Critical exponents and amplitude ratios from electrical resistivity measurements of dysprosium

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Resistive anomalies in helical antiferromagnets are shown to reflect the temperature dependence of the specific heat sufficiently close to the Néel temperature. The temperature dependence of superzone energy gaps does not enter. Numerical analysis of electrical-resistance data for a helical antiferromagnet, dysprosium, provides quantitative support for theoretical estimates of critical exponents and amplitude ratios obtained by renormalization-group ϵ expansions and by field-theoretical methods and suggests that dysprosium is described by a model whose order parameter has $n=4$ degrees of freedom.

I. INTRODUCTION

There has been a great deal of interest in resistive anomalies at magnetic critical points following the realization that quasielastic electronic scattering cross sections contain information about the same spin-fluctuation correlation function which determines the temperature dependence of equilibrium properties, such as the internal energy, at the critical point. For example, based on an approximate treatment of the de Gennes-Friedel model,[1] the contribution to the total resistivity in the μth crystallographic direction from itinerant electrons weakly scattering from a set of localized spins $\{\vec{S}_{\vec{R}}\}$ located at lattice points $\{\vec{R}\}$, when normalized to the high-temperature spin-disorder limit, is given for $T > T_c$ by

$$\frac{\rho^{\mu}(T)}{\rho_0^{\mu}} = \sum_{\vec{R}} \phi_{\text{el}}^{\mu}(\vec{R})\Gamma(\vec{R},T) \quad , \tag{1}$$

where $\Gamma(\vec{R},T) = \langle \vec{S}_{\vec{R}} \cdot \vec{S}_{\vec{0}} \rangle / S(S+1)$ and $\phi_{\text{el}}^{\mu}(\vec{R})$ is an electronic coherence factor with a *finite* length scale fixed by the electronic mean-free path,[2] l, or the relevant Fermi-surface caliper,[3] $2k_F^{\mu}$. By contrast, the length scale of the spin-correlation function is $\xi(T) = \xi_0 t^{-\nu} \rightarrow \infty$ as $t = (T-T_c)/T_c \rightarrow 0$. Since electronic length scales provide a cutoff in Eq. (1), and since $\Gamma(\vec{R},T)$ also determines the internal energy, Fisher and Langer[2] concluded that $d\rho^{\mu}(T)/dT \sim t^{-\alpha}$ for $t \rightarrow 0+$ in ferromagnets. A similar conclusion was reached by Mannari[4] by somewhat different reasoning. This conclusion has been generalized to anisotropic ferromagnets and to the ordered state for $t \rightarrow 0-$ for both isotropic[5] and anisotropic[3] ferromagnets. It is generally believed that theory and experiment are in reasonable agreement for ferromagnets although there are cases where more detailed analysis is desirable.

The case of antiferromagnets and helical antiferromagnets, such as occur in the heavy rare-earth metals, has been more problematic. Geldart and Richard[6] concluded that $d\rho^{\mu}(T)/dT$ should reflect specific-heat temperature dependence for $t \rightarrow 0+$ for antiferromagnets as well as for ferromagnets, in contrast to the suggestion of Suezaki and Mori,[7] and also indicated that there should be a temperature regime somewhat above T_N, but still in the critical regime, where $\rho^{\mu}(T)$ should reflect the temperature dependence of "long-range" correlations. The existence of such a generalized Ornstein-Zernike,[5,8,9] or long-range regime, is due to the breakdown of the asymptotic short-distance expansion for $\Gamma(q,T)$ with increasing temperature at fixed $\vec{q}$, as will be seen below, and is not to be confused with the Ginzberg-Landau limit. The above prediction for $T \geq T_N$ has been confirmed by detailed calculations of Richard[10] and others,[11,12] although there is not full agreement in the literature concerning the possible long-range regime.[11] The theoretical situation for $T < T_N$ has been even less clear due to the lack of thorough treatment of effects such as superzone gaps induced by the long-range order, $M(t) \sim (-t)^{\beta}$.

On the experimental side, the above points have received partial support in some cases[13,14] but there are substantial inconsistencies. For example, there have been two experimental studies[13,15] of the c-axis resistivity of dysprosium. The interpretation of the results of the two studies led to diametrically opposed conclusions concerning (i) the relative roles of long-range versus short-range correlations for $t > 0$ and (ii) the role of the long-range-order parameter for $t < 0$. In our opinion, this disparity was not due to purely experimental problems but rather to the highly unsettled state of the theory and the lack of adequate theoretical guides to aid in the quite delicate analysis of data and consequent interpretation of results.

The major objectives of the present work have been twofold. First, the role of the long-range order below the Néel point has been clarified by explicitly deriving the microscopic Boltzmann transport equation describing current flow in the presence of the scattering from spin fluctuations near the critical point. The calculations are lengthy, so we give here only the final result and an outline of the reasoning which leads to it. The net result is that, even in the ordered state below T_N where the thermodynamically averaged sublattice magnetization is nonzero, the effective scattering cross section for $T \approx T_N^-$ involves only $\langle \vec{S}_{\vec{R}} \cdot \vec{S}_{\vec{0}} \rangle$ and not $\langle \vec{S}_{\vec{R}} \cdot \vec{S}_{\vec{0}} \rangle - \langle \vec{S}_{\vec{R}} \rangle \cdot \langle \vec{S}_{\vec{0}} \rangle$; when treated with sufficient accuracy *all* effects due to the long-range order disappear *both* in the cross section and the electronic band structure so one is left with a result very much like Eq. (1) in which the electronic coherence factor has a finite length scale so that $d\rho^\mu(T)/dt \sim |t|^{-\alpha}$ for both $t \rightarrow 0+$ and $t \rightarrow 0-$. The above is in contrast to a common point of view in which the nonzero thermodynamically averaged long-range order below T_N is considered to give rise to a new periodic potential and thereby to new electron energies and velocities which necessarily reflect the temperature dependence of the order parameter. This consequently applies also to the purely electronic coherence function $\phi_{\rm el}^\mu(\vec{R})$ in Eq. (1). Having taken account of the average part of the perturbation, the electrons are then considered to scatter only from deviations from this average so that $\Gamma(\vec{R}\, T) \propto \langle (\vec{S}_{\vec{R}} - \langle \vec{S}_{\vec{R}} \rangle) \cdot (\vec{S}_{\vec{0}} - \langle \vec{S}_{\vec{0}} \rangle) \rangle$ is also to be used in Eq. (1). This line of reasoning is valid for some purposes at temperatures below T_N but must fail very near T_N as it is based on the assumption that the action of the localized spin system on the electronic subsystem is dominated by the mean field $\langle \vec{S}_{\vec{R}} \rangle$. Since the relevant time scales of the electrons are not longer than those of the localized spin system, such a picture can be useful only if the localized spin system itself is well described by mean-field theory, i.e., outside the critical region. For $T \approx T_N$, it is the fluctuating part of the spin field which has the dominating influence on the electrons.

To obtain meaningful results, one must recognize that $|\langle \vec{S}_{\vec{R}} \rangle \cdot \langle \vec{S}_{\vec{0}} \rangle| << |\langle \vec{S}_{\vec{R}} \cdot \vec{S}_{\vec{0}} \rangle|$ for fixed $\vec{R}$ and $T \rightarrow T_N^-$; so a more correct procedure is to calculate the conductivity as a power expansion in the magnetization while avoiding decoupling of spin correlation functions. With this guide, a Boltzmann equation can be derived for the charge current in which the leading temperature dependence is clearly due to short-distance correlations. The deviation proceeds along the general lines established by Baym and Kadanoff but will be given elsewhere[16] in detail as it is rather lengthy.

II. NUMERICAL RESULTS

The above results show that the electric resistivity reflects the temperature dependence of the internal energy so that we can now turn, with some confidence, to our second major objective, which is a quantitative comparison of renormalization-group (RG) predictions for critical exponents and amplitude ratios for the internal energy (or, equivalently, the specific heat) with results of detailed nonlinear least-squares analysis of published resistivity data[13] for c-axis Dy. The results are as follows. The data near T_N are well described by the short-distance expansion, provided we include a *correction* to scaling[17]; amplitude ratios and the leading critical exponent are determined and found to be in good agreement with theory. In particular, the value of α is consistent with that calculated by RG[18] and field-theoretical[19] methods and supports the view that Dy is to be described by a spin model with $n=4$ degrees of freedom. The asymptotic short-distance expansion (SDE) is shown to break down a few degrees above T_N, and the temperature range for $T-T_N>4$ K is found to be consistently described by long-range spin correlations. The amplitude ratio for the leading singular terms in this regime is also found to be in good agreement with that numerically computed from hcp lattice sums.

To examine data near T_N, consider fitting the measured resistance data in the range $176 \lesssim T \lesssim 182.5$ K to the short-distance-expansion form

$$R(T)=C_0+C_1 t+A\,\theta(t)\,t^{1-\alpha}+A'\theta(-t)(-t)^{1-\alpha'} , \quad (2)$$

where $\theta(t)=1(0)$ for $t>0$ (<0). We found $\alpha'-\alpha=0.016\pm0.025$ and equally good fits were obtained with the constraint $\alpha'=\alpha$, even when a regular $C_2 t^2$ correction term was included. It is thus consistent to assume that the leading exponents are indeed equal, as expected, for $t \rightarrow 0\pm$ so this constraint was subsequently adopted and the sensitivity of the fitted parameters in Eq. (2) was studied as a function of the temperature range of the fit. It was found that α was very sensitive to the range, which indicates that important correction terms must be added to Eq. (2). *A priori,* such terms could be t^2, $t^{2-\alpha}$, or $t^{1-\alpha+\Delta_1}$, where Δ_1 is the correction to scaling exponent[17] $\Delta_1=\omega\nu$. Consequently, Eq. (2) was replaced by

$$R(T)=C_0+C_1 t+A\,\theta(t)\,t^{1-\alpha}(1+Ft^{\Delta_1}) + A'\theta(-t)(-t)^{1-\alpha} , \quad (2')$$

and fits were made over a variety of temperature intervals and for a range of values of Δ_1. Based on the usual statistical criteria and the requirement that plots of the fitted versus experimental $R(T)$ should have minimal systematic structure, the best fits were obtained for the interval $176<T<184.5$ K. In these fits, the crossover-correction exponent Δ_1 was held fixed at $\Delta_1=0.55$. We have not been able to find a simultaneous fit of all parameters in $R(T)$; this problem is numerically ill conditioned and would require more accurate data to be soluble. The value $\Delta_1=0.55$ is the RG result for the correction for scaling exponent at $n \simeq 3-4$ and is insensitive to n. These results, which do not improve upon inclusion of t^2 or $t^{2-\alpha}$ corrections, are summarized as follows: $T_N = 180.50 \pm 0.01$ K, $\alpha = -0.20 \pm 0.05$, $C_0 = 118.37 \pm 0.04$ mΩ, $C_1 = -1194 \pm 191$ mΩ, $A = 2781 \pm 222$ mΩ, $A' = -1750 \pm 77$ mΩ, $F = -1.5 \pm 0.4$, and rms error per point $= 28.6$ $\mu\Omega$. The range of this fit is $4 \times 10^{-5} \lesssim |t| \lesssim 2.5(2.2) \times 10^{-2}$ for $t < 0$ ($t > 0$). The confidence intervals given are 95% intervals (corresponding to ± 1.96 standard deviations) obtained in the usual way. We have not attempted to find any F' correction term for $T < T_N$, since the number of measured points in this region is rather low.

To compare RG predictions, rewrite Eq. (1) for the c axis as

$$\frac{\rho^c(T)}{\rho_0^c} = \int^{\text{BZ}} \frac{d^3q}{8\pi^3} \phi_{\text{el}}^c(\vec{q}) \Gamma(q,T) \quad , \tag{3}$$

where $\Gamma(\vec{q},T)$, the Fourier lattice transform of $\Gamma(\vec{R},T)$, has been determined by Fisher and Aharony,[9] for small q and t, to second order in ϵ as

$$\Gamma(\vec{q},T) = \Gamma(\vec{q},T_c)[1 + \tilde{C}_1(q)t + \tilde{C}_2(q)t^{1-\alpha} + \cdots] \ . \tag{3'}$$

The first point is that the experimentally determined $\alpha = -0.20 \pm 0.05$ (to be identified with the specific-heat exponent) is consistent with the second-order RG calculation of Mukamel and co-workers[18] ($\alpha \simeq -0.17$) but does not seem to be consistent with available fixed-field theoretical estimates of α for $n = 3$ systems.[20,21] Thus, the conclusion that Dy is described by a spin model with $n = 4$ degrees of freedom is *tentatively* supported.[18] (However, note also the discussion in Sec. III where reference is made to series-expansion results for α for $n = 3$ systems.) Second, the contribution to C_1 from the slope of the phonon background can be estimated[22] to be $C_1^{\text{ph}} \simeq 37.1$ mΩ so the observed value $C_1 \simeq -1194$ mΩ is completely dominated by the linear term arising from spin fluctuations. Consequently, the observed amplitude ratio $A/C_1 \simeq -2.33$ is given to high accuracy by $\tilde{C}_2(q)/\tilde{C}_1(q) = -(q\xi_0)^{\alpha/\nu}(\gamma-1)/(\gamma+1+\alpha) \simeq -2.24$ for $q \simeq 2k_F^c$; the agreement is obviously excellent.[23] The third quantity to compare with RG predictions is A/A', the negative of the specific-heat amplitude ratio given to order ϵ^2 by Bervillier,[24] from which we estimate $A/A' \simeq -1.7$ which is also seen to be in very satisfactory agreement with the experimental value $A/A' = -1.60$ (with $\simeq 10\%$ probable error). From these results, it is evident that experimental results for Dy and theoretical predications are in excellent agreement in this regime.

However, as the upper limit of the temperature range of the fit was increased significantly beyond 184.5 K, the quality of fits decreased markedly and could *not* be improved by adding further regular terms. We conclude that the short-distance expansion is beyond its limits of validity for $T > 185$ K and that the important values of $1/(q\xi)$ in Eq. (3) are no longer small enough to apply the asymptotic expansion [Eq. (3′)] so that $\Gamma(\vec{q},T)$ is better described in this regime by a long-range spin-correlation function. This was first tested by fitting the data in the range 185 K $< T < T_{\max}$, for a range of values of $T_{\max}$ from 190 to 230 K, to $R(T) = C_0 + C\,t + At^Z$ with T_N fixed at 180.50 K. The results were $C_0 \simeq 118$ mΩ, $C_1 \simeq 60$ mΩ, $A \simeq -50$ mΩ, and $Z \simeq 0.6$ and are *very* different from those of the previous fit; in particular, note that C_1 does *not* reflect the strong negative linear contribution from spin fluctuations and that the leading exponent Z is less than unity. However, the deduced value of Z was found to be sensitive to the temperature range of the fit which indicates that a correction term must be added. To gain some idea about the required correction term, return to Eq. (1) and invoke the long-range representation: $\Gamma(\vec{R},T) = 1$ for $\vec{R} = \vec{0}$ and

$$\Gamma(\vec{R},T) = C_0(a/R)(\kappa a)^{\eta} \exp(-\kappa R) \cos\vec{Q}\cdot\vec{R}$$

for $R > a$ where C_0 is a constant (≈ 0.25), a is the average lattice constant, and $\vec{Q}$ is the magnetic-ordering vector. Since η is expected to be small,[18] we impose the temporary constraint $\eta = 0$. Next, since $\phi_{\text{el}}^c(\vec{R})$ in Eq. (1) provides a cutoff, we may expand the exponential in $\Gamma(\vec{R},T)$ to obtain $\rho^c(T)/\rho_0^c = S_0^c + S_1^c t^{\nu} + S_2^c t^{2\nu} + \cdots$, which implies that the data for $T \gtrsim 185$ K should be described by

$$R(T) = C_0 + C_1 t + A(t^{\nu} + Ft^{2\nu}) \quad , \tag{4}$$

where $T_N = 180.5$ K and, since spin fluctuations in this regime do *not* contribute a linear term in t, C_1 is due to the regular phonon slope given previously as $C_1 \simeq 37.1$ mΩ. With these constraints, data in the range $185 < T < 190$ K were fitted to Eq. (4) with the results: $C_0 = 117.60 \pm 0.02$ mΩ, $A = 43.8 \pm 5.6$ mΩ, $F = -1.0 \pm 0.3$, and $\nu = 0.68 \pm 0.05$ with rms deviation of 5 $\mu\Omega$ per point. Note that this estimate of ν is consistent with that obtained in the short-distance regime, $\nu = \frac{1}{3}(2-\alpha) = 0.73 \pm 0.02$ and with RG predictions. Also, this amplitude ratio is consistent with numerical evaluation of hcp lattice sums which gave $-3 < F = S_2^c/S_1^c \lesssim -1$ for all combinations of model parameters.[25] These results clearly give strong support to the idea that the important spin fluctuations in this regime are of long-range character.

The generalization of Eq. (4) when η is not constrained to be zero is easily determined from Eq. (1) but it is difficult to obtain accurate independent estimates of all parameters. As an alternative, we constrained T_N and C_1 as before and obtained fits for a range of fixed positive values of η to determine the range of η which resulted in values of ν *consistent* with the SDE value of ν in the sense that 95% confidence limits overlapped. With these criteria, the range of allowed values of η was found to be $0 \lesssim \eta \lesssim 0.04$ which is consistent with the RG prediction[18] $\eta \approx 0.02$.

III. SUMMARY AND DISCUSSION

In our opinion, the above results provide strong support for the present interpretation of resistive anomalies at magnetic critical points and also indicate that detailed RG and field-theoretical calculations of critical exponents and amplitude ratios are consistent with experiment in both the short-distance and generalized Ornstein-Zernike or long-range temperature regimes. Furthermore, it is clear that these results are consistent with the view that Dy is to be described by a spin model with $n=4$ degrees of freedom (as opposed, e.g., to $n=3$, which we feel is not favored for the data). As this is a major point of our conclusion, we will discuss it in some detail.

To appreciate in simplest terms that Dy might be described by such a model,[26] recall that critical fluctuations for such spiral-ordering systems occur at wave vectors $\pm\vec{Q}$, where $\vec{Q}$ is the usual magnetic-ordering vector. Since the magnetic moments order in the basal plane, $\langle M_Z \rangle = 0$, there are thus only two (not three) degrees of freedom for each $\pm\vec{Q}$, with the result that there are four degrees of freedom of the order parameter. (These may be regarded as corresponding to left- and right-"handed" spirals with each having x and y components.) If we accept tentatively that Dy corresponds to a $n=4$ system, the next question to consider is that of the effective interactions permitted by symmetry. Then the RG methods can be applied to determine the stable fixed point of interest and its associated critical exponents. This was done to order ϵ^2 in Ref. 18, where it was shown that the only stable fixed point exhibited tetragonal symmetry, even though the critical exponents coincided with those of a fully isotropic $n=4$ spin model. This coincidence is peculiar to $n=4$, and we are not aware of any apparent reason to expect it to remain to higher order in ϵ. However, if our objective is to obtain reasonable theoretical estimates for critical exponents for $n=4$ systems, the following points should be noted. The critical exponents of simple systems appear to give most reliable results when asymptotic series are truncated at order ϵ^2. When anisotropy is taken into account, critical exponents calculated to order ϵ^2 are rather close to their corresponding isotropic counterparts (one striking example of this is indeed provided by the anisotropic Dy model and another by the usual isotropic model with cubic anisotropy[27]). In view of these facts, we feel that reasonable theoretical estimates for critical exponents of Dy are provided by the ϵ^2 expansion of Ref. 18 with an indication of the importance of higher-order terms being provided by field-theoretical calculations for the isotropic $n=4$ model.[19,21] We emphasize, as discussed further below, our objective is to determine whether experimental data for Dy favors $n=4$ degrees of freedom of the order parameter and our basic premise is that the specific-heat exponent, in particular, is much more sensitive to n for $n \simeq 3$ or 4 than it is to perturbations due to anisotropic interactions.[28] Of course, this need not apply to amplitude ratios or to the magnitude of the exponent, Δ_1, giving the leading correction to asymptotic scaling in the short-range regime. (It is for this reason that we also generated fits to the data for a range of values of Δ_1. Results were quoted above for $\Delta_1 = 0.55$. For $\Delta_1 = 0.30$, parameters are altered somewhat, of course. Specifically, α becomes -0.25 ± 0.06, which overlaps the $\Delta_1 = 0.55$ value.) However, based on the above facts, we shall assume that comparison of theory and experiment for the leading critical exponents may be based on estimates provided by isotropic-spin models.

Our conclusion that experimental evidence for Dy is consistent with $n=4$ as regards critical properties is based on two points. First, if superzone gaps should dominate for $T \rightarrow T_N^-$, the leading critical exponent for the resistivity would be 2β, which is $\simeq 0.76 \pm 0.03$ for $n=4$ (Ref. 21) or $\simeq 0.72 \pm 0.02$ for $n=3$.[19] However, the leading exponent is instead found to be $\simeq 1.20 \pm 0.05$ with 95% confidence limits. Clearly, the view that short-range correlations dominate $\rho(T)$ for $T \rightarrow T_N^-$ is favored; so we accept $\alpha \simeq -0.20 \pm 0.05$ as an estimate of the specific-heat exponent of Dy. The second point is then to attempt to decide whether this value for α favors $n=4$ as opposed to other possibilities. Restricting attention to isotropic continuous-spin models, the second-order ϵ expansion gives -0.10 and -0.17 for α, $n=3$ and 4, respectively. The field-theoretical estimates are -0.115 ± 0.015 and -0.21 ± 0.02 for $n=3$ and 4, respectively.[19,21] The essential point, as indicated above, is that α is very sensitive to n and that there is almost a factor of two difference between $n=3$ and $n=4$ estimates; this sensitivity to n is not apparent in other critical exponents or in amplitude ratios. For this reason, we conclude that fits to the experimental data which yield $\alpha = -0.20 \pm 0.05$ for $\Delta_1 = 0.55$ (or $\alpha = -0.25 \pm 0.06$ for $\Delta_1 = 0.30$) with 95% confidence limits (based, of course, on the usual assumptions concerning statistical distribution of errors in the data) are consistent with $n=4$ but probably not consistent with $n=3$ if corresponding RG and field-theoretical estimates of α are adequate guides (see below). We have also done fits with α *fixed* to the $n=3$ and $n=4$ values (-0.115 and -0.211) and found that $\alpha = -0.211$ gives the better fit, with an rms error per point 6% lower than for

$\alpha=-0.115$ for a variety of temperature ranges of the fit. It may be noted parenthetically that if one believed certain factors such as uniaxial anisotropy to be irrelevant for such systems as Dy, one might consider also the possibility that $n=6$. For the $n=6$ isotropic-spin model, $\alpha \simeq -0.27$ to second order in ϵ, and we expect more precise field-theoretical estimates to yield $\alpha \approx -0.38$ on the basis of extrapolation from smaller n (Ref. 19); we therefore consider $n=6$ also to be unlikely for Dy and this possibility will not be discussed further.

It should also be noted that the above discussion is based on the premise that continuous-spin models are appropriate for comparison of theory and experiment. There is not universal agreement concerning this point. In fact, information concerning $n=3$ systems, in particular, is also provided by analysis of high-temperature expansions and by other experiments on systems thought to correspond to $n=3$. For example, for large-spin, high-temperature series expansions[29] $\alpha=-0.14\pm0.04$ for $n=3$; note that the error bars overlap the field-theoretical estimates.[19] Also note that there remain persistent and unresolved questions concerning the possibility of violations of hyperscaling[30] (i.e., scaling relations with explicit reference to dimensionality), with the implied need that RG and field-theoretical work should require reconsideration. Beyond expressing the expectation that the addition of further terms in both the high-temperature and field-theoretically-derived series expansions (with also possible refinements in methods of analysis) *may* resolve these fundamental questions concerning the relevance of various simplified models to real magnetic systems, we have no further comments concerning this question which is quite beyond the scope of this work. Next, it should be pointed out that there exists a variety of experimental data on $n=3$ materials and that, e.g., Kornblit and Ahlers,[31] consider the best experimental estimate of α for $n=3$ systems with isotropic short-range interactions to be provided by $RbMnF_3$, for which $\alpha=-0.14\pm0.01$ has been estimated. It seems to us to be quite possible that the probable error estimates for this system may be too small and that experimental results for α for $n=3$ systems will ultimately be found to be much closer, in fact, to field-theoretical estimates than would be judged from presently available results. This view is based in part on our own detailed analysis of published specific-heat data[32] for Ni in a temperature range where any inhomogeneity smearing and interference from long-range dipolar interactions are expected to have minimal effects. The results of this analysis are consistent with RG and field-theoretical predictions for the leading critical exponent and amplitude ratio. Finally, it may be noted that there has been experimental work directly concerned with study of the specific heat of Dy.[33,34] Unfortunately, the data points of Ref. 33 are too few to permit analysis of the present type with reasonable confidence limits, while the conclusions of Ref. 34 are not valid as their method of numerical analysis of data is known to be seriously inadequate.[35,36] Consequently, we have no other independent experimental estimates of the specific-heat exponent of possible $n=4$ system.

In conclusion, our appraisal of all of the above results leads us to believe (i) that the long-range order, as reflected in superzone gaps, does not enter quasistatic transport properties for $T \rightarrow T_N^{\pm}$ and that the dominant temperature dependence of such properties is due to short-range spin correlations; (ii) that the critical properties of simple spiral-helical antiferromagnets such as Dy are probably described by a spin model with $n=4$ degrees of freedom of the order parameter; and (iii) that experiment and theory are in encouraging quantitative agreement concerning critical exponents and amplitude ratios. However, the accuracy of this analysis of available data is still not adequate to resolve more detailed questions such as the precise role of various possible anisotropic corrections to critical exponent and amplitude ratios for these systems. We wish to encourage further experimental and theoretical work in this area.

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[20]For isotropic $n=3$ systems, $\alpha \simeq -0.10$ to second order in ϵ. The more precise estimates of Ref. 19 yield $\alpha=-0.115 \pm 0.015$. It is difficult to see how the experimental result $\alpha=-0.20 \pm 0.05$ could be consistent with either isotropic or anisotropic $n=3$ models.

[21]It is interesting that the exponents $\alpha=-0.20 \pm 0.05$ $[\nu=\frac{1}{3}(2-\alpha)=0.73 \pm 0.02]$ are in excellent agreement with the results $\alpha=-0.21 \pm 0.02$, $\nu=0.74 \pm 0.01$ obtained by the methods of Ref. 19 for the fully isotropic $n=4$ model [B. G. Nickel (private communication)]. Other critical exponents are also available and are consistent with scaling.

[22]Nonmagnetic lutetium has the same crystal structure and essentially the same Debye temperature as Dy so C^{ph} for Dy is obtained from the slope of the resistivity of Lu at $T=180.5$ K. See D. W. Boys and S. Legvold, Phys. Rev. 174, 377 (1968).

[23]The dominant q's which enter Eq. (3) are of order $2k_F^c$ which is estimated from the average c-axis Fermi-surface caliper given by band-structure calculations. See the review by A. J. Freeman, in *Magnetic Properties of Rare Earth Metals,* edited by R. J. Elliot (Plenum, London, 1972), p. 245.

[24]C. Bervillier, Phys. Rev. B 14, 4964 (1976).

[25]The model parameters are the Fermi-surface anisotropy, density of current carriers, the electron mean free path and the magnetic-ordering vector.

[26]The following discussion is based on the results of Ref. 18 and is included here for the convenience of readers who may not be familiar with the details of that work.

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[28]See also M. E. Fisher, Rev. Mod. Phys. 46, 597 (1974).

[19]High-temperature series-expansion data for Heisenberg models are reviewed by G. S. Rushbrooke, G. A. Baker, Jr., and P. J. Wood, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1974), p. 246.

[30]See G. A. Baker, Jr., Phys. Rev. B 15, 1552 (1977) for recent discussion.

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