Two-point correlation function near the coexistence surface

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Differential renormalization-group equations for the one-particle-irreducible vertex functions are solved analytically to provide information about the scaling properties of the correlation function at small and moderate wave vectors everywhere in the critical region. For Ising-like spin systems, this extends previous work on the anomalous dimension crossover function to nonzero magnetization. The susceptibility-correlation-length relative-amplitude ratio is correctly given to $O(\epsilon)$, $\epsilon=4-d$; moreover, a fully nonlinear crossover-amplitude function is obtained. For general n-component spins, the longitudinal part of the correlation function, $\Gamma_{2,L}$ is calculated to $O(\epsilon)$ in crossover form for small and moderate k. The asymptotic singularity $\Gamma_{2,L} \sim k^{\epsilon}$ is automatically incorporated. The result is exact in the spherical limit $n \rightarrow \infty$.

I. INTRODUCTION

The use of nonlinear solutions of differential renormalization-group equations has proved to be a powerful method for the calculation of thermodynamic functions for critical phenomena in crossover exponentiated form.[1-3] This approach provides results which represent a summation of an infinite class of Feynman diagrams; thus, instead of a series of logarithmic terms, an expression with power-law behavior is obtained. A natural auxiliary benefit is that the first correction-to-scaling behavior is generally automatically incorporated in a crossover equation form (the crossover being between simple-Landau and true-critical behavior; other more complex crossovers are also obtainable). Most recently,[1] what was called the anomalous dimension crossover function, which relates the susceptibility to the correlation length, was calculated for general order O multicritical points in the disordered phase. For the ordinary critical case, the results are in essential agreement with earlier calculations based on equations of the Callan-Symanzik type due to Lawrie[4] and Bruce and Wallace.[5] This function carries the effects of a nonzero value of the critical exponent η. In view of the small size of this exponent, the crossover behavior involving η may well be unobservably small.

However, in the presence of a nonzero magnetization (in particular on the coexistence surface), the relationship between the susceptibility and correlation length involves more than η. These M-dependent contributions to the anomalous dimension crossover function are relatively large. Formally, they are $O(\epsilon)$ rather than $O(\epsilon^2)$, where $\epsilon=4-d$. At $d=3$, they may be 5–10 times as large. In this paper, we extend the earlier results to the ordered phase of an ordinary Ising-like critical point.[6]

For general n-component spin models, both the transverse and longitudinal susceptibilities diverge on the coexistence surface. To make the corresponding extension, an exponentiated nonlinear crossover form of the general n wave-vector-dependent longitudinal susceptibility is calculated to $O(\epsilon)$ with emphasis on the behavior near the coexistence surface and restricted to moderate k. If the four-spin coupling constant u is set equal to its fixed-point value, the present result is the same as that of Schäfer and Horner[7] to the order given here. If the power-law singularities are expanded into logarithms, the result reproduces the longitudinal part of the $O(\epsilon)$ full-susceptibility tensor of Brezin *et al.*,[8] as well as the conjectured partial exponentiation of Mazenko.[9] Crossover to mean-field behavior is included, in contrast to Refs. 7–9. The calculation gives an exact result in the spherical limit $n \rightarrow \infty$. In the Ising case, $n=1$, the $O(\epsilon)$ part of the anomalous dimension crossover function is recovered.

For small k a somewhat different exponentiation from that of Achiam and Kosterlitz[10] [who use a double-matching technique to obtain the correlation function for all k in an implicit rather than (as here) explicit] is obtained.

In Sec. II, some general properties of the anomalous dimension crossover function (henceforth, D) are discussed for $M \neq 0$, with applications to the scaling of the irreducible vertex functions. An exact nonlinear scaling result for D is given.

In Sec. III, the calculation of D to leading order is detailed for the ordinary critical point, $n=1$. The one-particle-irreducible 1PI renormalization-group generator[11] is used. The calculation serves to illustrate how the (1PI) equations are used to obtain nonlinear results.

In Sec. IV the longitudinal two-point function (whose inverse is the susceptibility) is calculated to $O(\epsilon)$. Although a strictly rigorous nonlinear solution cannot be obtained analytically, the approximations made should provide good exponentiations in some regions of the wave vector k. As mentioned above, the result is exact in the spherical limit and reproduces the results of Sec. III for $n=1$.

In Appendix A, an alternate definition of quasiglobal eigenfunctions of the renormalization group is introduced. Although the new definition has no immediate consequences for the present calculation, it does clarify the relationship between the alternate derivations of D given in Ref. 1. For calculations to higher order in perturbation theory, the present definition should be useful.

In Appendix B, a few technical details needed in the derivation of D used in Sec. III are established.

II. ORDERED PHASE: GENERAL RESULTS

The basic idea used in Ref. 1 was to maintain a specified form of the partially renormalized correlation function $\Gamma_2(k)$ so that for all values of the renormalization parameter l

$$\Gamma_2(k,l)=\Gamma_2(k=0,l)+k^2+O(k^4) \quad . \tag{2.1}$$

This is done by choosing an appropriately defined rescaling factor $\eta(l)$; at a fixed point $\eta(l)$ reduces to the value of the critical-point exponent η for the critical behavior controlled by that fixed point. Any thermodynamic or correlation functions can be expressed in terms of the unscaled physical magnetization M_k, nonlinear scaling fields and the anomalous dimension crossover function D

$$D\equiv\exp\int_0^\infty \eta(l)\,dl \quad . \tag{2.2}$$

For example, the physical value of the correlation function is given by

$$\Gamma_2(k)=D[\lim_{l\to\infty}\Gamma_2(k=0,l)\exp(-2l)+k^2+O(k^4)] \quad . \tag{2.3}$$

In Eq. (2.3), the indicated limit just the inverse square of the correlation length ξ^{-2}, which is an exact nonlinear scaling field. The inverse susceptibility $[=\Gamma_2(k=0)]$ is therefore $\chi^{-1}=D\xi^{-2}$. As detailed in Ref. 1, the correct [to $O(\epsilon^2)$] expression for D is given by (in the disordered phase)

$$D=Y^{-\eta/\epsilon}\exp[(Y-1)(1-\bar{u})] \quad , \tag{2.4a}$$

$$Y^{-1}=1+\bar{u}(\xi^{\epsilon}-1) \quad . \tag{2.4b}$$

In Eq. (2.4), $\bar{u}=u/u^*$, and u is the value of the four-spin coupling constant and u^* its fixed-point value, $u^*=\frac{1}{3}\epsilon$.

In the presence of a finite magnetization ($n=1$), a magnetization-dependent value of η may be defined in a similar fashion so that

$$\Gamma_2(\phi_0,k,l)=\Gamma_2(\phi_0,k=0,l)+k^2+O(k^4) \quad ,$$

where ϕ_0 is a scaled magnetization variable (defined below). The same procedure is now applied to calculate D.

Before giving the actual calculation, a discussion of the general features of the approach will clarify the method and results. Renormalization-group generators such as the one-particle-irreducible generator give an expression for the renormalization of the Helmholtz functional A; this can be expressed in terms of the physical fields M_k or in terms of rescaled fields. We define the "minimally rescaled" field ϕ_k by

$$M_k=\phi_k(l)\exp\left[-\tfrac{1}{2}\int_0^l \eta(\phi_0(y),y)\,dy\right] \quad , \tag{2.5}$$

where $\eta(\phi_0,l)$ is the magnetization-dependent rescaling factor to be defined below. The function $\phi_0(l)$ satisfies

$$\frac{\partial\phi_0(l)}{\partial l}=\tfrac{1}{2}\eta(\phi_0(l),l)\phi_0(l); \qquad \phi_0(l=0)=M \quad . \tag{2.6}$$

The use of a magnetization-dependent η leads to a nonlinear scaling of the fields[12] {the usual fully scaled field $s_k=\phi_k\exp[\frac{1}{2}(d-2)l]$}.

The free-energy functional given by

$$A=\sum_p\frac{\hat{\Gamma}_p(k_1\cdots k_p,\phi_k=0)}{p!}\phi_{k_1}\cdots\phi_{k_p} \tag{2.7}$$

can also be written in terms of the M_k:

$$A=\sum_p\frac{\hat{\Gamma}_p(k_1\cdots k_p,\phi_k=0)}{p!}M_{k_1}\cdots M_{k_p}D^{p/2} \quad , \tag{2.8a}$$

where D is the magnetization-dependent anomalous dimension crossover function.

$$D=\exp\int_0^\infty \eta(\phi_0(y),y)\,dy \quad . \tag{2.8b}$$

The physical vertex functions are defined by functional derivatives with respect to the M_k

$$\Gamma_p=\frac{\delta^pA}{\delta M_{k_1}\cdots\delta M_{k_p}} \quad . \tag{2.9}$$

Applying the functional derivatives to Eqs. (2.8) there will be contributions of δ functions $\delta_{k_i,0}$ from the derivatives of D which cancel corresponding terms in the renormalization-group equations for $\hat{\Gamma}_p$. For k_i infinitesimally different from zero, the δ functions are absent

$$\Gamma_p(k_1\cdots k_p,M_k)=D^{p/2}\hat{\Gamma}_p(k_1\cdots k_p,\phi_k) \quad . \tag{2.10}$$

Extending the arguments given in Ref. 1, it is easy to see that the minimally rescaled vertex functions are exact nonlinear scaling fields in $\phi \equiv \phi_0$ ($l=\infty$) of canonical dimension: for example, $\hat{\Gamma}_2(k=0, t=0) \simeq \phi^{4/(d-2)}$. This exact nonlinear scaling of the $\hat{\Gamma}_p$ in ϕ may be useful in the consideration of alternate exponentiations of perturbative results for the vertex functions or other thermodynamic quantities. From this scaling property and the corresponding property shown for the vertex functions $\tilde{\Gamma}_p$ of Ref. 1 scaled with a magnetization-independent $\eta(l)$ it follows that $D(\phi)/D(\phi=0)$ is an exact nonlinear scaling invariant. The vertex functions scaled with $D(0)$ are singular at the critical temperature while those of the present calculation are not; for example, the $k=0$ two-point function $\tilde{\Gamma}_2$ behaves as $M^{4/(d-2)} \times (t/M^{1/\beta})^{\eta\nu}$ as $t \to 0$. The term in brackets is a scaling invariant, so that this two-point function is still of canonical dimension. The use of $D(\phi)$ simply absorbs the temperature singularity so that the minimally rescaled (and physical) vertex functions may be finite and nonzero at $t=0$. An exact calculation in either approach would, of course, give the same physical vertex functions; however, in low-order approximations an erroneous behavior may be introduced. This is illustrated in the calculation of the equation of state; compare the renormalized zero-loop approximation and the fully exponentiated solutions given in Ref. 2.

III. CALCULATION OF D FOR THE ORDINARY CRITICAL POINT

The discussion of Sec. II applies to all critical systems to any order in perturbation theory. In this section, the leading behavior for the ordinary Ising critical point will be calculated. As the use of the 1PI renormalization equations may be unfamiliar, the method will be given in some detail. The renormalization-group equation for $\hat{\Gamma}_2$ is

$$\frac{\partial\hat{\Gamma}_2}{\partial l} = -\eta\hat{\Gamma}_2 - \frac{1}{2}\eta\frac{\phi\partial}{\partial\phi}\hat{\Gamma}_2 + \exp(-dl)\int\frac{d\Omega}{\Omega}\frac{\hat{\Gamma}_4(\vec{\Omega}e^{-l}, -\vec{\Omega}e^{-l}, \vec{k}, -\vec{k})}{\hat{\Gamma}_2(\Omega e^{-l})}$$
$$- \exp(-dl)\int\frac{d\Omega}{\Omega}\int d^d\vec{g}\,\frac{\hat{\Gamma}_3(\vec{\Omega}e^{-k}, \vec{g}, k)^2\delta(\vec{g}+\vec{k}+\vec{\Omega}e^{-l})\Theta(|g|e^l-1)}{\hat{\Gamma}_2(\vec{g})\vec{\Gamma}_2(\vec{\Omega}e^{-l})} \quad . \tag{3.1}$$

The first two terms of Eq. (3.1) are a consequence of the rescaling of the magnetization variable ϕ. The integrals represent the infinitesimalization of the one-loop diagrams contributing to the two-point function. All the vertex functions in Eq. (3.1) are functions of ϕ. Similar equations can be written for the three- and four-point functions but they will not be needed. Requiring that the k^2 term be invariant leads to

$$\eta = \eta_4 + \eta_{3,3} \ , \tag{3.2}$$

where η_4 and $\eta_{3,3}$ are the k^2 parts of the corresponding integrals. Note that Eqs. (3.1)–(3.2) are exact; the approximations arise in the expressions used for the three- and four-point functions.

In the ordered phase $\hat{\Gamma}_3=0$ and the k-dependent parts of $\hat{\Gamma}_4$ are $O(\epsilon^2)$ and expressible in terms of the $O(\epsilon)$ solution for the k-independent Γ_4. In the ordered phase, $\hat{\Gamma}_3 \sim \phi_0\Gamma_4$ and $\hat{\Gamma}_3^2$ is $O(\epsilon)$. Therefore, $\eta_{3,3}$ is $O(\epsilon)$ and, as will be shown below, η_4 remains $O(\epsilon^2)$. In general, the $O(\epsilon)$ perturbation induced by $\eta_{3,3}$ would disrupt the $O(\epsilon)$ solutions for the vertex functions. As shown in Appendix B, the asymptotic behavior of the renormalization-group equations is not significantly altered. This is consequence of the lack of any slow-transient or slow-growth behavior in $\eta_{3,3}$ which at $\epsilon=0$ would have induced logarithmic singularities.[13]

Within these approximations, η_4 can be evaluated as before

$$\eta_4 \equiv \frac{\eta}{u^{*2}}\Gamma_4^2\frac{\exp(2\epsilon l)}{(1+\hat{\Gamma}_2\exp 2l)^2} + O(\epsilon^3) \ , \tag{3.3}$$

where Γ_4 is the solution for the four-point function given in Ref. 2, and $\eta = \frac{1}{54}\epsilon^2$. Similarly, $\eta_{3,3}$ can be written

$$\eta_{3,3} = \frac{-\exp(\epsilon l)\hat{\Gamma}_4^2\phi_0^2}{(1+\hat{\Gamma}_2\exp 2l)} \times \left[\int\frac{d\Omega}{\Omega}\frac{\Theta(|\vec{\Omega}+\vec{k}e^l|-1)}{\hat{\Gamma}_2\exp(2l)+|\vec{\Omega}+\vec{k}e^l|^2}\right]_{k^2\,\mathrm{part}} \ . \tag{3.4}$$

This function behaves like $\exp[(2+\epsilon)l]$ for small l and like $\exp(-dl)$ for large l. Its integral over l is finite even at the critical point and does not depend strongly on ϵ. This is in contrast to η_4 which has an initially slow growth, proportional to $\exp(2\epsilon l)$ and the integral of which diverges at the critical point. For $\eta_{3,3}$ the slow variation in $\hat{\Gamma}_4$ is completely swamped by the more rapidly changing behavior of the angular integral. Therefore, in Eq. (3.4) the vertex functions will be set equal to their asymptotic limits. Furthermore, the integral of $\eta_{3,3}$ over l can be extended to $l=-\infty$ since the $\exp(2l)$ factor provides the necessary rapid damping. The effect of all the approximations is a factor of ϵ smaller than the final result obtained. The integrals are now easily performed:

$$D = Y^{-\eta/\epsilon}\exp[(1-\bar{u})(Y-1)\eta/\epsilon + \tfrac{1}{12}\hat{\Gamma}_4^2\phi^2\xi^{2+\epsilon}] \ . \tag{3.5}$$

Of course, to the order calculated, the last factor could be replaced by $\Gamma_4^2 M^2/\Gamma_2$, or any number of expressions equivalent to $O(\epsilon)$. The form used in Eq. (3.5) is an exact nonlinear invariant. Combined with the approximations made for η_4, the total expression in Eq. (3.5) does satisfy the exact invariance property described in Sec. II.

At the fixed point, D simplifies to give the asymptotic relation between the susceptibility and correlation length.

$$\chi^{-1} = \xi^{(2-\eta)}\exp(+\tfrac{1}{12}\phi^2\xi^{2-\epsilon}u^{*2}) \ . \tag{3.6}$$

Using the equation of state,[2] the argument of the exponential may be evaluated (it is finite everywhere in the critical region). The universal amplitude ratio of $\chi^{-1}\xi^{2-\eta}$ is thus found to be

$$\frac{A_-}{A_+} = 1 + \tfrac{1}{12}\epsilon + O(\epsilon^2) \ , \tag{3.7a}$$

$$A_\pm \equiv \lim_{t\to 0\pm} \chi^{-1}\xi^{2-\eta} \ , \tag{3.7b}$$

in agreement with earlier calculations.[8,14]

The specific form of the exponentiation of Eq. (3.5) is not particularly suggestive. The argument of the exponential represents only the amplitude of the leading singularity and thus cannot be checked by any simple asymptotic scaling property. In Sec. IV, an alternate approach to the longitudinal correlation function for all values of n provides (in the limit $n=1$) a different form of the $O(\epsilon)$ part of the anomalous dimension crossover function, identical to $O(\epsilon)$, but with a more plausible exponentiation.

The corrections-to-scaling effects (crossover effects) in Eq. (3.5) are now $O(\epsilon)$, being generated by the $\hat{\Gamma}_4^2\phi^2$ term. The $O(\epsilon)$ part of the crossover may be written

$$D = 1 + \tfrac{1}{12}\Gamma_4^2 M^2\xi^{2+\epsilon} + O(\epsilon^2) \ . \tag{3.8}$$

In Eq. (3.8) the distinctions between Γ_4 and $\hat{\Gamma}_4$ on the one hand, and ϕ^2 and M^2 on the other, have been dropped. Γ_4 is given in Ref. 2 as $\Gamma_4 = uY$, with Y given in Eq. (2.4b). The numerical factor of 12, although correct in the ϵ expansion, might be considered as an adjustable constant in comparison with experiment. To $O(\epsilon^2)$ the form of Eq. (3.8) would, of course, also change, but a detailed calculation would be cumbersome in the present approach.

IV. LONGITUDINAL CORRELATION FUNCTION

As is well known,[8] the ϵ expansion for the thermodynamic functions of the n-component spin model has some difficulties of interpretation in the ordered phase. This is a consequence of the divergence of the longitudinal susceptibility on the coexistence surface. Denoting the longitudinal part of the two-point function as $\Gamma_{2,L}(k,h)$, the asymptotic behavior is given by

$$\Gamma_{2,L}(k=0,h) \sim h^{\epsilon/2} \ , \tag{4.1a}$$

$$\Gamma_{2,L}(k,h=0) \sim k^{\epsilon} \ . \tag{4.1b}$$

An exponentiated expansion is one approach to reexpress the logarithms of the ϵ expansion in a form which contains Eqs. (4.1). For $k=0$, a solution for the equation of state was given in Ref. 2 which embodied Eq. (4.1a) and which was exact in the spherical limit, $n\to\infty$. An alternate exponentiation due to Nelson[3] incorporated Eq. (4.1a) but did not reproduce the spherical limit. This indicates that different *a priori* reasonable methods may be expected to provide distinct exponentiations. Complete criteria for judging exponentiation are presently lacking. When possible, comparison with higher-order ϵ expansions or exact results may be used as partial checks.[2] In this section, we extend the $k=0$ to finite (small) k for the longitudinal correlation function. A fully nonlinear solution of the differential renormalization-group equations does not seem feasible; however, in the special cases of $n=1$ and $n=\infty$, the approximations made reduce to the anomalous dimension crossover function of Sec. III (in a different exponentiation) and the exact spherical result, respectively. The result is given in nonlinear crossover form, containing the effects of the first corrections-to-scaling variable. For $u=u^*$, the result is similar to that of Schäfer and Horner[7] and when ϵ expanded gives the $O(\epsilon)$ result of Brézin *et al.*[8] The partly exponentiated form conjectured by Mazenko[9] is also contained within the present result and may be recovered by a partial expansion in ϵ. However, since a fully nonlinear solution is not used, the exponentiation is not uniformly good. In particular, the large k region is excluded (the asymptotic properties in this region have been given by Brézin *et al.*[15]).

We begin with the renormalization-group equation of the 1PI generator

$$\frac{\partial A}{\partial l} = \exp(-dl)\int\frac{d\vec{\Omega}}{\Omega}\int \mathrm{tr}_{(ij)}[A^{ij}_{\vec{s},\vec{s}} - A^{ir}_{\vec{s},\vec{g}}(A^{-1})^{rt}_{\vec{g},\vec{g}'}A^{tj}_{\vec{g}',\vec{s}}] \ , \tag{4.2a}$$

where

$$A^{ij}_{\vec{p}\vec{q}} = \frac{\delta^2 A}{\delta M^i_{\vec{p}}\,\delta M^j_{-\vec{q}}} = \Gamma_2^{ij}(p,-q) \ . \tag{4.2b}$$

In Eq. (4.2a), $\vec{s}$ denotes a shell wave vector, $\vec{s}=\vec{\Omega}\exp(-l)$ and $\vec{g}$ a "greater-than-shell" wave vector, $|g|>\exp(-l)$; $(i,j,r,t,\ldots)$ denotes the components of $\vec{M}_k$; sums over repeated indices and integrals over $\vec{g}$ and $\vec{g}'$ are implied.

To obtain the equation for the two-point function $\Gamma_2^{ik}(k,-k)$, two functional derivatives of Eq. (4.2a) are taken. The result is simplified by evaluating for a uniform magnetization M to enforce momentum conservation. To lowest order, we may also neglect the k dependence of the three- and four-point vertex functions (this restricts us to relatively small k in a sense that will be made clear). Separating the two-point function into transverse and longitudinal parts

$$\Gamma_2^{ij}(k)=\Gamma_{2,T}(k)(\delta_{ij}-\hat{M}_i\hat{M}_j)+\hat{M}_i\hat{M}_j\Gamma_{2,L} \tag{4.3}$$

(where $\hat{M}_i$ is a unit vector in the direction of the magnetization), we have the following renormalization-group equations

$$\dot{\Gamma}_{2,T}(k)=\tfrac{1}{3}\exp(-dl)\int\frac{d\vec{\Omega}}{\Omega}\left[\left(\frac{1}{\Gamma_{2,L}(\vec{s})}+\frac{n+1}{\Gamma_{2,T}(\vec{s})}\right)\Gamma_4 -\tfrac{2}{3}M^2\Gamma_4^2\Theta(\exp(l)|\vec{s}+k|-1)\left(\frac{1}{\Gamma_{2,L}(\vec{s})\Gamma_{2,T}(\vec{s}+\vec{k})}+L\rightleftarrows T\right)\right], \tag{4.4a}$$

$$\dot{\Gamma}_{2,L}(k)=\exp(-dl)\int\frac{d\vec{\Omega}}{\Omega}\left[\left(\frac{1}{\Gamma_{2,L}(\vec{s})}+\frac{\frac{1}{3}(n-1)}{\Gamma_{2,T}(\vec{s})}\right)\Gamma_4-\tfrac{2}{9}M^2\Gamma_4^2\Theta(\exp(l)|\vec{s}+\vec{k}|-1)\times\left(\frac{9}{\Gamma_{2,L}(\vec{s})\Gamma_{2,L}(\vec{s}+\vec{k})}+\frac{n-1}{\Gamma_{2,T}(\vec{s})\Gamma_{2,T}(\vec{s}+\vec{k})}\right)\right]. \tag{4.4b}$$

As $k\to 0$, these equations are equivalent to those used previously. In that limit the sole effect of the step functions averaged over all angles is to introduce a factor of $\frac{1}{2}$. The product of transverse and longitudinal propagators is reduced by noting that in this approximation $\Gamma_{2,L}=\Gamma_{2,T}+\Gamma_4 M^{2/3}$ and that $\Gamma_{2,L}(s)=s^2+\Gamma_{2,L}$, and $\Gamma_{2,T}=s^2+\Gamma_{2,T}$ to lowest order. When no argument is indicated, the value at $k=0$ is meant.

By using the lowest-order forms and neglecting the l dependence of the vertex functions, Eq. (4.4) can be trivially integrated. It is useful to recall how that integration is done. The k-dependent parts involve propagators at two different momenta and effective masses

$$I(k,m_1^2,m_2^2)=\int\frac{d^dp}{(p^2+m_2^2)[(\vec{p}+\vec{k})^2+m_1^2]}\ . \tag{4.5}$$

Feynman parameters are used to combine the propagators

$$I(k,m_1^2,m_2^2)=\int_0^1 d\alpha\int\frac{d^dp}{\{\alpha(p^2+m_2^2)+(1-\alpha)[(\vec{p}+\vec{k})^2+m_1^2]\}^2}\ . \tag{4.6}$$

Changing variables, we set $\vec{q}=\vec{p}+(1-\alpha)\vec{k}$; the integral can now be done analytically (cf. Appendix A of Ref. 2). The leading behavior (imposing a cutoff of unity on the q integration)

$$I(k,m_1^2,m_2^2)=\frac{1}{\epsilon}[\Pi(k,m_1^2,m_2^2)-1]\ , \tag{4.7}$$

where the function Π is[16]

$$\Pi\equiv\Gamma(1+\tfrac{1}{2}\epsilon)\Gamma(2-\tfrac{1}{2}\epsilon)\times\int[\alpha(1-\alpha)k^2+\alpha M_1^2+(1-\alpha)M_2^2]^{-\epsilon/2}d\alpha\ . \tag{4.8}$$

The use of the ϵ expansion of Eqs. (4.7) and (4.8) yields the $O(\epsilon)$ results of Brézin *et al.*[8] We will use the notation $\Pi(k,m^2)=\Pi(k,m^2,m^2)$ subsequently.

This method cannot be directly applied to Eq. (4.4) when Eq. (4.4) is considered nonlinearly. By infinitesimalizing the renormalization process, the integral over all momenta [such as p in Eq. (4.5)] is broken down into an angular integral over a shell, with masses ($\Gamma_{2,L}$ and $\Gamma_{2,T}$) and vertices which depend on the magnitude of the shell momentum [$=\exp(-l)$]. Thus, although the propagators may be combined with Feynman parameters, the change of variable is not simply done. The vertex functions are constants on shells concentric about the original origin; on the other hand, the convenient integration variable is offset with respect to that origin. Finally the step function complicates any integration. A solution of Eq. (4.4) using all the information contained in the nonlinear equations would be very difficult.

To proceed, we look for a limit in which simplifications may be made. By the use of k-independent vertices, the large-k region is already excluded. The longitudinal equation (4.4b) is somewhat simpler since it does not involve mixed transverse and longitudinal propagators. If $k^2 << \Gamma_2$ (more precisely $\bar{\Gamma}_2$, see below), the effect of the offcenter integrals may be assumed to be small; the integrals over the renormalization-group parameter l for the transverse propagator are always cutoff by the longitudinal propagator, even in the limit of zero magnetic field. Thus the solution to Eq. (4.4a) may be approximated by

$$\Gamma_{2,T}(k,l) = \Gamma_{2,T}(l) + k^2 + O(\epsilon k^2/\bar{\Gamma}_2) \quad , \tag{4.9}$$

where $\Gamma_{2,T}(l)$ can be taken from Ref. 2. This approximation will affect the final result only in $O(\epsilon^2 k^2/\bar{\Gamma}_2)$ terms.

The longitudinal equations are to be handled by means of an approximation of the value of $\Gamma_4(l)$. The k-independent value is shifted an $O(\epsilon^2)$ amount to correspond to the shells defined by the k-dependent propagator terms. This of course does not affect the $O(\epsilon)$ result at all. At $O(\epsilon)$ the effective value of l at which the contribution is cutoff is governed by the Π function alone. Thus the *leading* part of the exponentiated behavior is also governed by Π. Since we are interested in $k << 1$ the step function can be replaced by $\frac{1}{2}$. If the Γ_4 were constant this would be exact, so again we are introducing a nonleading $O(\epsilon^2)$ error. The total error depends on the relative sizes of k^2, h/M, and $\Gamma_{2,L}$. For the longitudinal propagators, the error introduced will again be $O(\epsilon^2 k^2/\Gamma_{2,L})$. The error in the transverse terms is more complicated. For $h/M << k^2$, the error will be $O[\epsilon^2(h/M/k^2)]$; for $h/M >> k^2$, the error induced is $O[\epsilon^2 k^2/(h/M)]$; for intermediate values of k, we can only state that the error is $O(\epsilon^2)$, but no additional factor enters to further reduce the error. Small errors at $O(\epsilon^2)$ may be important for reliable exponentiation. Within these deformed trajectory approximations the longitudinal correlation function is given by

$$\Gamma_{2,L}(k^2) = \frac{h}{M} + k^2 + \frac{1}{3}\,\frac{uM^2}{1-\bar{u}+\bar{u}[(n-1)/(n+8)]\Pi(k,h/M)+\bar{u}[9/(n+8)]\Pi(k,\bar{\Gamma}_2)} \quad . \tag{4.10}$$

The function $\bar{\Gamma}_2$ is the inverse longitudinal susceptibility shorn of transverse effects.[2] The equation of state governing h/M and $\bar{\Gamma}_2$ is given in Eqs. (3.7) and (3.11) of Ref. 2. Setting $\bar{u}=1$ and ϵ expanding, the result of Brézin *et al.*[7] is obtained. By setting $h=0$ and expanding the longitudinal Π function the result of Mazenko[9] is recovered.

Still for $\bar{u}=1$, this result reproduces to the given order the result of Ref. 7. This agreement can be improved by adjusting the amplitudes $(n-1)/(n+8)$ and $9/(n+8)$ using the results for the $k=0$ limit as discussed below.

We note that in the spherical limit (4.10) is exact (as is the corresponding equation of state). For the Ising limit, the results of Sec. III can be reproduced by the expansion of the Π function in k^2

$$\Gamma_{2,L}(k^2) = \frac{h}{M} + k^2 + \frac{\frac{2}{3}uM}{1-\bar{u}+\bar{u}[\Gamma_2^{-\epsilon/2} - \frac{1}{12}\epsilon(k^2/\Gamma_2)\Gamma_2^{-\epsilon/2} + O(k^4)]}$$

$$= \Gamma_2 + k^2[1+\Gamma_4^2(\tfrac{1}{12}M^2)k^2\xi^{2+\epsilon}] = D(M)(\xi^{-2}+k^2) \quad . \tag{4.11}$$

As discussed in Ref. 2, nonlinear crossover expressions such as Eq. (4.10) can be tentatively extended by exponent improvement; that is, low-order expansions of the critical-point exponents are replaced by higher-order expressions or by the exponent itself (to be determined by other means). In this manner, the equation of state given in Ref. 2 for the Ising case, $n=1$, can be made to agree to $O(\epsilon^2)$ with the ϵ-expansion results. In the same spirit, amplitudes may be corrected by comparing the ϵ expansions of the exponentiated forms with other results. If the form of Eq. (4.10) is retained, a comparison with the equation of state[17] suggests that the factors $(n-1)/(n+8)$ and $9/(n+8)$ be replaced by y, and $(1-y)$ with

$$y = \frac{n-1}{n+8}\left[1+\frac{\epsilon}{2(n+8)}\left(16-\frac{60}{n+8}\right)\right] \quad . \tag{4.12a}$$

The corresponding value of y for the equation of state is

$$y_{\rm es}=\frac{n-1}{n+8}\left[1+\frac{\epsilon}{2(n+8)}\left(25-\frac{60}{n+8}\right)\right] . \quad (4.12b)$$

Note that Eq. (4.10) contains three different types of crossover. First, the crossover between Landau and true critical behavior is controlled by $\bar{u}$, $\bar{u}=1$ being the fully critical limit of this crossover. Second, transverse and longitudinal effects compete [with a crossover amplitude of $y/(1-y)$]. Finally, $k=0$ and $k\neq 0$ compete within the transverse Π function. We also note that the solution of the nonlinear differential equations in lowest-order crossover form correspond to an approximation similar to the simple screening approximation.

The approach which gives results closest to the calculation given here is that of Schäfer and Horner.[7] Both approaches follow naturally from their $k=0$ limits for which the $O(\epsilon^2)$ results are available for comparison. Therefore, a comparison will be made principally for the equation of state. Reference 7 includes a partial calculation of the two-loop diagrams neglected in Ref. 1 and here. These alter the effective value of y from $(n-1)/(n+8)$ to that given by Eq. (4.12) and incorporate (for the equation of state) the second leading term of the quadrature integral given by Brézin *et al.*[8] and discussed in Ref. 1. Beyond these emendations there is still a difference in approach. Both methods introduce an effective mass for longitudinal effects; the factor $\bar{\Gamma}_2$ which enters the present and previous calculations arises from the natural cutoff of the differential equations by longitudinal propagation. By its construction $\bar{\Gamma}_2$ has no Goldstone effects. In fact, it is analytic in the scaling variable $x(\equiv t/M^{1/\beta})$ near the coexistence surface. The mass term m^2 used in Ref. 7 is linear in the magnetic field near the coexistence surface and hence nonanalytic in x. It is finite at $h=0$, but its higher derivatives with respect to M will be singular. These differences do not affect their agreement at $O(\epsilon^2)$. Both match the ϵ-expansion result[8] [if Eq. (4.12) is used]. The two results are identical at $n=1$ and $n=\infty$.

Schäfer and Horner base their approach on the Ward identities [$0(n)$ invariance] and an analysis of the diagram series. Here and in Ref. 1 there are no diagrams but rather a set of differential equations. The leading behavior of the solutions of these equations do not depend upon the approximations used here. For example, replacing the factors of $M\Gamma_4$ with a wave-vector-dependent Γ_3 will not disturb the structure of (4.4b). The dominant effects are given (for $h\rightarrow 0$) by the propagator terms. They determine the l at which the integration is effectively cut off, which in turn governs the nature of the singularities. A more detailed analysis appears to be difficult. The present calculation, unlike that of Ref. 7, does have the virtue of giving the crossover to mean-field behavior ($u\neq u^*$).

APPENDIX A: ADVANCED QUASIGLOBAL OPERATORS

In Ref. 2 several methods of calculating D were discussed. Since the same methods are needed for Secs. II and III of the present work, the relationship between these methods is further clarified in this Appendix.

In Appendix A of Ref. 1 quasiglobal eigenfunctions (eigenoperators) for the 1PI generator were given. They have the property of reducing to the linearized eigenfunctions at both the Gaussian and infinite Gaussian fixed points. For all sufficiently small Hamiltonians, these represent the $l\rightarrow\mp\infty$ limits of the renormalized Hamiltonian trajectory. That is, noncritical Hamiltonians run away as $l\rightarrow\infty$ to the infinite Gaussian fixed point (critical Hamiltonians of course approach some nontrivial fixed point). For differential generators, the Hamiltonian trajectory may be continued to negative values of l; if the initial Hamiltonian is sufficiently close to the Gaussian fixed point (within the suitably defined separ-surface) then the continuation of the trajectory approaches the Gaussian fixed point for $l\rightarrow-\infty$. In any case, an expansion of the ϵ type always places the Hamiltonian near the Gaussian fixed point, so that the linearized operators at the nontrivial fixed point can be calculated in an ϵ expansion about the Gaussian operators. The use of the quasiglobal operators generally simplifies the analysis of the global nonlinear behavior in the $l\rightarrow\infty$ limit at which the thermodynamic functions may be recovered.

The renormalization-group equations have terms linear in the 1PI vertices with propagators formed by the two-point function $r(k,l)$. The use of the previously defined quasiglobal eigenfunctions would remove these terms entirely if the two-point function were independent of the renormalization parameter l. The terms which remain for l dependent r may or may not be important for the particular problem considered. By requiring that these terms be completely removed, a new definition of quasiglobal operators is obtained. They are defined by

$$O_f=\exp(-Y)\int_{k_i\cdots k_m}\int f(k_i\cdots k_m)s_{k_i}\cdots s_{k_m} , \quad (A1)$$

where $f(k_1\cdots k_m)$ is a homogeneous function of its arguments and (for the k^2 propagator case) Y is given by

$$Y\equiv\int_0\frac{d^dk}{\Omega k^2}\frac{1}{r[\hat{k},l-\ln(k)]}\frac{\delta^2}{\delta s_k\delta s_{-k}} . \quad (A2)$$

Note that the two-point function is evaluated at an advanced value of $l(\ln(k)<0)$, so that $r(r,l)$ must be specified for all values of l. The solution of the differential equations for the parameters other than $r(k,l)$ are solved with $r(k,l)$ assumed known and then $r(k,l)$ is calculated self-consistently. If there is no nontrivial variation with l we may write $r(k,l) = k^2+r_0\exp(2l)$ and Eq. (A2) would look more familiar

$$Y=\int_0^1 \frac{d^dk}{\Omega(k^2+r_0e^{2l})}\frac{\delta^2}{\delta s_k \delta s_{-k}} \quad . \tag{A3}$$

In the general case the operator Y is more complicated but the asymptotic solutions of the differential equations are not made more difficult. Some care must be taken, however, to include all the important quadratic terms. For example, for the ordinary critical point the use of these operators reduces the equation for the reduced temperature to

$$\dot{t}=2t+O[u^2/(1+t)^2] \quad , \tag{A4}$$

showing that the initial value of this variable t differs only at $O(\epsilon^2)$ from its asymptotic limit (which is the inverse susceptibility). The u^2 must be taken into account or an improper exponentiation results.

The use of these operators changes the form of the operator-defined $\eta(l)$ given in Ref. 2. With the present definition of Y, the only remaining distinction between the two expressions given in Eqs. (2.5) and (3.1) of the earlier work is that between $r(l)$ and the new $t(l)$.

A similar modification can be employed to give the advanced quasiglobal operators for the general propagator case. With the definitions of Ref. 1 we define

$$Y=\int_0^1 \frac{d\Omega}{\Omega}\frac{\omega^{\lambda_0-1}d\omega}{\omega r(\hat{e},l-\ln\omega)}\frac{\delta^2}{\delta s_{q_i}\delta s_{-q_i}} \quad ;$$

$$\lambda_0=\sum d_i/\sigma_i, \quad \vec{q}_i=\omega^{1/\sigma_i}\hat{e}_i \quad .$$

For non-Ising systems, the function f is a tensor in the spin-component indices, and Y is the corresponding Laplacian.

APPENDIX B: EFFECTS OF $\eta(M)$ ON $O(\epsilon)$ CALCULATIONS

We wish to show that the physical vertex functions are not significantly changed when calculated with an $O(\epsilon)$ $\eta(M)$. Such a change might be expected since the derivation of the approximate 1PI equations essentially assumed that the coefficient of the momentum-dependent term (here, k^2) was always $1+O(\epsilon^2)$. In the presence of a nonzero magnetization, however, there are ϵ^2M^2 corrections to the leading behavior, which if M^2 is treated as $O(\epsilon^{-1})$ as is required in the equation of state, are definitely $O(\epsilon)$. Moreover there are $O(\epsilon)$ k^4 and higher momentum-dependent contributions. In this Appendix, we show that the lack of a slow-growth term in the expression for $\eta(M)$ precludes any change in the vertex functions even at $O(\epsilon^2)$.

To illustrate this, we may ignore the k^4 and higher terms; their analysis precisely parallels the k^2 case. The equation for the four-point function is then

$$\frac{\partial\hat{\Gamma}_4}{\partial l}=-2\eta(\phi,l)\hat{\Gamma}_4-\tfrac{1}{2}\eta(\phi,l)\phi\frac{\partial\hat{\Gamma}_4}{\partial\phi} -\frac{3\exp(\epsilon l)\hat{\Gamma}_4^2}{(l+\hat{\Gamma}_2\exp 2l)^2} \quad . \tag{B1}$$

Undoing the minimal scaling, the corresponding equation for Γ_4 eliminates the first two terms of Eq. (B1):

$$\frac{\partial\Gamma_4}{\partial l}=\frac{-3\exp\left[\epsilon l-2\int_0^l\eta(\phi(y),y)\,dy\right]\Gamma_4^2}{(1+\hat{\Gamma}_2\exp 2l)^2} \quad , \tag{B2}$$

where, of course,

$$\Gamma_4(M)=\hat{\Gamma}_4\exp\left[2\int_0^l\eta(\phi(y),y)\,dy\right] \quad .$$

This differs from the expression of Ref. 2 by the presence of the η term and by the distinction between Γ_2 and $\hat{\Gamma}_2$ in the propagators. The solution of Eq. (B2) is

$$\Gamma_4=u\hat{Y} \quad ,$$

$$\hat{Y}^{-1}=1+3u\int_0^l\frac{\exp\left[\epsilon l-2\int_0^l\eta(\phi(y),y)\,dy\right]}{(1+\hat{\Gamma}_2\exp 2l)^2} \quad , \tag{B3}$$

where u is the initial value of Γ_4. The asymptotic behavior of the integral may be estimated by terminating the integral at l_ξ, $\hat{\Gamma}_2\exp(2l_\xi)=1$. To estimate the effect of $\eta(M)$, the exponential is expanded

$$\int_0^{l_\xi}\exp(\epsilon s)\left[1-2\int_0^s\eta(\phi(y),y)\,dy+\cdots\right]ds =\frac{\xi^\epsilon-1}{\epsilon}-\tfrac{1}{12}\hat{\Gamma}_4^2\phi^2\xi^{2+\epsilon}\xi^\epsilon \quad , \tag{B4}$$

where the $O(\epsilon)$ expression for $\eta_{3,3}$ has been used and smaller terms dropped. In a strict ϵ expansion the first term is $O(1)$ and the second is $O(\epsilon)$ leading to $O(\epsilon^2)$ changes in the equation of state. However, in the exponentiated ϵ expansion, the coefficients of ξ^ϵ are $O(\epsilon^{-1})$ and $O(\epsilon)$, leading to $O(\epsilon^3)$ changes in the exponentiated form, and hence the direct effects of the η term may be neglected. The distinction between $\xi^{-2+\eta}$ and Γ_2 induced by the $\eta(M)$ term does not change the equation of state even at $O(\epsilon^2)$. If Γ_2 is replaced by $\xi^{-2+\eta}$ in Eq.

(3.14) of Ref. 2 (without the anomalous dimension crossover term) the equation of state is only changed in a strict ϵ expansion by a term proportional to $\epsilon^2 M^2$ which is removed by a change of scale. Only at $O(\epsilon^3)$ will distinctions need to be made.

A completely careful calculation would have to include the effects of the k^4 and higher terms. One formal approach is to define a wave-vector-dependent η such that the renormalized two-point function is always exactly in the form $\hat{\Gamma}_2(k) = \hat{\Gamma}_2(k=0) + k^2$. Then all the effects may be analyzed with the magnetization- and momentum-dependent anomalous dimension crossover function.

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