Lower critical field of an anisotropic type-II superconductor

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We consider the Ginzburg-Landau free energy of the anisotropic mass form in the presence of a magnetic field of arbitrary fixed direction. It is shown that the free energy may be transformed into the isotropic Ginzburg-Landau form with a κ that depends upon the direction of the magnetic induction $\vec{B}$ relative to the crystal lattice. The lower critical field $\vec{H}_{c1}$ is then found for arbitrary direction of $\vec{B}$. For highly anisotropic crystals the angular dependence of $\vec{H}_{c1}$ can exhibit a discontinuity or a cusp. The special case of a crystal with uniaxial symmetry is considered in detail.

I. INTRODUCTION

Recently, there has been a growing interest in the effects of anisotropy on the magnetic properties of type-II superconductors.[1–5] Although single-element superconductors do not exhibit a large degree of critical-field anisotropy, the layered and filamentary superconductors in particular may exhibit anisotropies of the upper critical field of more than an order of magnitude.[6,7] One would thus expect a similar, though slightly smaller, anisotropy of the lower critical field. However, to date there has not been any theoretical treatment of the anisotropy of $\vec{H}_{c1}$ that would apply to such highly anisotropic materials. As experiments on $(SN)_x$, $TaS_2(\text{pyridine})_{1/2}$, and other such materials are currently being performed,[8,9] it would appear useful to have some theoretical results to compare with the data.

For the layered and filamentary superconductors, the appropriate Ginzburg-Landau free energy is of the Lawrence-Doniach Josephson-coupled form.[10] For sufficiently low temperatures such that the coherence length perpendicular to the layers (filaments) is less than the interlayer (interfilament) spacing, the normal cores of the vortices fit between the layers (filaments) if the field is directed parallel to them. Clearly, this should have drastic effects upon $\vec{H}_{c1}$, since it costs very little energy to fit a vortex between the layers (filaments). This problem will be discussed in a further publication.

Near T_c, the coherence length perpendicular to the layers (filaments) extends over many layers (filaments), and the system acts as a bulk anisotropic superconductor. In this temperature region, the anisotropy of $\vec{H}_{c2}$ obeys a simple anisotropic-mass law.[11,12] We shall see that the anisotropy of $\vec{H}_{c1}$ obeys a more complicated law, arising primarily from the fact that, even in the bulk, the microscopic magnetic induction is not parallel to $\vec{H}_{c1}$.

In Sec. II, we discuss the anisotropic-mass Ginzburg-Landau free energy and write it in dimensionless form. In Sec. III, we show that the free energy may be mapped onto the usual isotropic Ginzburg-Landau free energy with a Ginzburg-Landau parameter $\tilde{\kappa}$ that depends upon the direction cosines of the local magnetic induction relative to the crystal symmetry axes. In Sec. IV, we calculate the upper and lower critical fields and examine in detail the lower critical field for the case of planar isotropy, which is applicable to layered and filamentary superconductors near T_c. Finally, in Sec. V, we discuss additional ramifications of our solution.

II. ANISOTROPIC-MASS MODEL

We assume that the Helmholtz free energy in the superconducting state, relative to that in the normal state in zero field, can be written in the anisotropic-mass form[13–18]

$$F_S=\int d^3r\left[\alpha|\psi|^2+\tfrac{1}{2}\beta|\psi|^4+\sum_{\mu=1}^{3}\frac{1}{2m_\mu}\left|\left(-i\hbar\partial_\mu+2\frac{ea_\mu}{c}\right)\psi\right|^2+\frac{\vec{b}^2}{8\pi}\right]\ , \tag{1}$$

where $\vec{b}(\vec{r})=\vec{\nabla}\times\vec{a}$ is the microscopic magnetic induction, $\vec{a}$ is the vector potential, e is the magnitude of the electronic charge, α and β are the usual Ginzburg-Landau parameters, $\psi(\vec{r})$ is the order parameter, $\partial_\mu\equiv\partial/\partial x_\mu$, x_μ is x, y, z for $\mu=1, 2, 3,$

respectively, and the integration extends over the volume V of the specimen. We assume that the direction of $\vec{b}$ is constant throughout the superconductor, which implies that $\vec{b}$ may be written as

$$\vec{b}(\vec{r}) = b(\vec{r})(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \qquad (2)$$

in spherical coordinates. In the absence of a field, we have $\psi = \psi_0 = (-\alpha/\beta)^{1/2}$ in equilibrium below T_c. We further define a penetration depth $\lambda(T)$ and a coherence length $\xi(T)$ via

$$\lambda(T) = \left(\frac{mc^2}{16\pi e^2 \psi_0^2}\right)^{1/2} \qquad (3a)$$

and

$$\xi(T) = \left(\frac{-\hbar^2}{2m\alpha}\right)^{1/2}, \qquad (3b)$$

where

$$m = (m_1 m_2 m_3)^{1/3}. \qquad (4)$$

We next express all quantities in dimensionless form, using the conventional normalization[19,20] in which length is measured in units of λ, magnetic field in units of $\sqrt{2}H_c = \phi_0/2\pi\xi\lambda$, vector potential in units of $\lambda\sqrt{2}H_c = \phi_0/2\pi\xi$, and energy density in units of $H_c^2/4\pi = \phi_0^2/(32\pi^3\xi^2\lambda^2) = \alpha^2/\beta$, where $\phi_0 = hc/2e$. The reduced free energy is then given by

$$F_s = \int d^3r \left[-|f|^2 + \tfrac{1}{2}|f|^4 + \sum_\mu \frac{m}{m_\mu} \left| \left(-\frac{i}{\kappa}\partial_\mu + a_\mu \right) f \right|^2 + b^2 \right], \qquad (5)$$

where

$$f = \psi/\psi_0 \qquad (6)$$

and $\kappa = \lambda/\xi$ is the effective Ginzburg-Landau parameter. We now write f in terms of a magnitude and a phase,

$$f = f_0 e^{i\gamma}, \qquad (7a)$$

and introduce the gauge-invariant quantity,

$$\vec{a}_0 = \vec{a} + \frac{1}{\kappa}\vec{\nabla}\gamma, \qquad (7b)$$

usually referred to as the superfluid velocity, to obtain

$$F_s = \int d^3r \left[-f_0^2 + \tfrac{1}{2}f_0^4 + \sum_\mu \frac{m}{m_\mu} \left(\frac{1}{\kappa^2}(\partial_\mu f_0)^2 + a_{0\mu}^2 f_0^2 \right) + b^2 \right]. \qquad (8)$$

Equation (8) and the equation

$$\vec{b} = \vec{\nabla} \times \vec{a}_0 \qquad (9)$$

comprise the starting point for our calculation. Equation (8) clearly reduces to the isotropic form if each of the m_μ equals m, but we are interested in the case in which not all of the m_μ are equal. For generality, we assume that all of the m_μ are different. In Sec. III, we show how Eq. (8) can be transformed to isotropic form.

III. TRANSFORMATION TO ISOTROPIC FORM

We note that Eq. (8) contains the mass-anisotropy factors m/m_μ in the gradient and vector potential terms, but not in the b^2 term. Although a single set of transformations that will transform this equation immediately to isotropic form can be found, it is instructive to break the transformation into three parts: an anisotropic scale transformation, a rotation, and finally an isotropic scale transformation. During each of these transformations, it is necessary to preserve Maxwell's equations.

We first make the anisotropic scale transformation,

$$x_\mu = (m/m_\mu)^{1/2} x'_\mu, \qquad (10a)$$

which implies

$$\partial_\mu = (m_\mu/m)^{1/2} \partial'_\mu. \qquad (10b)$$

This transformation is subject to the restriction

$$\sum_\mu \partial_\mu b_\mu = \sum_\mu \partial'_\mu b'_\mu = 0, \qquad (11)$$

which implies

$$b_\mu = (m/m_\mu)^{1/2} b'_\mu. \qquad (12)$$

Furthermore, using Eq. (4) we obtain

$$b'_\mu = \left(\frac{m_\mu}{m}\right)^{1/2} b_\mu = \sum_{\beta\gamma} \left(\frac{m_\mu m_\beta}{m^2}\right)^{1/2} \epsilon_{\mu\beta\gamma}\partial_\beta a_{0\gamma} = \sum_{\beta\gamma} \left(\frac{m}{m_\gamma}\right)^{1/2} \epsilon_{\mu\beta\gamma}\partial'_\beta a_{0\gamma}, \qquad (13)$$

where μ, β, and γ are all different. Since we require $\vec{b}' = \vec{\nabla}' \times \vec{a}'_0$ in the primed frame, we obtain

$$a_{0\mu} = (m_\mu/m)^{1/2} a'_{0\mu}. \qquad (14)$$

With this definition of m,

$$d^3r = d^3r' \qquad (15)$$

and the reduced free energy may now be written

$$F_s=\int d^3r'\left[-f_0^2+\tfrac{1}{2}f_0^4+\frac{1}{\kappa^2}(\vec{\nabla}'f_0)^2+a_0'^2f_0^2+\sum_\mu\frac{m}{m_\mu}b_\mu'^2\right] . \tag{16}$$

We observe that this scale transformation has removed the m/m_μ factors from the gradient and vector potential terms, but has introduced them into the magnetic energy term. If we are to perform an additional transformation we would like to transform the magnetic energy term

$$\sum_\mu\frac{m}{m_\mu}b_\mu'^2=cb''^2 , \tag{17}$$

without disturbing the symmetry of the other two relevant terms. Hence, we desire a transformation that also has the form

$$(\vec{\nabla}'f_0)^2\to(\vec{\nabla}''f_0)^2 \tag{18a}$$

and

$$a_0'^2\to a_0''^2 , \tag{18b}$$

if such a transformation can be found. Clearly, Eq. (18a) can be satisfied by a rotation. The problem then is to find a rotation that satisfies Eqs. (17) and (18b).

Let us consider a vector $\vec{x}$ in the primed and double-primed frames,

$$\vec{x}=\sum_\mu x_\mu'\hat{e}_\mu'=\sum_\mu x_\mu'\hat{e}_\mu=\sum_\mu x_\mu''\hat{e}_\mu'' , \tag{19}$$

where the second equality follows from the fact that the primed frame has the same axes as the unprimed frame. Setting

$$x_\mu'=\sum_\nu\lambda_{\nu\mu}x_\nu'' , \tag{20}$$

and choosing the $\lambda_{\nu\mu}$ such that $\vec{b}''$ is only in the $\hat{e}_z''$ direction,

$$b_\mu''=b''\delta_{\mu 3} , \tag{21}$$

we find that $a_{0\mu}'$ and $a_{0\nu}''$ also satisfy Eq. (20), and that

$$\sum\frac{m}{m_\mu}b_\mu'^2=\sum_{\mu\nu\gamma}\frac{m}{m_\mu}\lambda_{\nu\mu}\lambda_{\gamma\mu}\delta_{\nu 3}\delta_{\gamma 3}b''^2=\left[\sum_\mu\frac{m}{m_\mu}\lambda_{3\mu}^2\right]b''^2\equiv\frac{b''^2}{\alpha^2} . \tag{22}$$

Hence, Eqs. (17) and (18b) can be satisfied by the rotation described by $\lambda_{\nu\mu}$.

Since in a rotation the volume element transforms according to

$$d^3r'=d^3r'' , \tag{23}$$

the reduced free energy is now in the much simpler form,

$$F_s=\int d^3r''\left[-f_0^2+\tfrac{1}{2}f_0^4+\frac{1}{\kappa^2}(\vec{\nabla}''f_0)^2+a_0''^2f_0^2+\frac{b''^2}{\alpha^2}\right] . \tag{24}$$

This equation is nearly in the desired isotropic form, as we have transformed all of the mass parameters into one parameter α. Since the reduced free energy also contains the parameter κ, which is the only parameter of the isotropic Ginzburg-Landau reduced free energy, it is desirable to perform a simple scale transformation in order to put the free energy into a single-parameter form.

We therefore make the transformation

$$x_\mu''=\tilde{x}_\mu/\alpha , \tag{25a}$$

$$\partial_\mu''=\alpha\tilde{\partial}_\mu , \tag{25b}$$

$$a_{0\mu}''=\tilde{a}_{0\mu} , \tag{25c}$$

$$b_\mu''=\alpha\tilde{b}_\mu . \tag{25d}$$

It is easy to show that this transformation satisfies Maxwell's equations. We finally have

$$F_s=\alpha^{-3}\int d^3\tilde{r}\,[-f_0^2+\tfrac{1}{2}f_0^4+\tilde{\kappa}^{-2}(\tilde{\nabla}f_0)^2+f_0^2\tilde{a}_0^2+\tilde{b}^2] , \tag{26}$$

where

$$\tilde{\kappa}=\kappa/\alpha . \tag{27}$$

This equation is of the same form as the isotropic reduced Ginzburg-Landau free energy, with an effective $\tilde{\kappa}$ that depends, via α, upon the direction cosines of the magnetic induction with respect to the crystal lattice. It remains merely to find α.

We remark that since we have transformed the Ginzburg-Landau free-energy functional and not merely the Ginzburg-Landau equations, the transformations we have performed are not restricted in applicability to the mean-field properties of the superconductor. Rather, they even apply in the critical region.

The Ginzburg-Landau (mean-field) equations may easily be obtained from Eq. (26) by minimizing F_s with respect to variations in f_0 and $\tilde{a}_{0\mu}$, yielding

$$-\frac{1}{\tilde{\kappa}^2}\tilde{\nabla}^2f_0+\tilde{a}_0^2f_0=f_0(1-f_0^2) \tag{28a}$$

and

$$\tilde{\vec{\nabla}}\times\tilde{\vec{b}}=\tilde{\vec{\nabla}}\times(\tilde{\vec{\nabla}}\times\tilde{\vec{a}}_0)=-f_0^2\tilde{\vec{a}}_0 . \tag{28b}$$

It now remains only to calculate the $\lambda_{\nu\mu}$ and hence

α. Let us first calculate $\lambda_{3\mu}$. Let $\hat{b}$, $\hat{b}'$, and $\hat{b}''$ be unit vectors parallel to the untransformed, scale transformed, and rotated local magnetic induction, respectively. Hence, $\hat{b}''=\hat{e}_3''$ by Eq. (21). By Eq. (19), a vector $\vec{\zeta}=\zeta\hat{b}$ transforms to $\vec{\zeta}'=\zeta'\hat{b}'$, where

$$\vec{\zeta}'=\sum_\mu \hat{e}_\mu'\zeta_\mu'=\sum_\mu \hat{e}_\mu\left[\frac{m_\mu}{m}\right]^{1/2}\zeta_\mu$$

$$=\zeta\sum_\mu \hat{e}_\mu\left[\frac{m_\mu}{m}\right]^{1/2}\hat{b}\cdot\hat{e}_\mu \ , \quad (29)$$

which implies that

$$\zeta'=\zeta\left[\sum_\mu\frac{m_\mu}{m}(\hat{e}_\mu\cdot\hat{b})^2\right]^{1/2} \quad (30)$$

and

$$\hat{b}'\cdot\hat{e}_\mu=(\hat{b}\cdot\hat{e}_\mu)\left[\frac{m_\mu}{m}\right]^{1/2}\Bigg/\left[\sum_\mu\frac{m_\mu}{m}(\hat{e}_\mu\cdot\hat{b})^2\right]^{1/2} \ . \quad (31)$$

Now in the rotation,

$$\lambda_{3\mu}=\hat{b}''\cdot\hat{e}_\mu'=\hat{b}'\cdot\hat{e}_\mu'=\hat{b}'\cdot\hat{e}_\mu \ , \quad (32)$$

where the second equality holds because we have merely rotated the coordinate axes, holding the direction of $\hat{b}'$ fixed and thus equal to that of $\hat{b}''$, and the rotation preserves the length of the vector. Combining Eqs. (22), (31), and (32), we have

$$\alpha^{-2}=\sum_\mu\frac{m}{m_\mu}\lambda_{3\mu}^2$$

$$=\left[\sum_\mu\frac{m_\mu}{m}(\hat{e}_\mu\cdot\hat{b})^2\right]^{-1} \quad (33)$$

or

$$\alpha=\left[\sum_\mu\frac{m_\mu}{m}(\hat{e}_\mu\cdot\hat{b})^2\right]^{1/2} \ , \quad (34)$$

where the $\hat{e}_\mu\cdot\hat{b}$ are the direction cosines of $\vec{\mathrm{b}}$ as given by Eq. (2).

The entire $\lambda_{\nu\mu}$ matrix may now be found in a straightforward way. Let us choose $\hat{e}_1''=\hat{b}'\times\hat{e}_3'$ and $\hat{e}_2''=\hat{e}_3''\times\hat{e}_1''$. In the primed frame, $\hat{b}'$ is at an angle θ' with respect to $\hat{e}_3'$, and the projection of $\hat{b}'$ into the $\hat{e}_1'-\hat{e}_2'$ plane makes an angle ϕ' with respect to $\hat{e}_1'$. Hence, the rotation from the primed to the unprimed frame corresponds to a rotation of ϕ' about the $\hat{e}_3'$ axis and then a rotation of θ' about the $\hat{e}_2'$ axis. Hence,

$$\hat{e}_3''=\hat{e}_1'\sin\theta'\cos\phi'+\hat{e}_2'\sin\theta'\sin\phi'+\hat{e}_3'\cos\theta' \ ,$$

$$\hat{e}_2''=\hat{e}_1'\cos\theta'\cos\phi'+\hat{e}_2'\cos\theta'\sin\phi'-\hat{e}_3'\sin\theta' \ , \quad (35)$$

$$\hat{e}_1''=\hat{e}_1'\sin\phi'-\hat{e}_2'\cos\phi' \ .$$

We therefore have

$$\lambda_{\nu\mu}=\hat{e}_\nu''\cdot\hat{e}_\mu'$$

$$=\begin{bmatrix}\sin\phi' & -\cos\phi' & 0\\ \cos\theta'\cos\phi' & \cos\theta'\sin\phi' & -\sin\theta'\\ \sin\theta'\cos\phi' & \sin\theta'\sin\phi' & \cos\theta'\end{bmatrix} \ , \quad (36)$$

where from Eqs. (31) and (32),

$$\sin\theta'\cos\phi'=\frac{1}{\alpha}\left[\frac{m_1}{m}\right]^{1/2}\sin\theta\cos\phi \ , \quad (37a)$$

$$\sin\theta'\sin\phi'=\frac{1}{\alpha}\left[\frac{m_2}{m}\right]^{1/2}\sin\theta\sin\phi \ , \quad (37b)$$

$$\cos\theta'=\frac{1}{\alpha}\left[\frac{m_3}{m}\right]^{1/2}\cos\theta \ . \quad (37c)$$

In retrospect, we note that the three transformations we used to map the anisotropic Ginzburg-Landau free energy onto the isotropic Ginzburg-Landau free energy may all be combined into one transformation,

$$x_\mu=\frac{1}{\alpha}\left[\frac{m}{m_\mu}\right]^{1/2}\sum_\nu\lambda_{\nu\mu}\tilde{x}_\nu \ , \quad (38a)$$

$$\partial_\mu=\alpha\left[\frac{m_\mu}{m}\right]^{1/2}\sum_\nu\lambda_{\nu\mu}\tilde{\partial}_\nu \ , \quad (38b)$$

$$b_\mu=\alpha\left[\frac{m}{m_\mu}\right]^{1/2}\sum_\nu\lambda_{\nu\mu}\tilde{b}_\nu \ ,$$

$$=\alpha\left[\frac{m}{m_\mu}\right]^{1/2}\lambda_{3\mu}\tilde{b} \ , \quad (38c)$$

$$a_{0\mu}=\left[\frac{m_\mu}{m}\right]^{1/2}\sum_\nu\lambda_{\nu\mu}\tilde{a}_{0\nu} \ , \quad (38d)$$

where $\tilde{b}_\nu=\tilde{b}_3\delta_{\nu3}$ and $\tilde{a}_{03}=0$.

IV. UPPER AND LOWER CRITICAL FIELDS

Since the free energy for the anisotropic superconductor can be transformed into isotropic form, the calculation of $\vec{\mathrm{H}}(\vec{\mathrm{B}})$ is closely related to that for an isotropic superconductor, with the replacement of κ by $\tilde{\kappa}$. To show this, we begin by examining how fluxoid quantization manifests itself in the transformed frame. We continue to use the normalization in which magnetic flux is measured in units of $\sqrt{2}H_c\lambda^2=\lambda\phi_0/2\pi\xi$, such that the flux quantum ϕ_0 becomes $2\pi/\kappa$ in reduced units. Integrating Eq. (7b) around the axes of n singly quantized vortices, we obtain in the original frame

$$\Phi-\oint d\vec{\mathrm{l}}\cdot\vec{\mathrm{a}}_0=\frac{2\pi n}{\kappa} \ , \quad (39a)$$

where

$$\Phi=\int d\vec{S}\cdot\vec{b}=\oint d\vec{l}\cdot\vec{a} \quad . \tag{39b}$$

Applying the transformations of Eqs. (38a)–(38d) to this result, we obtain in the fully transformed frame

$$\tilde{\Phi}-\oint d\vec{\tilde{l}}\cdot\vec{\tilde{a}}_0=\frac{2\pi n}{\tilde{\kappa}} \quad , \tag{40a}$$

where

$$\tilde{\Phi}=\int d\vec{\tilde{S}}\cdot\vec{\tilde{b}}=\oint d\vec{\tilde{l}}\cdot\vec{\tilde{a}} \quad . \tag{40b}$$

As can be seen from Eqs. (40a)–(40b) and the Ginzburg-Landau equations (28a)–(28b), the properties of the vortex structure in the transformed frame can be calculated exactly as in the isotropic case, except that the usual Ginzburg-Landau parameter κ is replaced by $\tilde{\kappa}$. However, to relate these properties to the corresponding behavior in the original frame is complicated by the fact that the flux density $\vec{B}$ (averaged over a unit cell of the vortex lattice) is not in general parallel to $\vec{H}$. In thermodynamic equilibrium $\vec{H}(\vec{B})$ is defined via[19,20]

$$\vec{H}(\vec{B})=\tfrac{1}{2}\vec{\nabla}_{\vec{B}}F(\vec{B}) \quad , \tag{41}$$

where $F(\vec{B})$ is the Helmholtz free energy per unit volume of the superconducting state relative to the Meissner state,

$$F=F_S/V+\tfrac{1}{2} \quad . \tag{42}$$

Close to $\vec{H}_{c2}$, we can directly apply Abrikosov's results[19] to first find the Helmholtz free energy in the fully transformed frame and then use the fact that $\tilde{B}=B$, derived from Eqs. (33) and (38c), to obtain

$$F(\vec{B})=\tfrac{1}{2}+B^2+\frac{[\tilde{\kappa}(\hat{B})-B]^2}{1+[2\tilde{\kappa}^2(\hat{B})-1]\beta_A} \quad , \tag{43}$$

where $\beta_A=1.1596$ for the equilateral triangular lattice.[21] When the flux density is equal to

$$\vec{B}_{c2}=\hat{B}\tilde{\kappa}(\hat{B}) \quad , \tag{44}$$

bulk superconductivity is quenched ($f_0\rightarrow 0$), and the magnetic field as defined by Eq. (41) is also equal to[10,11,16–18,22,23]

$$\vec{H}_{c2}=\vec{B}_{c2}=\hat{B}\tilde{\kappa}(\hat{B}) \quad . \tag{45}$$

For widely separated vortices close to $\vec{H}_{c1}$,[19,20]

$$F\approx n\epsilon_1 \quad , \tag{46}$$

where $n=\kappa B/2\pi$ is the number of vortices per unit area and ϵ_1 is the energy per unit length stored in an isolated vortex line, where both n and ϵ_1 are expressed in reduced form. From Eqs. (26) and (42) it can be seen that ϵ_1 is related to the corresponding quantity $\tilde{\epsilon}_1$ in the fully transformed frame via

$$\epsilon_1 L=\alpha^{-3}\tilde{\epsilon}_1\tilde{L} \quad , \tag{47}$$

where L and $\tilde{L}$ are line lengths in the original and fully transformed frames. By taking $\tilde{x}_\nu=\tilde{L}\delta_{\nu 3}$ and using Eqs. (41a) and (33), we obtain $\tilde{L}=\alpha^2 L$, such that

$$F\approx B\tilde{\kappa}\tilde{\epsilon}_1/2\pi \quad . \tag{48}$$

Now $\tilde{\epsilon}_1$ depends only upon $\hat{B}$, the direction of $\vec{B}$, through its dependence on $\tilde{\kappa}$[19,20]:

$$\tilde{\epsilon}_1(\tilde{\kappa})=2\pi\int_0^\infty d\tilde{\rho}\tilde{\rho}[\tfrac{1}{2}(1-f_0^2)^2+\tilde{\kappa}^{-2}(\tilde{\nabla}f_0)^2+f_0^2\tilde{a}_0^2+\tilde{b}^2] \quad , \tag{49}$$

which can be reexpressed more compactly as[24]

$$\tilde{\epsilon}_1(\tilde{\kappa})=2\pi\int_0^\infty d\tilde{\rho}\,\tilde{\rho}(1-f_0^2) \quad . \tag{50}$$

From Eq. (41) we obtain in the limit $\vec{B}\rightarrow 0$

$$\vec{H}_{c1}(\theta,\phi)=\left[\hat{B}+\hat{\theta}\frac{\partial}{\partial\theta}+\hat{\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]H_{c1\parallel\vec{B}}(\theta,\phi) \quad , \tag{51}$$

where the component of $\vec{H}_{c1}$ parallel to $\vec{B}$ is

$$H_{c1\parallel\vec{B}}=\tilde{\kappa}\tilde{\epsilon}_1(\tilde{\kappa})/4\pi \quad . \tag{52}$$

In the large $\tilde{\kappa}$ limit,[25]

$$H_{c1\parallel\vec{B}}\approx(2\tilde{\kappa})^{-1}(\ln\tilde{\kappa}+0.497) \quad . \tag{53}$$

Since $\tilde{\kappa}$ depends explicitly upon θ and ϕ for an anisotropic superconductor, $\vec{H}_{c1}$ will be parallel to $\hat{B}$ only when both are along a symmetry direction. When the anisotropy is large, $\vec{H}_{c1}$ and $\hat{B}$ may be nearly perpendicular for some directions of $\hat{B}$ (or $\hat{H}_{c1}$). To illustrate the angular dependence of $\vec{H}_{c1}$, we shall consider in detail the special (but important) case of $m_x=m_y\neq m_z$.

Setting[26] $\epsilon\equiv m_x/m_z$, we write for $\epsilon<1$

$$\tilde{\kappa}=\kappa_<(\cos^2\theta+\epsilon\sin^2\theta)^{-1/2} \quad , \tag{54}$$

where $\kappa_<=\kappa(m/m_z)^{1/2}$. For $\kappa_< >> 1$ and $\epsilon<1$, we see that $\tilde{\kappa}>>1$ for all θ. For $\epsilon>1$ we write

$$\tilde{\kappa}=\kappa_>(\sin^2\theta+\epsilon^{-1}\cos^2\theta)^{-1/2} \quad , \tag{55}$$

where $\kappa_>=\kappa(m/m_x)^{1/2}$. The cases $\epsilon<1$ and $\epsilon>1$ correspond to materials with layered and filamentary structure, respectively. Equation (55) has the same dependence on angle as Eq. (54) with the replacements $\theta\rightarrow 90°-\theta$ and $\epsilon\rightarrow\epsilon^{-1}$. The component of $\vec{H}_{c1}$ perpendicular to $\hat{B}$ and parallel to $\hat{\theta}$ is given in the large-$\tilde{\kappa}$ limit by

$$H_{c1\perp\vec{B}}\approx\frac{\sin 2\theta}{4\tilde{\kappa}\alpha^2}\left[\frac{m_z-m_x}{m}\right](\ln\tilde{\kappa}-0.503) \quad , \tag{56}$$

which depends upon the sign of $1-\epsilon$. The components of $\vec{H}_{c1}$ and $H_{c1}=|\vec{H}_{c1}|$ itself are related by the following symmetry properties:

$$H_{c1\parallel\vec{B}}(\kappa,m_x/m,m_z/m,\theta)$$
$$=H_{c1\parallel\vec{B}}(\kappa,m_z/m,m_x/m,90^\circ-\theta)\ , \quad (57)$$

$$H_{c1\perp\vec{B}}(\kappa,m_x/m,m_z/m,\theta)$$
$$=-H_{c1\perp\vec{B}}(\kappa,m_z/m,m_x/m,90^\circ-\theta)\ , \quad (58)$$

$$H_{c1}(\kappa,m_x/m,m_z/m,\theta)=H_{c1}(\kappa,m_z/m,m_x/m,90^\circ-\theta)\ . \quad (59)$$

In Fig. 1, we have plotted the angular dependence of $\vec{H}_{c1}$ for $\kappa_<=\kappa(m/m_z)^{1/2}=10$, $\epsilon=10^{-1}$, normalized to its value at $\theta=0$. For materials with filamentary symmetry, this graph corresponds to $\epsilon=10$, $\kappa_>=\kappa(m/m_x)^{1/2}=10$, and $\theta\rightarrow 90^\circ-\theta$. We note that at the symmetry directions, $\theta=0^\circ$ and $\theta=90^\circ$, $\vec{H}_{c1}$ is parallel to $\vec{B}$, as expected. However, for $\theta\sim 70$–75°, $\vec{H}_{c1}$ is directed about 45° from $\vec{B}$. Hence, the magnitude of $\vec{H}_{c1}$ is more sharply angular dependent near $\theta=90^\circ$ than is $H_{c1\parallel\vec{B}}$. This strong angular dependence of H_{c1} near 90° is greatly enhanced as ϵ is decreased, as is shown in Fig. 2 for $\kappa_<=10$ and $\epsilon=10^{-2}$. In fact, for $\epsilon=10^{-2}$, $H_{c1\perp\vec{B}}$ is so much larger than $H_{c1\parallel\vec{B}}$ that $H_{c1}(\theta)$ has a peak at $\theta\sim 80^\circ$.

In Fig. 3, we have plotted the angular dependence of H_{c1} for $\kappa_<=10$ and $\epsilon=10^{-3}$. Since the angular dependence of $\vec{H}_{c1}$ is so strong near $\theta=90^\circ$, we have plotted the logarithm of $\vec{H}_{c1}/H_{c1}$ $(\theta=0)$ as a function of the logarithm of $90^\circ-\theta$. Note that at $\theta\sim 3^\circ$, H_{c1} has a peak, which is roughly 40% larger than its value at $\theta=0$. In the range $90^\circ\geq\theta\gtrsim 87^\circ$, H_{c1} varies by a factor of 20.

Since in Figs. 1–3 we note that, near 90°, $\vec{H}_{c1}$ is nearly perpendicular to B for $\epsilon\ll 1$, it would be instructive to plot $\vec{H}_{c1}$ versus the angle θ_H it makes with the $\hat{e}_z$ axis. In Fig. 4, we have plotted H_{c1} versus θ_H for materials with layered symmetry for $\kappa_<=10$, $\epsilon=10^{-1}$, 10^{-2}, and 10^{-3}. We observe that the curve for $\epsilon=10^{-1}$ is smooth and monotonic, reflecting the fact that $H_{c1}(\theta)$ vs θ does not have a peak. However, for $\epsilon=10^{-2}$ and 10^{-3}, $H_{c1}(\theta_H)$ is multiple valued for small θ_H. Comparing this figure with Figs. 2 and 3, we see that as θ increases roughly to the minimum in $H_{c1}(\theta)$, both $H_{c1}(\theta)$ and $H_{c1}(\theta_H)$ decrease monotonically. When this minimum is reached, however, an increase in θ corresponds to a decrease in θ_H. This decrease in θ_H with increasing θ continues until the maximum in H_{c1} is reached, after which θ_H increases very rapidly with a slight increase in θ.

In Fig. 5, we have plotted $H_{c1}(\theta_H)$ for filamentary systems for $\kappa_<=10$, $\epsilon=10^{-1}$, 10^{-2}, and 10^{-3}. Thus $\theta_H=90^\circ$ on the graph corresponds to the field parallel to the filaments. We note that even for $\epsilon=10^{-1}$, $H_{c1}(\theta_H)$ is multiple valued. As in the layered case, $\theta\sim 90^\circ$ corresponds to a relatively small angle for θ_H.

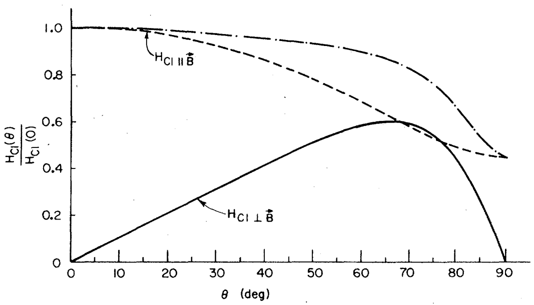

FIG. 1. Shown are the components of $\vec{H}_{c1}$ parallel and perpendicular to $\vec{B}$, and the magnitude of $\vec{H}_{c1}$, as a function of the angle θ that $\vec{B}$ makes with the $\hat{z}$ axis, for $\kappa_<=10$ and $\epsilon=10^{-1}$. The curves are all normalized by $H_{c1}(\theta=0)$.

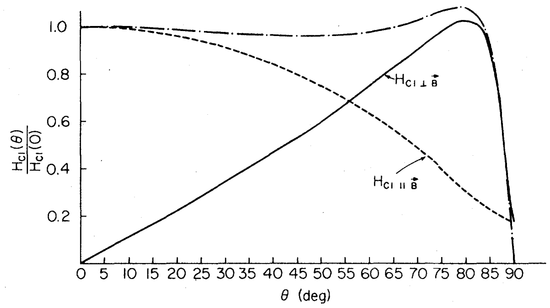

FIG. 2. Shown are the components of $\vec{H}_{c1}$ parallel and perpendicular to $\vec{B}$, and the magnitude of $\vec{H}_{c1}$, as a function of θ for $\kappa_<=10$ and $\epsilon=10^{-2}$.

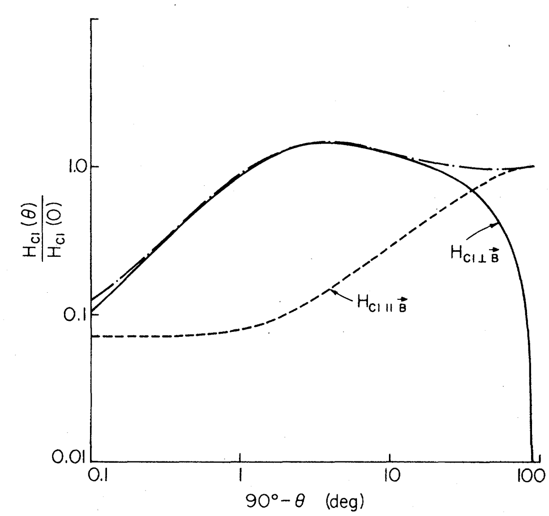

FIG. 3. Shown are the components of $\vec{H}_{c1}$ parallel and perpendicular to $\vec{B}$, and the magnitude of $\vec{H}_{c1}$, normalized by H_{c1} $(\theta=0)$, and plotted as a function of $90^\circ-\theta$, for $\kappa_<=10$ and $\epsilon=10^{-3}$. Note that this is a log-log plot.

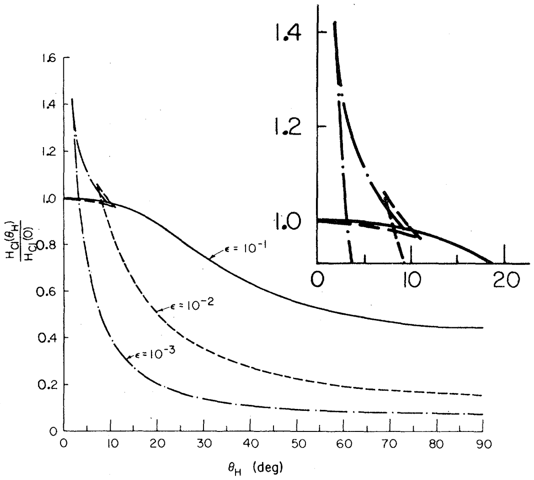

FIG. 4. Shown are plots of $H_{c1}(\theta_H)/H_{c1}(0)$ for materials with layered symmetry, for $\kappa_< = 10$, $\epsilon = 10^{-1}$, 10^{-2}, and 10^{-3}, as a function of θ_H, the angle $\vec{H}$ makes with the $\hat{z}$ axis. The inset is an enlargement of the upper-left-hand portion of the figure.

A comparison of Figs. 4 and 5 reveals that for highly anisotropic materials with either layered or filamentary symmetry, the vortices prefer to lie in the easy direction, for which H_{c1} is smallest, even if H is pointed in some other direction, as long as H is not too near to perpendicular to the easy direction. At angles θ_H for which H_{c1} has multiple values, the smallest H_{c1} is the physical one. Consider, for example, a slab specimen with surfaces parallel to the x-z plane in a parallel applied field $\vec{H}_a = H_a(\hat{z}\cos\theta_H + \hat{x}\sin\theta_H)$. For fixed θ_H, it first becomes energetically favorable for vortices to nucleate when H_a is equal to the smallest value of H_{c1}. This behavior can have dramatic consequences, as can be seen in Figs. 4 and 5. In Fig. 4, which corresponds to materials with layered symmetry, if the anisotropy is sufficiently large, H_{c1} vs θ_H can exhibit a cusp. In Fig. 5, which corresponds to materials with filamentary symmetry, if the anisotropy is not too large, H_{c1} vs θ_H can exhibit a discontinuity, reminiscent of a first-order phase transition; for large anisotropy, it exhibits a cusp. When demagnetization is taken into account, these anomalies are still present, as will be shown in a subsequent publication. These unusual features have never been observed experimentally. We hope that our predictions will stimulate experimental work on this topic.

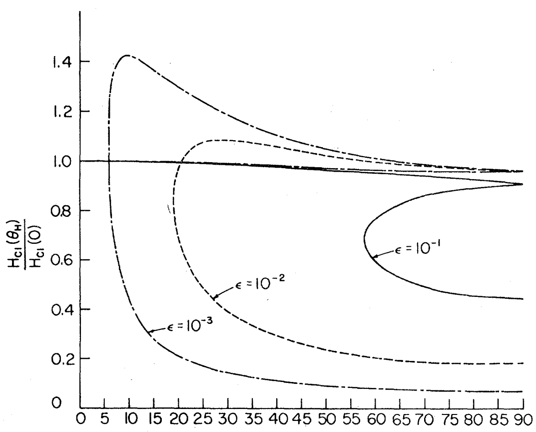

FIG. 5. Shown are plots of $H_{c1}(\theta_H)/H_{c1}(0)$ for materials with filamentary symmetry, for $\kappa_< = 10$, $\epsilon = 10^{-1}$, 10^{-2}, and 10^{-3} (corresponding here to $\kappa_> = 10$ and $m_z/m = 10^1$, 10^2, and 10^3) as a function of θ_H, where $90° - \theta_H$ is the angle $\vec{H}$ makes with the $\hat{z}$ axis.

V. DISCUSSION

We remark that since the Ginzburg-Landau free energy for the anisotropic-mass superconductor can be transformed exactly into isotropic form, $\vec{H}(\vec{B})$ can be found for all values and directions of $\vec{B}$. Since the isotropic Ginzburg-Landau free energy for a single vortex line has been calculated numerically[27] for arbitrary $\tilde{\kappa} > 2^{-1/2}$, these results can be extended for arbitrary $\tilde{\kappa} > 2^{-1/2}$, not just for $\tilde{\kappa} >> 1$. By using Eq. (52), since $\tilde{\epsilon}_1(\tilde{\kappa})$ is known for the isotropic case, the components of $\vec{H}_{c1}$ can be found in an elementary fashion.

We note that since $\tilde{\kappa}$ depends upon the direction cosines of $\hat{B}$ with the crystal lattice, the present treatment applies to materials[4,18,28] that are type I for the directions of $\hat{B}$ in which $\tilde{\kappa} < 2^{-1/2}$ and type II for directions in which $\tilde{\kappa} > 2^{-1/2}$.

Since in the transformed frame $\tilde{b}$ is a function only of $\tilde{\rho} = (\tilde{x}^2 + \tilde{y}^2)^{1/2}$, we may invert the transformation to obtain $b(\vec{r})$. We find that, for a vortex passing through the origin, the surfaces of constant local magnetic field magnitude obey the equation

$$\frac{m_1 m_2}{m^2}(x\sin\phi - y\cos\phi)^2 + \frac{m_3}{m\alpha^2}\left[\cos\theta\left(\frac{m_1}{m}x\cos\phi + \frac{m_2}{m}y\sin\phi\right) - z\sin\theta\left(\frac{m_1}{m}\cos^2\phi + \frac{m_2}{m}\sin^2\phi\right)\right]^2 = \text{const}\left(\frac{m_1}{m}\cos^2\phi + \frac{m_2}{m}\sin^2\phi\right)\Big/\alpha^2 , \tag{60}$$

which is clearly very complicated. We also note that in the mixed state, where the cores of the vortices form a triangular lattice in the transformed frame, the vortex lattice in the laboratory frame will be considerably distorted. Since in the transformed frame the equilateral triangular lattice is degenerate with respect to rotations, for a given θ and ϕ for $\vec{B}$, there will be an infinite number of possible lattices in the laboratory frame. This will be discussed further in a separate publication. Our finding that the anisotropic-mass Ginzburg-Landau theory leads to a degenerate vortex lattice is in disagreement with earlier work by Tilley,[17] who concluded that such a theory predicts locking of the vortex lattice onto the crystal lattice.

Finally, we note that the anisotropic-mass model for a superconductor has applications to other physical problems. For example, the Ginzburg-Landau free energy for a smectic liquid crystal is of the anisotropic-mass form, where the $2e\vec{a}\hbar/c$ is replaced by $(2\pi/d)\delta\vec{n}$, where $\delta\vec{n}$ is the local variation in the unit vector parallel to the local optical axis, and d is the spacing between the layers in the smectic phase.[29] However, as the analog of the magnetic energy term in the free energy is not of the form $(\vec{\nabla}\times\delta\vec{n})^2$, but rather of the form $\kappa_{22}(\vec{\nabla}\cdot\vec{\delta}n)^2+\kappa_{33}[(\hat{n}_0\cdot\vec{\nabla})\delta\vec{n}]^2$, where $\hat{n}_0$ is a unit vector parallel to the mean optical axis, the transformations we have performed will not preserve the form of these terms.

Recently, Hertz has discussed several gauge models for treating spin-glass frustration.[30] Our model would be similar to an anisotropic-mass extension of his Abelian model, but is only formally identical to it if one were to use the $n\rightarrow 0$ replica trick, which has important consequences. Nevertheless, the transformations we have employed would map the anisotropic-stiffness analog of his model into an isotropic-stiffness model, just as in the case we have considered.

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