2.26)$$

and

$$\Gamma_R^{(L,N)}(p;t,M,u)=Y(\rho,u)^N X(\rho,u)^L\tilde{t}^{\,-\nu(d-N(d-2+\eta)/2)-L}\times\Gamma_R^{(L,M)}(p\tilde{t}^{\,-\nu};1,x,u(\rho),\kappa=1) \ , \quad (2.27)$$

where

$$x=M(\rho)/\rho^{d/2-1}=Y(\rho,u)M\tilde{t}^{\,-\nu(d-2+\eta)/2} \quad (2.28)$$

is zero when $T>T_c$, $H=0$ and is given by the solution to the equation of state when $T<T_c$, $H=0$. The running coupling constant is easily found by solving Eq. (2.19) in the vicinity of the fixed point as

$$\frac{d\ln\rho}{du(\rho)}=\frac{1}{\omega[u(\rho)-u^*]}, \quad \omega=\beta'(u^*) \quad (2.29)$$

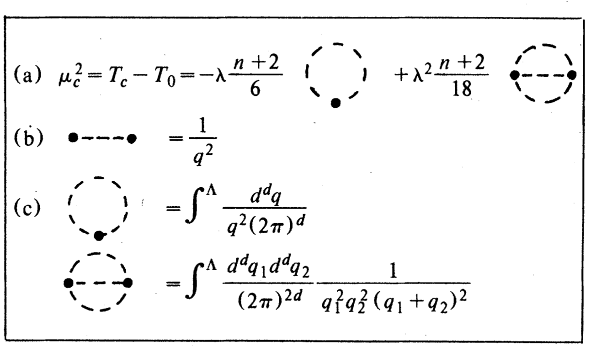


FIG. 1. (a) Feynman diagrams for μ_c^2. (b) Feynman propagator used in (a). (c) Feynman integrals for (a) with a cutoff Λ.

and therefore

$$u(\rho)=u^*+(u-u^*)\rho^{\omega} \ . \tag{2.30}$$

Finally we may now write the 1PI functions in the vicinity of the critical point including the correction to scaling arising from the confluent singularity produced by Eq. (2.30) as

$$\Gamma_R^{(L,N)}(p,t,M,u) = Y^N X^L \tilde{t}^{-\nu(d-N(d-2+\eta)/2)-L}\Gamma_R^{(L,N)}(p\tilde{t}^{-\nu};1,x,u^*,\kappa=1)$$
$$\times\left[1+(u-u^*)\tilde{t}^{\omega\nu}\left\{\frac{\partial\Gamma_R^{(L,N)}(p\tilde{t}^{-\nu};1,x,u,\kappa=1)/\partial u|_{u^*}}{\Gamma_R^{(L,N)}(p\tilde{t}^{-\nu};1,x,u^*,\kappa=1)}\right.\right.$$
$$\left.\left.-\frac{\frac{1}{2}N(\partial\eta(u)/\partial u)|_{u^*}+\nu(d-\frac{1}{2}N(d-2+\eta))(\partial\nu^{-1}(u)/\partial u)|_{u^*}}{\omega}\right\}\right] \ . \tag{2.31}$$

The second term in Eq. (2.31) comes from expanding the scales $X(\rho,u)$ and $Y(\rho,u)$ around $u(\rho)=u^*$, and X and Y are defined by $X=\lim_{\rho\to 0,u(\rho)\to u^*}$ and $Y=\lim_{\rho\to 0,u(\rho)\to u^*}Y(\rho,u)$, respectively. It is important to note that the only remnant of microscopic behavior is in the u dependence of the scale factors, X and Y, and in the confluent singular term $(u-u^*)$. Any dimensionless quantity independent of these factors will therefore be universal provided it can be shown to be independent of renormalization scheme.

III. SPECIFIC HEAT

In this paper we are concerned with the specific heat on the coexistence curve $H=0$, in particular with the ratio of the amplitudes of the confluent singular term above and below the transition. In this section we show how the specific heat may be calculated using the functional formalism outlined in Sec. II. We start from the generating functional of the connected Green's functions $F(H)=\ln Z(H)$

$$Z(H)=\int D(\phi)\exp\left\{-\int d^dx\left[\frac{1}{2}\{[\vec{\nabla}\vec{\phi}(\vec{x})]^2+\mu^2\phi^2(\vec{x})\}+\frac{\lambda}{4!}\left(\sum_i\phi_i^2(\vec{x})\right)^2-\vec{H}(\vec{x})\vec{\phi}(\vec{x})\right]\right\} , \tag{3.1}$$

$$\bar{\phi}(\vec{x})=\frac{\delta F(H)}{\delta H(\vec{x})} , \tag{3.2}$$

using a Legendre transform[10,11] we may define the potential $\Gamma(\bar{\phi})$,

$$\Gamma(\bar{\phi})+F(H)=\int\bar{\phi}(\vec{x})\cdot\vec{H}(\vec{x})\,d^dx , \tag{3.3}$$

$$\vec{H}(\vec{x})=\frac{\delta\Gamma(\phi)}{\delta\bar{\phi}(\vec{x})} , \tag{3.4}$$

which has been shown to be the generating function for the 1PI functions.[13] $\Gamma(\bar{\phi})$ can be calculated explicitly both above and below T_c in terms of a loop expansion. The relationship with thermodynamics is made by noting that, up to a factor of (kT), $-\ln Z(H)$ is the magnetic Gibbs' potential and therefore $\Gamma(\bar{\phi})$ is the Helmholtz free energy. The usual Legendre tranformation is simply a special case, uniform H, of the functional relationship given above. In the critical region the most singular term in the specific heat at constant H may therefore be written

$$C_H=\left.\frac{\partial^2F(H)}{\partial T^2}\right|_H , \tag{3.5}$$

here factors of T_c, which do not affect the subsequent discussion have been absorbed into C_H. When $T>T_c$ and $H=0$, $\bar{\phi}$, is constant, and therefore from Eqs. (3.3) and (3.5) we have immediately

$$C_H(T>T_c)=-\left.\frac{\partial^2\Gamma(\bar{\phi},\mu^2)}{\partial T^2}\right|_{\bar{\phi}=0} , \tag{3.6}$$

which, using Eq.(2.9) can be written

$$C_H(T>T_c)=-\left.\frac{\partial^2\Gamma(\bar{\phi},\mu^2)}{\partial(\mu^2)^2}\right|_{\bar{\phi}=0}$$
$$=-\Gamma^{(2.0)}(\bar{\phi}=0,\mu^2) \ . \tag{3.7}$$

On the other hand, when $T<T_c$, $\bar{\phi}=\bar{\phi}(\mu^2)$ at fixed H. First we must find the magnetization $\bar{\phi}$ by solving the the equation of state on the coexistence curve, then, using the Legendre transform (3.3) and (3.5), it follows that:

$$C_{H=0}(T<T_c)=-\frac{d^2\Gamma(\bar{\phi}(\mu^2),\mu^2)}{d(\mu^2)^2} \ . \tag{3.8}$$

The connection with renormalized perturbation theory is now readily made from Eq. (2.3)

$$\Gamma_R^{(2,0)}(M,t)=\frac{\partial^2\Gamma_R(M,t)}{\partial t^2}$$
$$=Z_{\phi^2}^2\Gamma^{(2,0)}(\bar{\phi};\mu^2)$$
$$-Z_{\phi^2}^2\Gamma^{(2,0)}(p,-p;\lambda,\Lambda)\Big|_{p^2=\kappa^2}\ , \qquad (3.9)$$

which may be integrated to give the renormalized potential as

$$\Gamma_R(M,t)=\Gamma(\bar{\phi}=Z_\phi^{1/2}M,\,\delta\mu^2=Z_{\phi^2}t)$$
$$-\tfrac{1}{2}t^2Z_{\phi^2}^2\Gamma^{(2,0)}(p,-p;\lambda,\Lambda)\Big|_{p^2=\kappa^2}\ . \qquad (3.10)$$

The calculation, details of which are given in Sec. IV, proceeds by calculating the potential Γ via a loop expansion. Then introducing a renormalized specific heat

$$C_{H=0}^R(T>T_c)=-\frac{\partial^2\Gamma_R(M=0,t)}{\partial t^2}$$
$$=-\Gamma_R^{(2,0)}(M=0,t) \qquad (3.11)$$

and

$$C_{H=0}^R(T<T_c)=-\frac{d^2\Gamma_R(M(t),t)}{dt^2}\Big|_{H=0}\ , \qquad (3.12)$$

where $M(t)$ is obtained from

$$H=0=\frac{\partial\Gamma_R(M,t)}{\partial M}\ . \qquad (3.13)$$

The specific heat may be obtained from

$$C_H^R(M,t)=Z_{\phi^2}^2C_H(\bar{\phi},\mu^2)$$
$$+Z_{\phi^2}^2\Gamma^{(2,0)}(p,-p;\lambda,\Lambda)\Big|_{p^2=\kappa^2}\ . \qquad (3.14)$$

We conclude this section with a discussion of the critical behavior which is handled in the usual way via the renormalization-group equation. For $T>T_c$ as we have seen Eq. (2.13), $C_H^R(T>T_c)$ satisfies an inhomogeneous renormalization-group equation.

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}-\left(\frac{1}{\nu(u)}-2\right)\left(2+t\frac{\partial}{\partial t}\right)\right]C_H^R(T>T_c)=\kappa^{-\epsilon}B(u)=\left[\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}-2\left(\frac{1}{\nu(u)}-2\right)\right]$$
$$\times Z_{\phi^2}^2\Gamma^{(2,0)}(p,-p;\lambda,\Lambda)\Big|_{p^2=\kappa^2}\ . \qquad (3.15)$$

The right-hand side of Eq. (3.15) will be calculated in Sec. IV. If we write $C_H^R=C_h+C_p$ where C_h and C_p are the homogeneous solution and particular integral of Eq. (3.15), respectively, then the homogeneous solution can be derived as in Eqs. (2.17) to (2.31). We have

$$C_h(t,M=0,u,\kappa)=X^2\tilde{t}^{-\alpha}C_h(1,0,u^*,\kappa=1)\left[1+(u-u^*)t^{\omega\nu}\left(\frac{\partial C_h/\partial u|_{u^*}}{C_h(u^*)}-\frac{(2-\alpha)}{\omega}\frac{\partial\nu^{-1}(u)}{\partial u}\Big|_{u^*}\right)\right]\ , \qquad (3.16)$$

where κ has been set equal to one for convenience. Here we note that as

$$\frac{\partial C_h}{\partial u}\Big|_{u^*}=\frac{\partial C_H^R}{\partial u}\Big|_{u^*}-\frac{\partial C_p}{\partial u}\Big|_{u^*}\ , \qquad (3.17)$$

the evaluation of the amplitude of the confluent singular term will only involve $\partial C_p/\partial u|_{u^*}$ rather than the complete particular solution. This calculation will be carried out in Sec. IV. From Eqs. (3.15) and (3.16) we can express $\lim_{t\to 0+}C_H(t,M=0,u,\kappa)$ as

$$C_H^R(t\to 0+)=A^+\tilde{t}^{-\alpha}(1+D^+\tilde{t}^{\omega\nu})+B^+\ , \qquad (3.18)$$

where

$$\tilde{t}=Xt\ , \qquad (3.19)$$

$$A^+=X^{2-\alpha}C_h(1,0,u^*,\kappa=1)\ , \qquad (3.20)$$

$$D^+=(u-u^*)\left(\frac{\partial C_h/\partial u|_{u^*}}{C_h(u^*)}-\frac{(2-\alpha)}{\omega}\frac{\partial\nu^{-1}(u)}{\partial u}\Big|_{u^*}\right)X^{\omega\nu}, \qquad (3.21)$$

$$B^+=C_p(\kappa,u)\ . \qquad (3.22)$$

For $T<T_c$ we must show that $C_H^R(T<T_c,M)$ sat-

isfies the same renormalization-group equation Eq. (3.15). As noted

$$C_H^R(T<T_c,M)=Z_{\phi^2}^2 C_H(\bar\phi,\mu^2)+Z_{\phi^2}^2\Gamma^{(2,0)}(p,-p;\lambda,\Lambda)|_{p^2=\kappa^2} . \quad (3.23)$$

Expanding each side about $M=t=0$ and $\bar\phi=0$, $\mu^2=\mu_c^2$, respectively,

$$\sum_{N,J} C_H^{R(N,J)}\frac{t^N M^J}{N!J!}=Z_{\phi^2}^2 C^{(N,J)}\frac{(\delta\mu^2)^N\bar\phi^J}{N!J!} , \quad (3.24)$$

where $C^{(N,J)}$ are the expansion coefficients of the right-hand side of Eq. (3.23). Noting that $\delta\mu^2=Z_{\phi^2}t$ and $\bar\phi=Z_\phi^{1/2}M$, we have

$$C^{(N,J)}=Z_{\phi^2}^{-(2+N)}Z_\phi^{-J/2}C_H^{R(N,J)} . \quad (3.25)$$

Taking the logarithmic derivative of both sides of the equation with respect to scale change

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}-\tfrac{1}{2}J\eta(u)-\left(\frac{1}{\nu(u)}-2\right)(2+N)\right]C_H^{R(N,J)}=\delta_{N,0}\delta_{J,0}\kappa^{-\epsilon}B(u) \quad (3.26)$$

and therefore

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}-\tfrac{1}{2}\eta(u)M\frac{\partial}{\partial M}-\left(\frac{1}{\nu(u)}-2\right)\left(2+t\frac{\partial}{\partial t}\right)\right]C_H^R(T<T_c,M)=\kappa^{-\epsilon}B(u) . \quad (3.27)$$

We have shown that the specific heat below the transition satisfies the same renormalization-group equation as $C_H^R(T>T_c,M=0)$, it therefore follows that:

$$C_H^R(T<T_c)=A^-\tilde t^{-\alpha}(1+D^-\tilde t^{\omega\nu})+B^- , \quad (3.28)$$

in which

$$\tilde t=Xt,\quad A^-=X^{2-\alpha}C_h(1,x,u^*,\kappa=1) ,$$

$$D^-=(u-u^*)\left(\frac{\partial C_h/\partial u|_{u^*}}{C_h(u^*)}-\frac{(2-\alpha)}{\omega}\frac{\partial\nu^{-1}(u)}{\partial u}\bigg|_{u^*}\right)X^{\omega\nu},\quad B^-=C_p(\kappa,u)=B^+ . \quad (3.29)$$

x is the solution of the equation of state on the coexistence curve.

From Eqs. (3.18) to (3.29) we find the ratio of the amplitudes of the confluent singular terms for $H=0$ is

$$\frac{D^+}{D^-}=\frac{\partial C_h(1,0,u,1)/\partial u|_{u^*}^{(T>T_c)}/C_h(1,0,u^*,1)(T>T_c)-[(2-\alpha)/\omega]\partial\nu^{-1}(u)/\partial u|_{u^*}}{\partial C_h(-1,x,u,1)/\partial u|_{u^*}^{(T<T_c)}/C_h(-1,x,u^*,1)(T<T_c)-[(2-\alpha)/\omega]\partial\nu^{-1}(u)/\partial u|_{u^*}} . \quad (3.30)$$

A^+/A^- has been calculated by Brezin *et al.*[14] and Bervillier[5] to order ϵ and ϵ^2, respectively, the calculation of D^+/D^- to order ϵ^2 will be given in Sec. IV. From Eq. (3.30) we see that D^+/D^- is dimensionless and independent of X and Y; it only depends on u^*. Hence if we can show that D^+/D^- is independent of renormalization scheme we will have established its universality.

For a renormalization scheme l_1 ($>$, $<$ implies $T>,<T_c$)

$$C_{1>}(t_1,u_1,\kappa)=X_1^2\tilde t_1^{-\alpha}C_{1>}(t_1=1,0,u_1^*,\kappa=1)\left[1+(u_1-u_1^*)\tilde t_1^{\omega\nu}\left(\frac{\partial C_{1>}/\partial u_1|_{u_1^*}}{C_{1>}(u_1^*)}-\frac{(2-\alpha)}{\omega}\frac{\partial\nu^{-1}(u_1)}{\partial u_1}\bigg|_{u_1^*}\right)\right] ,$$

$$C_{1<}(t_1,u_1,\kappa)=X_1^2(-\tilde t_1)^{-\alpha}C_{1<}(t_1=-1,x,u_1^*,\kappa=1)\left[1+(u_1-u_1^*)(-\tilde t_1)^{\omega\nu}\left(\frac{\partial C_{1<}/\partial u_1|_{u_1^*}}{C_{1<}(u_1^*)}-\frac{(2-\alpha)}{\omega}\frac{\partial\nu^{-1}(u_1)}{\partial u_1}\bigg|_{u_1^*}\right)\right], \quad (3.31)$$

For a different renormalization scheme l_2 as

$$C_2(t_2)Z_{\phi^2}^{-2}(l_2)=C_{\rm bare}(\lambda,\mu^2)=C_1(t_1)Z_{\phi^2}^{-2}(l_1) , \quad (3.32)$$

$$C_2(t_2)=C_1(t_1)Z_{21}^2 , \quad (3.33)$$

$$Z_{21}=Z_{\phi^2}(l_2)/Z_{\phi^2}(l_1) , \quad (3.34)$$

and therefore

$$C_{2>}(t_2,u_2,\kappa)=Z_{21}^2X_1^2\tilde{t}_1^{-\alpha}C_{1>}(t_1=1,0,u_1^*,\kappa=1)\left[1+(u_1-u_1^*)\tilde{t}_1^{\omega\nu}\left(\frac{\partial C_{1>}/\partial u_1|_{u_1^*}}{C_{1>}(u_1^*)}-\frac{2-\alpha}{\omega}\frac{\partial\nu^{-1}(u_1)}{\partial u_1}\bigg|_{u_1^*}\right)\right],$$

$$C_{2<}(t_2,u_2,\kappa)=Z_{21}^2X_1^2(-\tilde{t}_1)^{-\alpha}C_{1<}(t_1=-1,x,u_1^*,\kappa=1)$$
$$\times\left[1+(u_1-u_1^*)(-\tilde{t}_1)^{\omega\nu}\left(\frac{\partial C_{1<}/\partial u_1|_{u_1^*}}{C_{1<}(u_1^*)}-\frac{(2-\alpha)}{\omega}\frac{\partial\nu^{-1}(u_1)}{\partial u_1}\bigg|_{u_1^*}\right)\right].$$

All effects of change in the renormalization scheme are absorbed by a change in the two scales X and Y; which proves the universality of D^+/D^-.

IV. CALCULATIONAL DETAILS

The relationship between $\Gamma_R(M,t)$ and $\Gamma(\bar{\phi},\mu^2)$ is given by Eq. (3.10). The bare potential $\Gamma(\bar{\phi},\mu^2)$, which has been calculated by loop expansion,[10,11] is shown in Fig. 2; the renormalization constants Z_ϕ, Z_{ϕ^2}, and λ as a function of g are listed for reference in Fig. 3; and the subtraction term $Z_{\phi^2}^2\Gamma^{(2,0)}(p,-p;\lambda,\Lambda)|_{p^2=\kappa^2}$, as a power series in g, is given in Fig. 4. The renormalized potential $\Gamma_R(M,t)$ is then obtained from Eq. (3.10) by replacing $\delta\mu^2$ by $Z_{\phi^2}t$ and $\bar{\phi}$ by $Z_\phi^{1/2}M$. The Feynman diagrams for $\Gamma_R(M,t)$ are shown in Fig. 5. When $T>T_c$, $M=0$ if $H=0$. When $T>T_c$ and $H=0$, $M=M(t)$ is found by solving the equation of state on the coexistence curve $H_i=\partial\Gamma(M,t)/\partial M_i=0$. The Feynman diagrams for $\Gamma_R(M(t),t)|_{H=0,t<0}$ are given in Fig. 6. The specific heats $C_{H=0}(t>0)$ and $C_{H=0}(t<0)$ are obtained by taking the second derivative of $\Gamma_R(M=0,t>0)|_{H=0}$ and $\Gamma_R(M(t),t<0)|_{H=0}$, respectively. The Feynman diagrams for $C_{H=0}(t<0)$ calculated in dimensionally regularized form are listed in Fig. 7. The leading singular term in the specific heats and hence A^+/A^- are found by evaluating $C_{H=0}(t><0)$ at $u=u^*$ and subtracting $C_p(u^*)$. We note that the particular integral satisfies the inhomogeneous renormalization-group equation

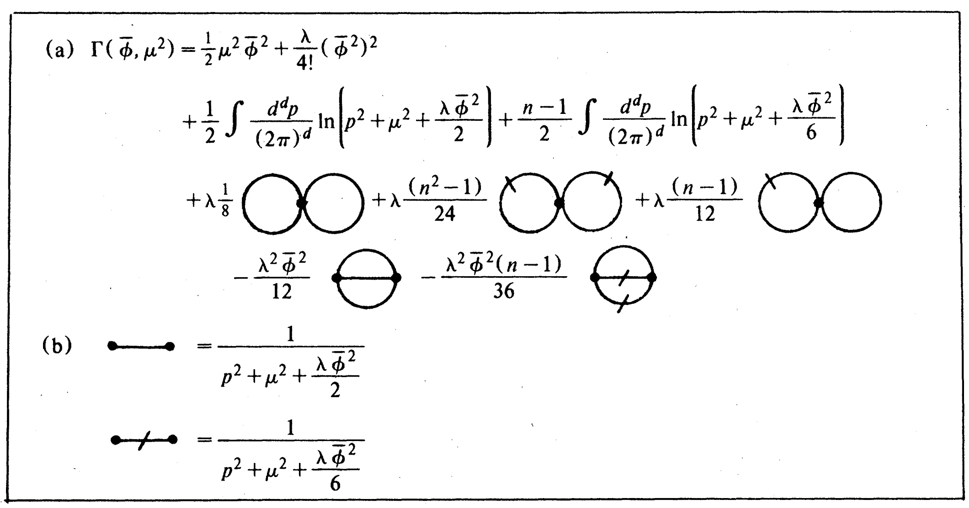
$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}-2\left(\frac{1}{\nu(u)}-2\right)\right]C_p=B(u)\kappa^{-\epsilon}\quad,\tag{4.1}$$

where

$$\kappa^{-\epsilon}B(u)=\left[\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}-2\left(\frac{1}{\nu(u)}-2\right)\right]$$
$$\times Z_{\phi^2}^2\Gamma^{(2,0)}(\bar{p},-\bar{p};\lambda,\Lambda)|_{p^2=\kappa^2}\ .\tag{4.2}$$

Defining a dimensionless particular solution by $C_p(u)=\kappa^{-\epsilon}\tilde{C}_p(u)$ we find

$$\left(d+\beta(u)\frac{\partial}{\partial u}-\frac{2}{\nu(u)}\right)\tilde{C}_p(u)=B(u)\ .\tag{4.3}$$



(a) $\Gamma(\bar{\phi},\mu^2)=\frac{1}{2}\mu^2\bar{\phi}^2+\frac{\lambda}{4!}(\bar{\phi}^2)^2+\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\ln\left(p^2+\mu^2+\frac{\lambda\bar{\phi}^2}{2}\right)+\frac{n-1}{2}\int\frac{d^dp}{(2\pi)^d}\ln\left(p^2+\mu^2+\frac{\lambda\bar{\phi}^2}{6}\right)$ $+\lambda\frac{1}{8}$ [diagram] $+\lambda\frac{(n^2-1)}{24}$ [diagram] $+\lambda\frac{(n-1)}{12}$ [diagram] $-\frac{\lambda^2\bar{\phi}^2}{12}$ [diagram] $-\frac{\lambda^2\bar{\phi}^2(n-1)}{36}$ [diagram]

(b) [propagator] $=\dfrac{1}{p^2+\mu^2+\frac{\lambda\bar{\phi}^2}{2}}$

[slashed propagator] $=\dfrac{1}{p^2+\mu^2+\frac{\lambda\bar{\phi}^2}{6}}$

FIG. 2. (a) Feynman diagrams for the free energy $\Gamma(\bar{\phi},\mu^2)$ in terms of bare parameters. (b) Feynman propagators used in (a).

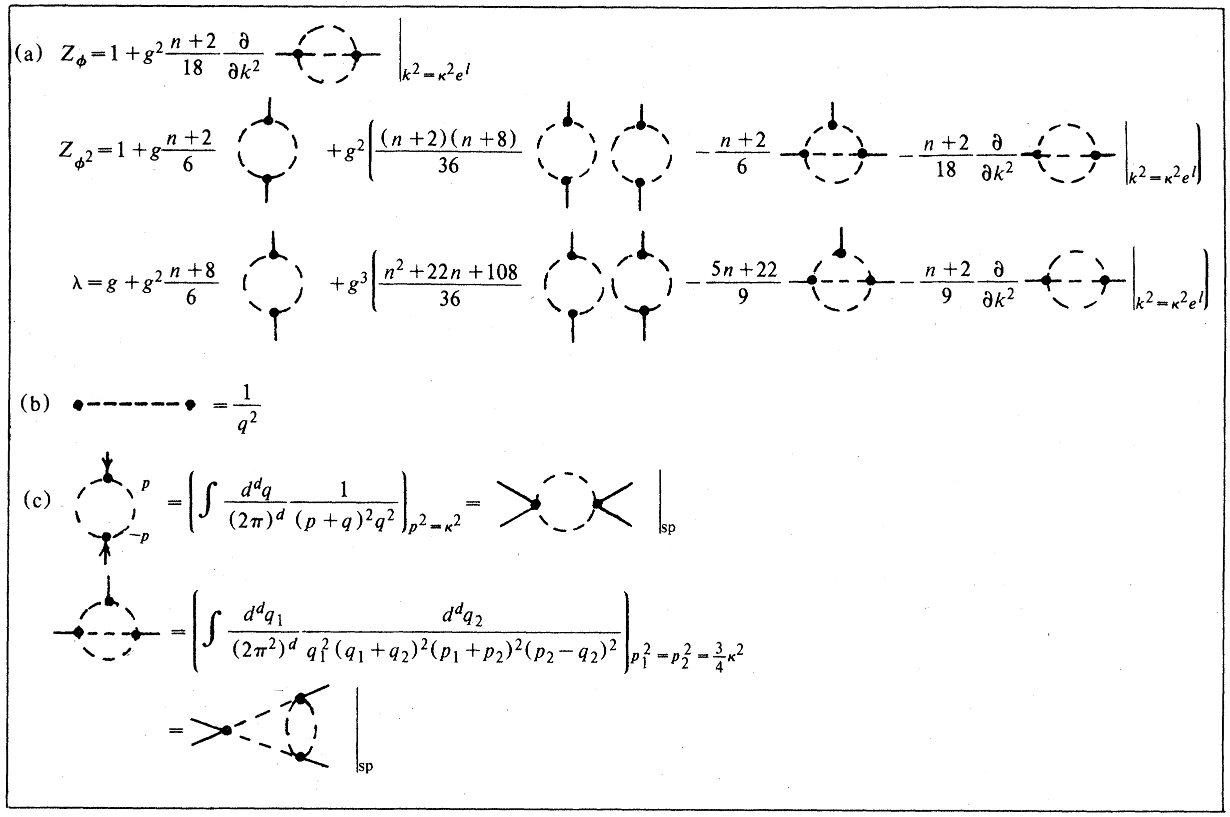


FIG. 3. (a) Renormalization constants expressed in terms of Feynman diagrams. (b) There is only one Feynman propagator in the massless theory. (c) The equivalence between the Feynman diagrams.

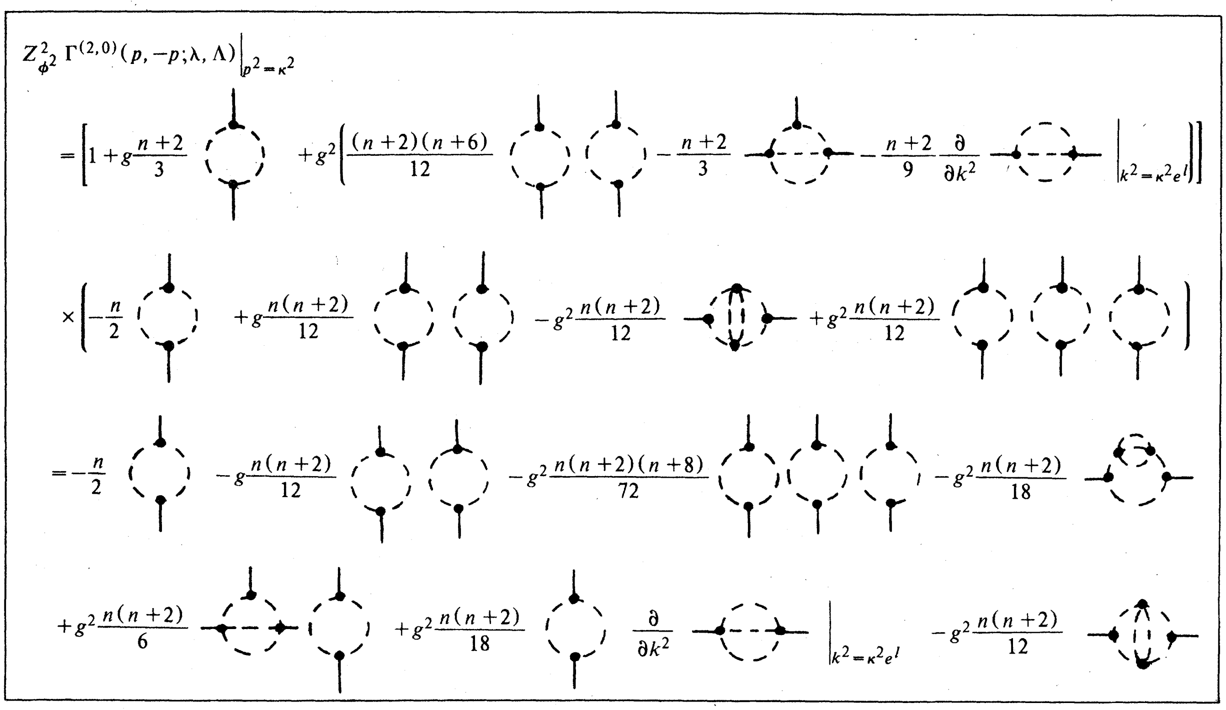


FIG. 4. Subtraction term in Eq. (2.3).

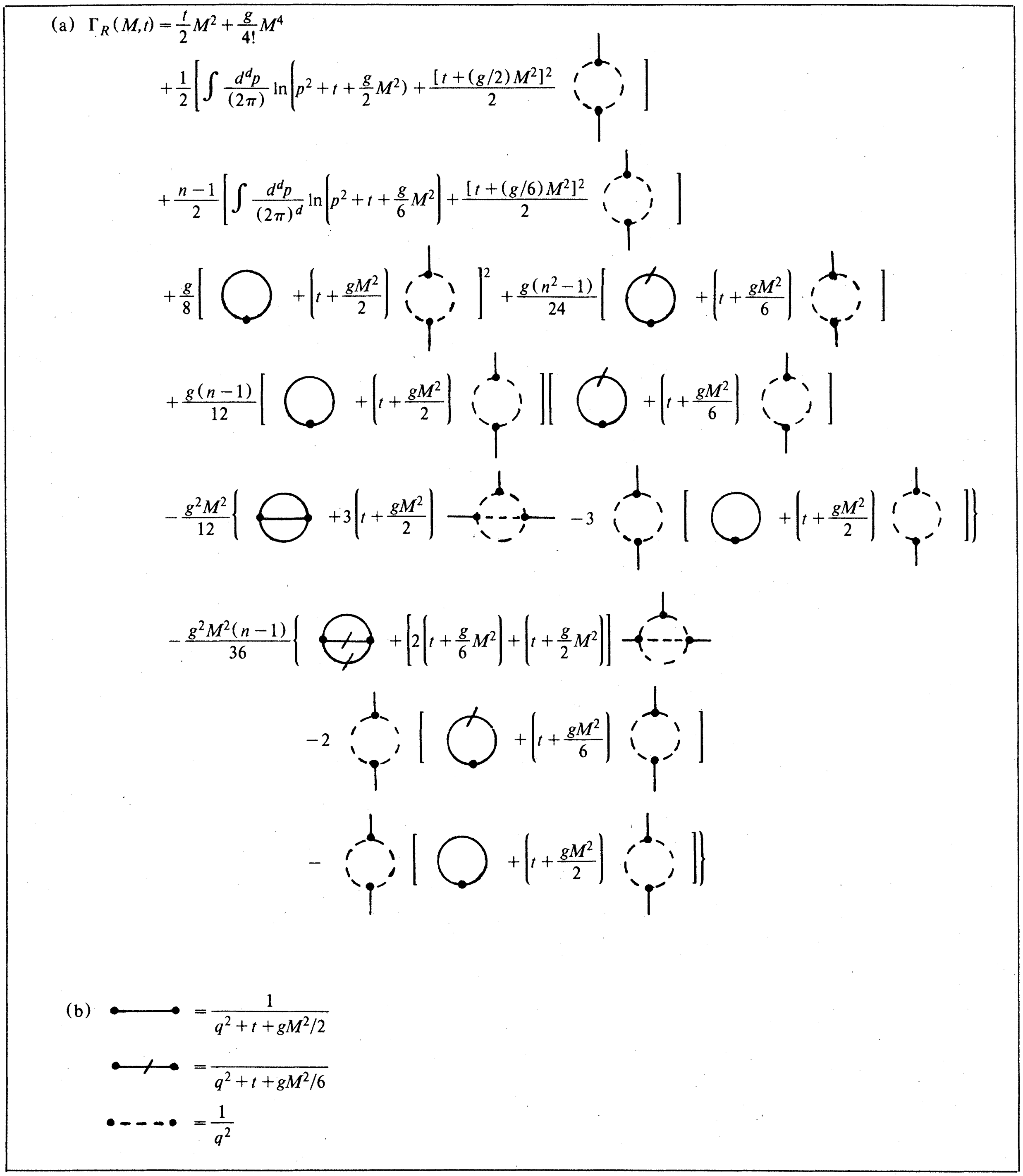


FIG. 5. (a) Renormalized free energy $\Gamma_R(M,t)$. If $t>0$, $H=0$, then $M=0$. (b) The Feynman propagators used in (a).

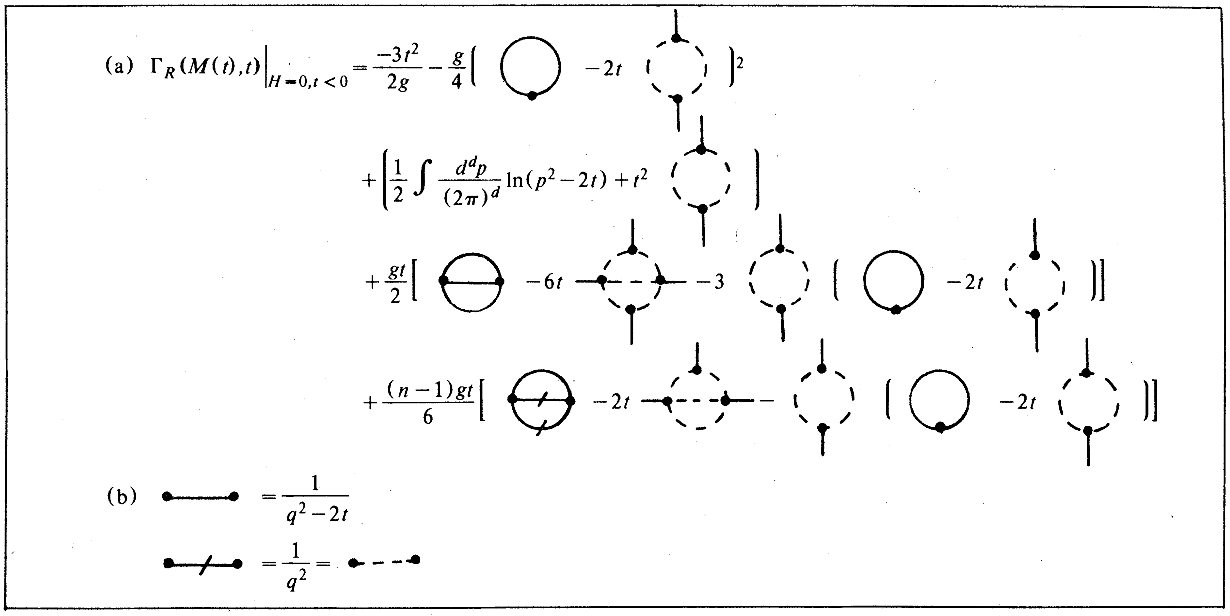


FIG. 6. (a) Renormalized free energy $\Gamma_R(M(t),t)|_{H=0,t<0}$, which is a function of t only. (b) The Feynman propagators used in (a).

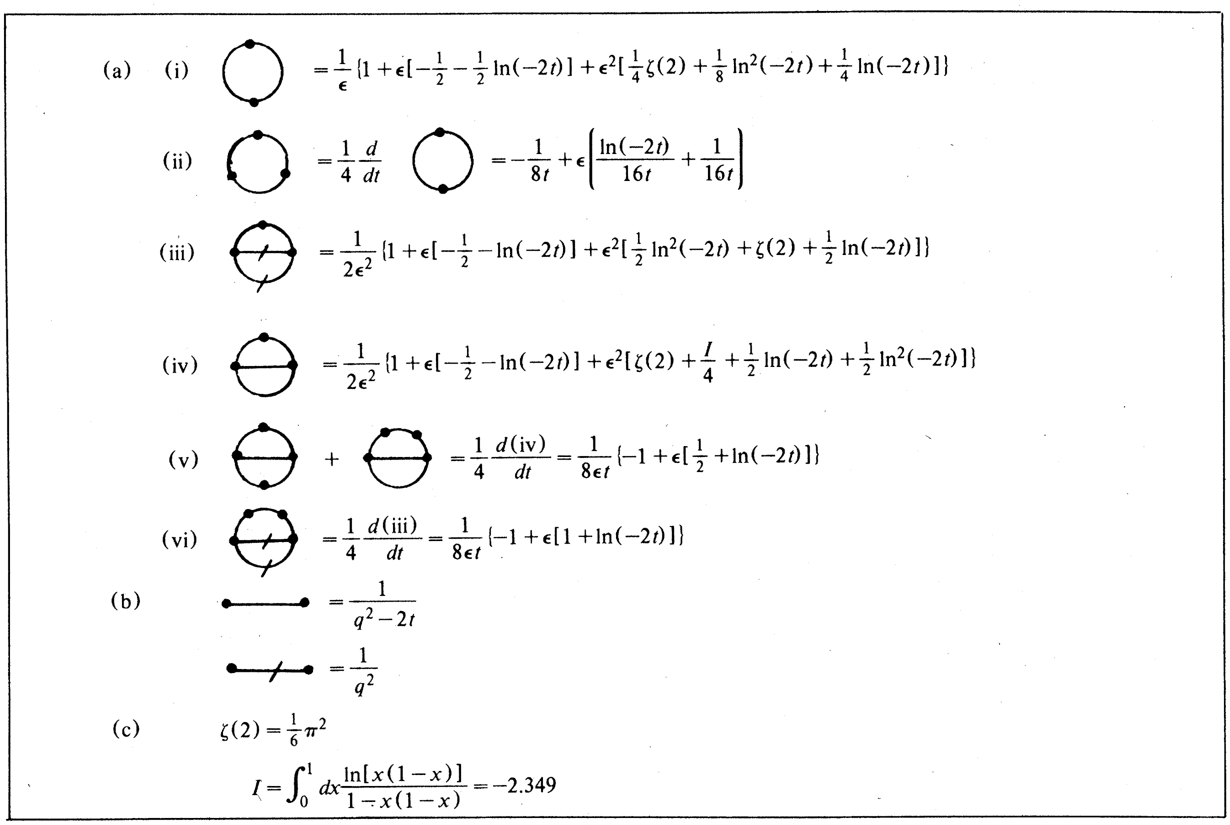


FIG. 7. (a) Feynman diagrams in $C_{H=0}(t<0)$ calculated in dimensionally regularized form. (b) The Feynman propagators for (a). (c) The explicit values for $\zeta(2)$ and I.

At the fixed point $\beta(u^*)=0$ and therefore it follows from Eq. (4.3) that

$$\tilde{C}_p(u^*)=-\frac{\nu}{2-d\nu}B(u^*) \ . \tag{4.4}$$

Knowing $B(u)$ and $[1/\nu(u)-2]$; and $Z_{\phi^2}^2\Gamma^{(2,0)}(\vec{p}_1,-\vec{p};\lambda,\Lambda)|_{p^2=\kappa^2}$ from Fig. 4, the inhomogeneous term $B(u)$ and hence $B(u^*)$ are found from Eq. (4.2). $B(u)$ is expressed in terms of Feynman diagrams in Fig. 8 and

$$B(u^*)=\tfrac{1}{2}n\left[1+\tfrac{1}{2}\epsilon+\epsilon^2\left[\tfrac{1}{2}+12\frac{(n+2)}{(n+8)^2}\left(-\tfrac{1}{3}+\tfrac{1}{4}J-\tfrac{1}{4}I\right)\right]\right] \tag{4.5}$$

and

$$B'(u^*)=\frac{2n(n+2)}{(n+8)}\epsilon\left(-\tfrac{1}{3}+\tfrac{1}{4}J-\tfrac{1}{24}I\right) \ , \tag{4.6}$$

where

$$J=\int_0^1\frac{dx}{x}\int_0^1 dy\,\ln\left[1-x+\left[\frac{3x}{4y}\right](1-x+xy)\right]=0.7494 \ . \tag{4.7}$$

We then find

$$\frac{A^+}{A^-}=2^{\alpha}\left(\tfrac{1}{4}n\right)\left[1+\epsilon+\epsilon^2\left[\frac{(3n^4+74n^3+708n^2+3264n+6400)}{2(n+8)^4}+\frac{4-n}{2(n+8)}\zeta(2)-\frac{3(5n+22)}{(n+8)^2}\zeta(3)+\frac{9(4-n)}{4(n+8)^2}I\right]\right]$$
$$+O(\epsilon^3) \ , \tag{4.8}$$

which agrees with Bervillier.[5] In Eq. (4.8) α is the specific-heat exponent as a power series in ϵ, $\zeta(2)$ and $\zeta(3)$ are Riemann ζ functions and

$$I=\int_0^1 dx\,\ln[x(1-x)]/[1-x(1-x)]=-2.349 \ . \tag{4.9}$$

As we pointed out in Sec. III, in order to compute the amplitude of the confluent singular term in the specific heat, we must calculate $\partial C_p/\partial u|_{u^*}$. See Eq.

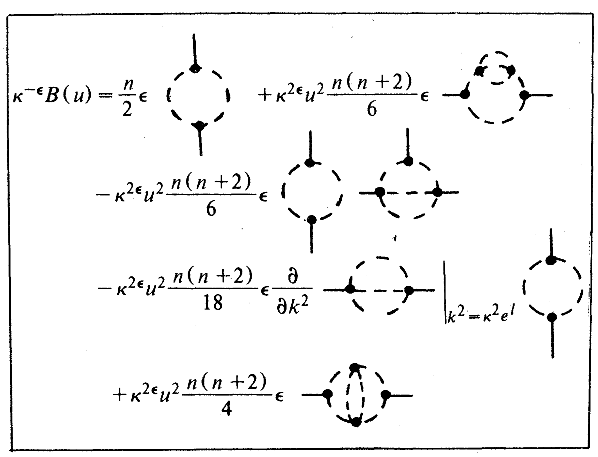


FIG. 8. Feynman diagrams for the inhomogeneous term $B(u)\kappa^{-\epsilon}$.

(3.17). It follows from Eq. (4.3) that:

$$\partial C_p(u)/\partial u=\{B(u)+\hat{C}_p(u)[2/\nu(u)-d]\}/\beta(u) \ . \tag{4.10}$$

then using Eq. (4.4) we find

$$C_p'(u^*)=\{\nu B'(u^*)+[2/(2-d\nu)B(u^*)\partial\nu(u)/\partial u|_{u^*}]\}/[\omega\nu-(2-d\nu)] \ . \tag{4.11}$$

Then using results for $C_{H=0}(t>0)$, $C_{H=0}(t<0)$ and Eqs. (3.16), (3.17), (3.30), (4.8), and (4.11), we find after considerable algebra

$$\frac{D^+}{D^-}=1+\epsilon\left[\frac{n+8}{2(n+2)}-\tfrac{1}{2}\ln 2\right]+\epsilon^2\left[\frac{3n^3-22n^2-88n+152}{4(n+8)(n+2)^2}-\zeta(2)-\frac{9}{2(n+8)}I-\frac{3(5n+22)}{(n+8)(n+2)}\zeta(3)-\frac{1}{4}\left[\frac{n^2-8n-68}{(n+8)^2}+\frac{n+8}{n+2}\right]\ln 2+\tfrac{1}{8}\ln^2 2\right] \ . \tag{4.12}$$

For liquid ^{4}He, $n=2$ and $d=3$; Eq. (4.12) gives

$$D^+/D^- = 1 + 0.9\epsilon - 3.84\epsilon^2 \; . \qquad (4.13)$$

Just as for the correction to scaling exponent the series converges poorly and the second-order term gives a large negative correction. This is consistent with general experience of ϵ expansions when three-loop diagrams have to be included and is due to the asymptotic nature of these expansions. Padé approximants suggest a value of $D^+/D^- \simeq 1.17$.

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