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Charged Bose gas in two dimensions. Zero magnetic field

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The dielectric response of the two-dimensional charged Bose gas is investigated in the random-phase approximation. Using this temperature-dependent dielectric function we find the energy spectrum and the damping for the quasiparticles. The electrostatic potential around a test charge at $T=0$ is displayed. We also study the thermodynamic functions of the gas at very low temperatures and high densities.

I. INTRODUCTION

Due to the interest in the behavior of real physical two-dimensional or quasi-two-dimensional systems which has been current for some years, it is of great intrinsic value to investigate the properties of specific model systems in two dimensions and to discover, whenever possible, how they differ from their counterparts in three dimensions.

The Coulomb interaction provides us with several examples, most of which have direct physical or model applications. Coulomb gases are particularly interesting and fruitful topics of study because of their capacity to support collective oscillations along with the fact that their long-range nature suggests that the system's properties may change dramatically with dimensionality. In this paper we study the dynamical response of a Bose-Coulomb gas in two dimensions.

The dynamical properties of a classical three-dimensional plasma both free of and subject to a magnetic field are well known.[1] Lindhard[2] first studied the degenerate electron plasma in the absence of a magnetic field and later Quinn *et al.*[3] examined this system in the presence of an external magnetic field.

Interest in the two-dimensional electron gas was then stimulated by observations of the behavior of electrons on semiconductor inversion layers[4] and layers of electrons on superfluid ^{4}He films.[5] Stern[6] first discussed the plasma oscillations in a two-dimensional electron gas. He obtained a dispersion relation at the absolute zero of temperature. Fetter[5,7] has subsequently studied the plasma oscillations of the Fermi gas at various temperatures, in particular in the classical regime.[7] The effect of an external magnetic field on the two-dimensional electron gas has also been investigated by various authors.[8]

The behavior of the three-dimensional charged Bose gas in zero magnetic field has been extensively studied by many workers.[9,10] The dielectric response of the three-dimensional system in the presence of a magnetic field has been examined by Hore and Frankel.[11] A charged Bose gas combines the Bose condensation phenomenon with the usual problems arising from the long-range Coulomb forces and so provides a particularly rich subject for investigation. It is therefore of interest to study the charged Bose gas in two dimensions.

In addition to the inherent worth of studying this particular two-dimensional many-body problem, the results obtained in this paper should also be of specific interest for superconducting thin films. In this context we note that it has recently been suggested that a new type of superconductivity may occur in electron-hole liquids[12] and in particular in thin films. According to the ideas of Schafroth[13] a gas of charged bosons may serve as a model for a superconductor.

In three dimensions the Coulomb interaction has the form $V(r) \propto 1/r$. However in the lower dimensions there are two well-defined Coulomb systems, namely the "restricted three-dimensional" gas and the "pure Coulomb" gas. In two dimensions the former consists of charges that interact via the three-dimensional $1/r$ Coulomb potential, but which are confined to motion in a plane. The second system is composed of particles which interact through the logarithmic two-dimensional Coulomb potential. It is the "restricted Coulomb" gas which models real films of charged particles. The gas with the logarithmic Coulomb potential, while its behavior is certainly of particular intrinsic interest as a many-body problem involving particles with very long-ranged interactions, is also valuable as a model of two-dimensional superfluidity.[14]

In the present paper we investigate the behavior of the two-dimensional charged Bose gas where the charges interact via the $1/r$ potential. In a second paper we shall examine the response of this system in the presence of a magnetic field. This study should have particular relevance for the charged Bose gas as a model of superconducting thin films. Our results for the logarithmic potential will be published in a third paper.

As is well known, the three-dimensional Bose gas undergoes Bose-Einstein condensation at nonzero temperatures. However the ideal Bose gas in two dimensions does not condense for temperatures above absolute zero and indeed there are theorems[15] which insist that long-range order does not exist at finite temperature in a general class of interacting two-dimensional systems. Whether or not a gas of particles with long-range interactions in two dimensions, like the Coulomb interaction, has a phase transition at any finite temperature, or specifically whether a Bose-Einstein condensate exists at all at $T=0$, has not as yet been rigorously established.

We here follow the approach of Hore and Frankel[10] in their three-dimensional treatment of the charged Bose gas, which we refer to hereafter as HF. They use a dielectric function formalism in the random-phase approximation.

This approach essentially treats the charged system by perturbing around the ideal gas behavior. The ideal gas is condensed at $T=0$. In three dimensions Foldy[16] has shown, using a method of Bogoliubov, that the ground state of the charged Bose gas indeed has macroscopic occupation of the single-particle ground state. This indicates that the random-phase approximation as used in HF is valid.

In the third paper of this series we shall return to the "restricted three-dimensional" charged Bose gas and show that this system likewise condenses at zero temperature, thus providing the justification for the use of the random-phase approximation in this paper. In that same paper we also show that the method of Bogoliubov does not indicate that a condensate exists in the "pure (logarithmic potential) two-dimensional" charged Bose gas at $T=0$. Therefore it is questionable whether, for this system, the use of the random-phase approximation is at all justified. We shall return to study this problem in detail in our third paper.

The present paper is organized as follows: after first obtaining a closed-form expression for the dielectric function in Sec. II, we find the quasiparticle energy spectrum and the form of the damping of the quasiparticles in Sec. III. The nature of the thermodynamic functions of the gas at low temperatures and high densities is obtained in Sec. IV, while the form of the electrostatic potential around a test charge immersed in the gas at zero temperature is investigated in Sec. V.

We find that the thermodynamic functions exhibit an interesting behavior around $T=0$, which does not occur in the three-dimensional gas. Possible explanations and implications are discussed.

II. DIELECTRIC FUNCTION

Consider a one component gas of N spinless bosons each with mass m and charge e confined to an area Ω at temperature T. We suppose the presence of a uniform background of particles of opposite charge to maintain charge neutrality.

The dielectric function, $\epsilon(\vec{q},\omega)$, can be derived by examining the response of the system in equilibrium to a small disturbance.[17] The result in two dimensions is

$$\epsilon(\vec{q},\omega)=1+\sum_{\vec{p}}\frac{2\pi e^2}{\hbar q\,\Omega}\left(\frac{F_0(\vec{p})-F_0(\vec{p}-\vec{q})}{\omega-\dfrac{\hbar}{m}\vec{p}\cdot\vec{q}+\dfrac{\hbar q^2}{2m}}\right) . \tag{1}$$

The variables $\vec{q}$ and ω are, respectively, the wave number and the frequency of oscillations of the gas about equilibrium. Associated with oscillations of frequency ω are quasiparticles of energy $\hbar\omega$; $\vec{p}$ represents the values of wave number accessible to a free particle in Ω. $F_0(\vec{p})$ is the equilibrium distribution of the bosons.

We make the random-phase approximation and take for $F_0(\vec{p})$ the ideal-Bose-gas distribution function,[18]

$$F_0(\vec{p})=1/(z^{-1}e^{\hbar^2p^2/2mk_BT}-1) , \tag{2}$$

where k_B is Boltzmann's constant, z is the fugacity, and for bosons

$$0\leq z\leq 1 .$$

Equation (1) then yields

$$\epsilon(\vec{q},\omega)=1+\frac{2\pi e^2}{\hbar q\,\Omega}\sum_{\vec{p}}\left[\left(\frac{1}{z^{-1}e^{\hbar^2p^2/2mk_BT}-1}-\frac{1}{z^{-1}e^{\hbar^2(\vec{p}-\vec{q})^2/2mk_BT}-1}\right)\Bigg/\left(\omega-\frac{\hbar}{m}\vec{p}\cdot\vec{q}+\frac{\hbar q^2}{2m}\right)\right] . \tag{3}$$

Since the ideal Bose gas does not condense at nonzero temperature in two dimensions there are, unlike for the three-dimensional calculation in HF, no singular terms in the summation in Eq. (3). Appendix A indicates how Eq. (3) can be written as follows:

$$\epsilon(\vec{q},\omega)=1+\frac{e^2m}{q^2\hbar^2A^{1/2}}\frac{\sqrt{\pi}}{2i}$$
$$\times\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}\{[1+\phi(iC_jD)]e^{-C_j^2D^2}$$
$$-[1+\phi(iC_jB)]e^{-C_j^2B^2}\}\ , \tag{4}$$

where

$$D=\frac{m\omega}{\hbar q}+\tfrac{1}{2}q,\quad B=\frac{m\omega}{\hbar q}-\tfrac{1}{2}q\ ,$$

$$C_j=A^{1/2}j^{1/2},\quad A=\frac{\lambda_T^2}{4\pi}=\frac{\hbar^2}{2mk_BT}\ ,$$

where λ_T is the thermal de Broglie wavelength and $\phi(x)$ is the error function[19] with

$$\phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\ .$$

III. QUASIPARTICLE ENERGY SPECTRUM

We now investigate $\epsilon(\vec{q},\omega)$ in the two asymptotic thermodynamic regions

$$\rho\lambda_T^2\to 0+$$

(high temperature and low density) and

$$\rho\lambda_T^2\to\infty$$

(low temperature and high density), where $\rho=N/\Omega$ is the areal number density.

The allowed values of ω are then found as solutions, $\omega=\omega(\vec{q})$, of[17]

$$\epsilon(\vec{q},\omega)=0\ . \tag{5}$$

First, however, we consider the gas at the absolute zero of temperature. Then

$$F_0(\vec{p})=\begin{cases}N, & \text{if } \vec{p}=0\\ 0, & \text{if } \vec{p}\neq 0\end{cases}$$

and Eq. (1) gives

$$\epsilon(\vec{q},\omega,T=0)=1-\frac{2\pi e^2\rho}{m}\frac{q}{(\omega^2-\hbar^2q^4/4m^2)}\ . \tag{6}$$

For the $T=0$ quasiparticle energy spectrum, Eq. (5) accordingly yields

$$\omega^2(q)=a_pq+\frac{\hbar^2q^4}{4m^2}\ , \tag{7}$$

where

$$a_p=\frac{2\pi e^2\rho}{m}\ .$$

In general, Eq. (5) is not easily solved; however asymptotic solutions can be obtained in a small-q limit. This is, of course, the region of most physical interest, since the oscillations are weakly damped and are, therefore, well defined as quasiparticles. It is well established[5-7] that

$$\omega(q)\sim(a_pq)^{1/2}$$

for two-dimensional charged gases in the small-q limit, so

$$D^2A\simeq B^2A\simeq ma_p/qk_BT>>1\ ,$$

provided T is not too large. We study this region using the results in the appendices as follows.

Firstly, Eq. (B3) is used in Eq. (4) to obtain the following result for $z\simeq 0+$:

$$\epsilon(\vec{q},\omega)=1-\frac{a_pq}{(\omega^2-\sigma)}\sum_{p=0}^{\infty}\frac{(-1)^p\left[\dfrac{g_{1+p}(z)}{\rho\lambda_T^2}\right]}{\Gamma(\frac{1}{2}-p)/\sqrt{\pi}}\ \frac{\left[\dfrac{2\theta}{\omega^2}\right]^p\sum_{n=0}^{p}\begin{pmatrix}2p+1\\2n+1\end{pmatrix}\left[\dfrac{\sigma}{\omega^2}\right]^n}{(1-\sigma/\omega^2)^{2p}}$$
$$+i\frac{2\pi e^2m}{q^2\hbar^2\lambda_T}\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}\sinh\left[j\frac{\hbar\omega}{2k_BT}\right]\exp\left[-\tfrac{1}{2}j\frac{(\omega^2+\sigma)}{\theta}\right]\ ,$$

where we have defined, for convenience,

$$\theta=q^2k_BT/m$$

and

$$\sigma=\hbar^2q^4/4m^2\ .$$

This asymptotic expansion is good for

$$\omega^2>>\theta,\sigma\ .$$

The fugacity is now eliminated by Eq. (E3). After some algebra, the following expression for $\epsilon(\vec{q},\omega)$ is found:

$$\epsilon(\vec{q},\omega)=1-\frac{a_p q}{(\omega^2-\sigma)}\left[1+3\frac{\theta}{\omega^2}\frac{(1+\frac{1}{3}\sigma/\omega^2)}{(1-\sigma/\omega^2)^2}+15\left(\frac{\theta}{\omega^2}\right)^2\frac{[1+2\sigma/\omega^2+\frac{1}{5}(\sigma/\omega^2)^2]}{(1-\sigma/\omega^2)^4}+105(\theta/\omega^2)^3\right.$$
$$\times\frac{[1+5\sigma/\omega^2+3(\sigma/\omega^2)^2+\frac{1}{7}(\sigma/\omega^2)^3]}{(1-\sigma/\omega^2)^6}+945\left(\frac{\theta}{\omega^2}\right)^4\left[1+\frac{28}{3}\frac{\sigma}{\omega^2}+14\left(\frac{\sigma}{\omega^2}\right)^2+4\left(\frac{\sigma}{\omega^2}\right)^3+\frac{1}{9}\left(\frac{\sigma}{\omega^2}\right)^4\right]$$
$$\left.\times\left\{1+O\left(\frac{\theta}{\omega^2}\left[1+O\left(\frac{\sigma}{\omega^2}\right)\right]\right)\right\}\right][1+O(\rho\lambda_T^2)]$$
$$+i\frac{a_p\lambda_T m^2}{\hbar^2 q^2}\left\{\sinh\left(\frac{\hbar\omega}{2k_BT}\right)\exp\left[-\frac{1}{2}\left(\frac{\omega^2+\sigma}{\theta}\right)\right][1+O(\rho\lambda_T^2)]+O\left(\rho\lambda_T^2\sinh\left(\frac{\hbar\omega}{k_BT}\right)\exp\left[-\left(\frac{\omega^2+\sigma}{\theta}\right)\right]\right)\right\}, \quad (8)$$

where we have used[19]

$$\Gamma(\tfrac{1}{2})=(\pi)^{1/2}\ ;\ \ \Gamma(\tfrac{1}{2}-p)=(-1)^p2^p(\pi)^{1/2}(2p-1)!!,\ \ p\geq 1\ .$$

This expansion is valid when

$$\rho\lambda_T^2\ll 1\ \text{ and }\ \omega^2\gg\theta,\sigma\ .$$

Now we use Eqs. (C3) and (D2) in Eq. (4) to give the following expression for $z\simeq 1-$:

$$\epsilon(\vec{q},\omega)=1-\frac{a_p q}{(\omega^2-\sigma)}\sum_{p=0}^{\infty}\frac{1}{\Gamma(\frac{1}{2}-p)/(\pi)^{1/2}}\left(\frac{[\chi(p)-\ln(-\ln z)]}{p!}\frac{(-\ln z)^p}{\rho\lambda_T^2}+\sum_{\substack{\alpha=0,\\ \alpha\neq p}}^{\infty}\frac{(-1)^{\alpha+p}\zeta(1+p-\alpha)}{\alpha!}\frac{(-\ln z)^\alpha}{\rho\lambda_T^2}\right)$$
$$\times\frac{\left(\frac{2\theta}{\omega^2}\right)^p\sum_{n=0}^{p}\binom{2p+1}{2n+1}\left(\frac{\sigma}{\omega^2}\right)^n}{(1-\sigma/\omega^2)^{2p}}+i\frac{2\pi e^2 m}{q^2\hbar^2\lambda_T}\sum_{\substack{p=0,\\ j=1}}^{\infty}\frac{(\ln z)^p}{p!\,j^{1/2-p}}\sinh\left(j\frac{\hbar\omega}{2k_BT}\right)\exp\left[\frac{-j}{2}\frac{(\omega^2+\sigma)}{\theta}\right]\ .$$

This asymptotic expansion is good for

$$\omega^2\gg\theta,\sigma\ .$$

The fugacity is eliminated by use of Eqs. (E4) and (E5), yielding

$$\epsilon(\vec{q},\omega)=1-\frac{a_p q}{(\omega^2-\sigma)}\left\{1+O\left(\frac{e^{-2\rho\lambda_T^2}}{\rho\lambda_T^2}\right)+\frac{\pi^2}{2\rho\lambda_T^2}\left(\frac{\theta}{\omega^2}\right)\frac{(1+\frac{1}{3}\sigma/\omega^2)}{(1-\sigma/\omega^2)^2}[1+O(\rho\lambda_T^2e^{-\rho\lambda_T^2})]\right.$$
$$+\left[\frac{15\zeta(3)}{\rho\lambda_T^2}\left(\frac{\theta}{\omega^2}\right)^2\frac{[1+2\sigma/\omega^2+\frac{1}{5}(\sigma/\omega^2)^2]}{(1-\sigma/\omega^2)^4}+\frac{105\pi^4}{90\rho\lambda_T^2}\left(\frac{\theta}{\omega^2}\right)^3\frac{[1+5\sigma/\omega^2+3(\sigma/\omega^2)^2+\frac{1}{7}(\sigma/\omega^2)^3]}{(1-\sigma/\omega^2)^6}\right.$$
$$+\frac{945\zeta(5)}{\rho\lambda_T^2}(\theta/\omega^2)^4\left[1+\tfrac{28}{3}\frac{\sigma}{\omega^2}+14\left(\frac{\sigma}{\omega^2}\right)^2+4\left(\frac{\sigma}{\omega^2}\right)^3+\tfrac{1}{9}\left(\frac{\sigma}{\omega^2}\right)^4\right]$$
$$\left.\left.\times\left\{1+O\left(\frac{\theta}{\omega^2}\left[1+O\left(\frac{\sigma}{\omega^2}\right)\right]\right)\right\}\right][1+O(e^{-\rho\lambda_T^2})]\right\}$$
$$+i\frac{2\pi e^2 m}{q^2\hbar^2\lambda_T}[1+O(e^{-\rho\lambda_T^2})]\left\{\sinh\left(\frac{\hbar\omega}{2k_BT}\right)\exp\left[-\frac{1}{2}\left(\frac{\omega^2+\sigma}{\theta}\right)\right]+O\left[\sinh\left(\frac{\hbar\omega}{k_BT}\right)\exp\left[-\left(\frac{\omega^2+\sigma}{\theta}\right)\right]\right]\right\}\ . \quad (9)$$

We have used[19]

$$\zeta(0)=-\tfrac{1}{2},\quad \zeta(-1)=\tfrac{1}{12}\ ,\quad \zeta(2)=\tfrac{1}{6}\pi^2,\quad \zeta(4)=\tfrac{1}{90}\pi^4\ .$$

This expansion is valid when

$$\rho\lambda_T^2 \gg 1 \text{ and } \omega^2 \gg \theta,\sigma\ .$$

Asymptotic solutions to Eq. (5) can now be obtained by iteration. The resulting expression are, for

$$\omega=\omega_1+i\,\omega_2$$

(ω_1 and ω_2 are real), the following: for $\rho\lambda_T^2 \ll 1$ (low density, high temperature),

$$\omega_1^2(q)=a_pq\left[1+3\left(\frac{\theta}{a_pq}\right)+6\left(\frac{\theta}{a_pq}\right)^2+\left(\frac{\sigma}{a_pq}\right)+24\left(\frac{\theta}{a_pq}\right)^3+4\frac{\sigma\theta}{(a_pq)^2}\right.$$
$$\left.+180\left(\frac{\theta}{a_pq}\right)^4+O\left(\left(\frac{\theta}{a_pq}\right)^5+\frac{\sigma\theta^2}{(a_pq)^3}\right)\right][1+O(\rho\lambda_T^2)]\ ; \tag{10}$$

$$\omega_2(q)=-e^{-3/2}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{a_pq}{\theta}\right)^{3/2}\left(\frac{k_BT}{\hbar}\right)\sinh\left[\frac{\hbar(a_pq)^{1/2}}{2k_BT}\right]\exp\left[-\frac{1}{2}\left(\frac{a_pq+\sigma}{\theta}\right)\right]\ , \tag{11}$$

to leading order. For

$$\sigma/\theta \ll 1 \tag{12a}$$

and

$$\frac{\hbar\omega_1}{k_BT}=\left(\frac{\omega_1^2}{\theta}\right)\left(\frac{\sigma}{\omega_1^2}\right)^{1/2}\simeq\frac{\hbar(a_pq)^{1/2}}{k_BT}\ll 1\ , \tag{12b}$$

we may expand the hyperbolic sine function and neglect the term $\exp(-\sigma/\theta)$ to obtain, to leading order,

$$\omega_2(q)=-e^{-3/2}(\tfrac{1}{8}\pi)^{1/2}(a_pq)^{1/2}\left(\frac{a_pq}{\theta}\right)^{3/2}\exp\left[-\frac{1}{2}\left(\frac{a_pq}{\theta}\right)\right]\ . \tag{11$'$}$$

For $\rho\lambda_T^2 \gg 1$ (high density, low temperature),

$$\omega_1^2(q)=a_pq\left\{1+O\left(\frac{e^{-2\rho\lambda_T^2}}{\rho\lambda_T^2}\right)+\frac{1}{\rho\lambda_T^2}\left[\frac{\pi^2}{2}\left(\frac{\theta}{a_pq}\right)+\alpha_1\left(\frac{\theta}{a_pq}\right)^2\right]\left[1+O\left(\rho\lambda_T^2e^{-\rho\lambda_T^2}\right)\right]\right.$$
$$\left.+\frac{\sigma}{a_pq}+\frac{1}{\rho\lambda_T^2}\left[\alpha_2\left(\frac{\theta}{a_pq}\right)^3+\frac{2\pi^2}{3}\frac{\sigma\theta}{(a_pq)^2}+\alpha_3\left(\frac{\theta}{a_pq}\right)^4+O\left(\left(\frac{\theta}{a_pq}\right)^5+\frac{\theta^2\sigma}{(a_pq)^3}\right)\right][1+O(\rho\lambda_T^2e^{-\rho\lambda_T^2})]\right\}\ , \tag{13}$$

where

$$\alpha_1(\rho\lambda_T^2)=15\zeta(3)-\pi^4/4\rho\lambda_T^2\ ,$$

$$\alpha_1(\rho\lambda_T^2)=7\pi^4/6-180\zeta(3)+15\pi^2\zeta(3)/2\rho\lambda_T^2+\pi^6/4(\rho\lambda_T^2)^2\ ,$$

$$\alpha_3(\rho\lambda_T^2)=945\zeta(5)-[450\zeta^2(3)+90\pi^2\zeta(3)+7\pi^6/3]/\rho\lambda_T^2+105\zeta(3)\pi^4/2\rho\lambda_T^2-405\pi^8/1296(\rho\lambda_T^2)^3\ ;$$

$$\omega_2(q)=-\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{a_pq}{\theta}\right)^{1/2}\left(\frac{k_BT}{\hbar}\right)\left(\frac{1}{qa_0}\right)\sinh\left(\frac{\hbar(a_pq)^{1/2}}{2k_BT}\right)\exp\left[-\frac{1}{2}\left(\frac{a_pq}{\theta}+\frac{\sigma}{\theta}\right)\right]\ , \tag{14}$$

to leading order, where

$$a_0 = \hbar^2/me^2$$

is the Bohr radius. If Eqs. (12) hold, then we obtain

$$\omega_2(q) = -\left[\frac{\pi}{8}\right]^{1/2}\left[\frac{a_p q}{\theta}\right]^{1/2}(a_p q)^{1/2}\frac{1}{q a_0} \times \exp\left[-\frac{1}{2}\left[\frac{a_p q}{\theta}\right]\right] , \tag{14a}$$

to leading order.

The above expansions are all good for

$$\theta, \sigma << a_p q \quad . \tag{15a}$$

If we define wavenumbers q_1 and q_2 such that

$$\frac{\theta}{a_p q} = \frac{q}{q_1}, \quad q_1 = \frac{2\pi e^2 \rho}{k_B T}$$

and

$$\frac{\sigma}{a_p q} = \left[\frac{q}{q_2}\right]^3 ,$$

$$q_2 = \left[\frac{8\pi m e^2 \rho}{\hbar^2}\right]^{1/3} = \left[\frac{8\pi\rho}{a_0}\right]^{1/3} ,$$

then Eq. (15a) can be written

$$q << q_1, q_2 \quad . \tag{15b}$$

From Eq. (10) we see that the first quantum correction to the high-temperature, low-density classical frequency, in the small-q expansion, occurs as the fourth term. It is interesting to find out if this quantum correction can ever dominate the second classical term, within the validity of the expansion. The ratio of the quantum term to the classical term is

$$S = \frac{\sigma}{\theta} = \frac{(q\lambda_T)^2}{8\pi} \quad .$$

Since Eq. (15b) can be written

$$(q\lambda_T)^3, q\lambda_T << (\rho\lambda_T^2)\frac{\lambda_T}{a_0} \quad , \tag{15c}$$

it follows that for $\rho <<< a_0^{-2}$, q and λ_T may be chosen so that Eq. (15c) is satisfied along with the requirement $\rho\lambda_T^2 << 1$, but for $(q\lambda_T)^2 >> 1$ [e.g., $\rho \simeq 10^{-16} a_0^{-2}$, $\lambda_T \simeq 10^7 a_0$, and $q \simeq 10^{-6} a_0^{-1}$ satisfy these conditions]. Then $\sigma > \theta$ and $S > 1$.

The first two terms in Eq. (10) are exactly those given in Fetter's paper[5] on the classical electron gas.

We have evaluated only the leading classical term for the damping and this is also that obtained by Fetter.[5] It would be a simple matter to generate quantum corrections to the result using Eq. (8).

It is very significant that the low-temperature spectrum [Eq. (13)] has terms which are proportional to q^2 and to q^3 while the zero-temperature spectrum [Eq. (7)] has no such terms.

IV. THERMODYNAMIC BEHAVIOR AROUND $T = 0$

The nature of the thermodynamic functions of the gas at high densities and low temperatures is now investigated. This we accomplish by using the spectrum of Eq. (13) in the Landau[20] quasiparticle model. It is of great interest to determine to what extent the temperature corrections to the $T = 0$ excitation spectrum modify the thermodynamic functions calculated for $T \gtrsim 0$ using the zero-temperature spectrum, Eq. (7).

We consider here the internal energy per unit volume, $E(\rho, T)$, given by the Bose quasiparticle integral

$$E(\rho,T) = \frac{1}{(2\pi)^2}\int d^2q \frac{\hbar\omega_1(q)}{(e^{\hbar\omega_1(q)/k_B T} - 1)} = \frac{\hbar}{2\pi}\int_0^\infty dq\, q\,\omega_1(q)\sum_{j=1}^{\infty} e^{-j\hbar\omega_1(q)/k_B T} , \tag{16}$$

using Eq. (A1). The other thermodynamic function can easily be obtained by differentiation or integration of $E(\rho,T)$. If

$$\omega_1(q)\exp[-j\hbar\omega_1(q)/k_B T]$$

is expanded around $q = 0$ and put into Eq. (16), then the small-T behavior of E is correctly given, since only for values of q close to zero is there a significant contribution to the integral.

Writing the spectrum of Eq. (13) as

$$\omega_1^2(q) = a_p q + \gamma_1 T^2 q^2 + \gamma_2 T^3 q^3 + \gamma_3 q^4 + \gamma_4 T^4 q^4 + \gamma_5 T^2 q^5 + \gamma_6 T^5 q^5 + O(T^3 q^6) , \tag{17}$$

where the γ_i, $i = 1, \ldots, 6$, are independent of T and q, we find that

$$E(\rho,T) = \frac{\hbar}{\pi a_p^{1/2}}(T\gamma_7)^5\left[4!\zeta(5) - (7! - 6!)\frac{\zeta(7)}{2}\gamma_1 T^4\gamma_7^2 - (11! - 10!)\frac{\zeta(11)}{2}\gamma_3(T\gamma_7)^6 - (9! - 8!)\frac{\zeta(9)}{2}\gamma_2 T^7\gamma_7^4 + \left[\frac{10!}{8} - \frac{9!}{8} - 7!\right]\zeta(9)\frac{\gamma_1^2}{a_p}T^8\gamma_7^4 + O\left(\left[\gamma_5 + \frac{\gamma_4}{\gamma_7^2} + \frac{\gamma_1\gamma_3}{a_p}\right]T^{10}\gamma_7^8\right)\right] , \tag{18}$$

where

$$\gamma_7 = k_B^2/\hbar^2 a_p \quad .$$

Putting the values of the γ_i, $i=1,\ldots,7$, into Eq. (18), the energy can be written

$$E(\rho,T)=\frac{m^2(k_BT)^5}{4\pi^3\hbar^4e^4\rho^2}\left[(4!)\zeta(5)-\frac{(7!-6!)\zeta(7)}{32\pi}\frac{m^2(k_BT)^4}{\hbar^4e^4\rho^3}-\frac{(11!-10!)\zeta(11)}{128\pi^4}\frac{m^2(k_BT)^6}{\hbar^4e^8\rho^4}-\frac{15(9!-8!)\zeta(9)}{64\pi^5}\frac{m^3(k_BT)^7}{\hbar^6e^8\rho^5}+O\left[\frac{m^4(k_BT)^8}{\hbar^8e^8\rho^6}\right]\right] \quad . \tag{19}$$

Now using the zero-temperature spectrum, Eq. (7), to calculate the internal energy $E(\rho,T)$, that is, using only the first and the fourth terms of the finite-temperature spectrum, Eq. (17), we obtain, using Eq. (18),

$$E(\rho,T)=\frac{m^2(k_BT)^5}{4\pi^3\hbar^4e^4\rho^2}\left[(4!)\zeta(5)-\frac{(11!-10!)\zeta(11)}{128\pi^4}\frac{m^2(k_BT)^6}{\hbar^4e^8\rho^4}\right] \quad , \tag{20}$$

to leading order.

It can be seen from Eq. (18) that the first temperature correction term in Eq. (17) gives rise to the first-order term in the correct temperature expansion for the energy, Eq. (19). However the zero-temperature term in the spectrum, $\gamma_3 q^4$, contributes to the correct low-temperature expansion of $E(\rho,T)$ only to second order, as can be seen from a comparison of Eq. (19), the expansion obtained using the full spectrum [Eq. (17)] and Eq. (20), the expansion resulting from the zero-temperature spectrum [Eq. (7)]. Thus the temperature corrections to the $T=0$ quasiparticle spectrum have a major effect on the behavior of the thermodynamic functions.

This result is in sharp contrast with the situation in three dimensions as given by Hore and Frankel.[21] Using the full temperature-corrected quasiparticle energy spectrum of a dense charged Bose gas at low temperature, they have calculated the Helmholtz free energy, F, of the dense charged Bose gas around $T=0$. They found that

$$F=-\left[\frac{m^3\omega_p^5}{2\sqrt{2}\pi^4\hbar}\right]^{1/2}\left[\frac{k_BT}{\hbar\omega_p}\right]^{7/4}e^{-\hbar\omega_p/k_BT}\times\left\{\Gamma(\tfrac{3}{4})+\frac{1}{2}\Gamma\left[\frac{11}{4}\right]\left[\frac{k_BT}{\hbar\omega_p}\right]-\left[\tfrac{1}{32}\Gamma(\tfrac{15}{4})+\frac{3\sqrt{2}\zeta(\frac{5}{2})\Gamma(\frac{5}{4})}{\zeta(\frac{3}{2})}\left[\frac{\hbar\omega_p}{kT_c}\right]^{3/2}\right]\left[\frac{k_BT}{\hbar\omega_p}\right]^2+O((k_BT/\hbar\omega_p)^2)\right\} \quad ,$$

where T_c is the transition temperature of the ideal three-dimensional Bose gas and ω_p is the plasma frequency of the three-dimensional Coulomb gas.

In 1970 Fetter[9] use the zero-temperature excitation spectrum previously obtained by Foldy[16] to calculate the thermodynamic functions of the dense three-dimensional charged Bose gas around $T=0$. He obtained

$$F=-\left[\frac{m^3\omega_p^5}{2\sqrt{2}\pi^4\hbar}\right]^{1/2}\left[\frac{k_BT}{\hbar\omega_p}\right]^{7/4}e^{-\hbar\omega_p/k_BT}\left[\Gamma(\tfrac{3}{4})+\frac{1}{2}\Gamma\left[\frac{11}{4}\right]\left[\frac{k_BT}{\hbar\omega_p}\right]-\frac{1}{32}\Gamma\left[\frac{15}{4}\right]\left[\frac{k_BT}{\hbar\omega_p}\right]^2+O\left[\left[\frac{k_BT}{\hbar\omega_p}\right]^3\right]\right] \quad .$$

Clearly the temperature corrections to the quasiparticle energy spectrum here do not change the basic character of the thermodynamic functions. In the three-dimensional charged Bose gas the Foldy ground-state ($T=0$) spectrum gives the correct leading order terms in the low-temperature thermodynamic behavior. Low-temperature corrections to the spectrum produce only third- and higher-order corrections.

V. ELECTROSTATIC POTENTIAL

We now turn our attention to the electrostatic potential $V(\vec{r})$ about a charge Q immersed in the gas. For a two-dimensional system

$$V(\vec{q})=2\pi Q/q\epsilon(\vec{q},\omega=0) \quad , \tag{21}$$

where

$$V(\vec{r})=\frac{1}{(2\pi)^2}\int d^2q\, e^{i\vec{q}\cdot\vec{r}}V(\vec{q}) \quad . \tag{22}$$

For an isotropic gas the angular integration in Eq. (22) is trivial and yields

$$V(r)=Q\int_0^\infty dq\,\frac{J_0(qr)}{\epsilon(q,0)} \quad . \tag{23}$$

Here the defining relation[19] for the Bessel function of order zero,

$$2\pi J_0(x) = \int_{-\pi}^{\pi} \exp(-ix \sin\theta)\, d\theta \ ,$$

has been used.

At $T=0$ the potential can be evaluated exactly. From Eq. (6)

$$\epsilon(q, \omega=0, T=0) = 1 + q_2^3/q^3 \ . \tag{24}$$

Eq. (23) then gives

$$V(r) = Qq_2 \int_0^{\infty} dx \, \frac{x^3 J_0(Rx)}{1+x^3} \ , \tag{25}$$

where

$$R = q_2 r \ .$$

The integral can be evaluated exactly, as we show in Appendix F; its asymptotic form for large R is also obtained.

Equations (F4), (F2), and (F1), when put in Eq. (25) yield

$$V(r) = \frac{Q}{r}\left\{1 - \frac{2\pi}{3\sqrt{3}} R - \tfrac{3}{4} R^3 \ln R + \tfrac{3}{4}(1 - C + \ln 2) R^3 + O(R^5)\right\} \ , \tag{26}$$

for $R \ll 1$. Thus, close to the test charge, the bare Coulomb potential is recovered, as is expected.

Substituting Eqs. (F5), (F2), and (F1) into Eq. (25) obtains

$$V(r) = \frac{225Q}{rR^6}\left[1 - \frac{693^2}{R^6} + O\left(\frac{1}{R^{12}}\right)\right] \tag{27}$$

for $R \gg 1$, a rapid power law fall-off far from the test charge.

How $V(r)$ behaves in the intermediate region is not obvious; however because the series in Eq. (F4) is rapidly absolutely convergent it is an easy matter to ascertain this numerically. We find that $V(r)$ has the shape indicated in Fig. 1; curiously, there is a single wiggle into negative values between the small-argument Coulomb behavior and the large r asymptotic result.[22]

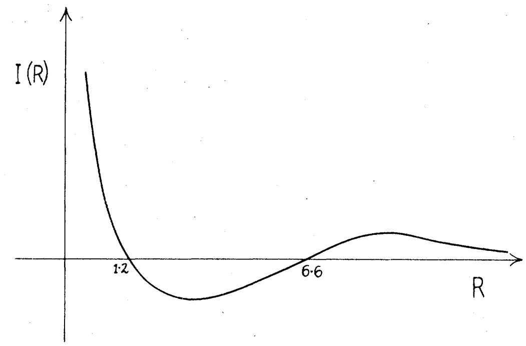

FIG. 1. Zero-temperature electrostatic potential is $V(r) = (Q/r)\cdot I(R)$. The shape of the curve is exaggerated. The minimum at $R=1.9$ has the value -0.096, while the maximum at $R=7.6$ has the value 4.2×10^{-4}. The dimensionless coordinate $R=q_2 r$ where the inverse length q_2 is given in Eq. (25).

VI. CONCLUDING REMARKS

We now discuss the behavior of the thermodynamic functions of the gas around $T=0$ which was displayed in Sec. IV. There the contribution to the low-temperature expressions for the thermodynamic functions resulting from the temperature corrections to the quasiparticle energy spectrum dominates that of the zero-temperature term in the spectrum.

It could be thought that this behavior is suggestive of some kind of "renormalization" of the system at, or just above, $T=0$, and that we have here a faint signaling of the presence of a weak phase transition in the two-dimensional charged Bose gas at a finite temperature.

This possibility seems, however, to be implausible. Hohenberg's theorem[15] states that long-range order is absent at finite temperature in a homogeneous two-dimensional quantum-mechanical system of particles which interact via a finite-ranged potential. While the Coulomb potential is itself long-ranged, the screening phenomenon produces an effective short-ranged potential.

It is most likely that the temperature behavior of the spectrum and resulting thermodynamics in the two-dimensional charged Bose gas, as contrasted with its counterpart in three dimensions, are a direct consequence of the suppression of condensation in the two-dimensional ideal Bose gas at all temperatures excepting absolute zero. This fact, along with the change in phase space available to the quasiparticles around $T=0$, we believe, are the major factors in producing the different low-temperature effects in two and three dimensions, respectively. It would be of great value to study this system further by investigating the behavior of more advanced approximation-perturbation schemes.

All of the aforementioned considerations lead one to consider the behavior of the "pure-Coulomb" two-dimensional charged Bose gas. The interactions in this system, being logarithmic, are far more long-ranged than the $1/r$ potential and it is not clear that studying this system within the random-phase approximation is correct. This system will be discussed at greater length in a forthcoming paper.

Notwithstanding all that we have already said, it would be most welcome to have some rigorous theorems proving the existence or nonexistence of a condensate in these two-dimensional Coulomb-Bose systems in the ground state and at finite temperature.

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APPENDIX A

As in HF we use the identity

$$\frac{1}{x^{-1}-1}=\sum_{j=1}^{\infty} x^j, \quad |x|<1 \tag{A1}$$

and the prescription, valid in the limit as $\Omega \to \infty$,

$$\sum_{\vec{p}} \to \frac{\Omega}{(2\pi)^2} d^2p$$

to obtain from Eq. (3) the following:

$$\epsilon(\vec{q},\omega)=1+\frac{e^2}{2\pi\hbar q}\sum_{j=1}^{\infty} z^j\left\{\int d^2p\left[\frac{\exp\left(\frac{-\hbar^2p^2}{2mk_BT}j\right)}{\omega-\frac{\hbar^2}{m}\vec{p}\cdot\vec{q}+\frac{\hbar q^2}{2m}}\right]-\int d^2p\left[\frac{\exp\left(\frac{-\hbar^2p^2}{2mk_BT}j\right)}{\omega-\frac{\hbar}{m}\vec{p}\cdot\vec{q}-\frac{\hbar q^2}{2m}}\right]\right\} .$$

The above integrals are easily evaluated using Cartesian co-ordinates p_x and p_y. Taking $\vec{q}$ as the p_x direction and performing the integration over p_y yields

$$\epsilon(\vec{q},\omega)=1+\frac{e^2m}{2\hbar^2q^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{1/2}\times\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}[Z(C_jB)-Z(C_jD)] , \tag{A2}$$

where

$$Z(\zeta)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\,\frac{e^{-x^2}}{x-\zeta}$$

is the plasma dispersion function of Fried and Conte.[23] Now for real values of its argument, $Z(\zeta)$ may be written[23]

$$Z(x)=i\pi^{1/2}e^{-x^2}[1+\phi(ix)] .$$

This result together with Eq. (A2) gives Eq. (4).

APPENDIX B

We here obtain asymptotic expansions of

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}\phi(ix^{1/2}j^{1/2})e^{-xj}$$

for both $x<1$ and $x>1$ in the limit as $z \to 0+$.

From Appendix B of HF

$$e^{-xj}\phi(ix^{1/2}j^{1/2})=ix^{1/2}j^{1/2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\,\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)}j^{-s}x^{-s}, \quad 0<c<1 . \tag{B1}$$

$\Gamma(x)$ is the γ function.[19]

Then

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}\phi(ix^{1/2}j^{1/2})e^{-xj}=ix^{1/2}\frac{1}{2\pi i}\int_{c-i\infty}^{c-i\infty} ds\,\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)}g_x(z)x^{-s}, \quad 0<c<1 ,$$

where

$$g_\alpha(z)=\sum_{j=1}^{\infty}\frac{z^j}{j^\alpha}$$

and this function has no poles, as a function of α, in the neighborhood of $z=0$.

The integral is easily evaluated using Cauchy's theorem. Closing the contour in the left half-s-plane obtains an expression for small x, while closing the contour in the right half-s-plane yields an expression appropriate for large x. The results are as follows

$x<<1$:

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}\phi(ix^{1/2}j^{1/2})e^{-xj}=i\sum_{p=0}^{\infty}\frac{(-1)^p}{\Gamma(\frac{3}{2}+p)}g_{-p}(z)x^{1/2+p} ; \tag{B2}$$

$x>>1$:

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}\phi(ix^{1/2}j^{1/2})e^{-xj}=i\sum_{p=0}^{\infty}\frac{(-1)^p}{\Gamma(\frac{1}{2}-p)}\frac{g_{1+p}(z)}{x^{1/2+p}} . \tag{B3}$$

APPENDIX C

We now derive expansions of

$$\sum_{j=1}^{\infty} \frac{z^j}{j^{1/2}} \phi(ix^{1/2}j^{1/2}) e^{-xj}$$

for $x < 1$ and $x > 1$ in the limit as $z \to 1-$.

As in Appendix B of HF we use the Mellin integral representation of

$$z^j = e^{-j \ln z}$$

and Eq. (B1) to obtain

$$\sum_{j=1}^{\infty} \frac{z^j}{j^{1/2}} \phi(ix^{1/2}j^{1/2}) e^{-xj} = \left(\frac{1}{2\pi i}\right)^2 ix^{1/2} \int_{d-i\infty}^{d+i\infty} dt \int_{c-i\infty}^{c+i\infty} ds\, \Gamma(t) \zeta(t+s) \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)} (-\ln z)^{-t} x^{-s} ,$$

$$0 < c < 1, \quad d > 0, \quad c + d > 1 .$$

$\zeta(x)$ is the Riemann ζ function.[19]

If the t contour is closed in the left half-plane the following expression appropriate for $z \simeq 1$ results:

$$\sum_{j=1}^{\infty} \frac{z^j}{j^{1/2}} \phi(ix^{1/2}j^{1/2}) e^{-xj} = ix^{1/2} \left\{ \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha}{\alpha!} (-\ln z)^\alpha \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)} \zeta(s-\alpha) x^{-s} \right.$$

$$\left. + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(s)[\Gamma(1-s)]^2}{\Gamma(\frac{3}{2}-s)} (-\ln z)^{s-t} x^{-s} \right\}, \quad 0 < c < 1 . \tag{C1}$$

Now closing the s contour in the left half-plane yields the small-x expansion

$$\sum_{j=1}^{\infty} \frac{z^j}{j^{1/2}} \phi(ix^{1/2}j^{1/2}) e^{-xj} = i \left[\sum_{\alpha,p=0}^{\infty} \frac{(-1)^{\alpha+p}\zeta(-\alpha-p)}{\alpha!\Gamma(\frac{3}{2}+p)} (-\ln z)^\alpha x^{1/2+p} + \frac{1}{(-\ln z)^{1/2}} \sum_{p=0}^{\infty} \frac{(-1)^p p!}{\Gamma(\frac{3}{2}+p)} \left(\frac{x}{-\ln z}\right)^{1/2+p} \right] . \tag{C2}$$

If the contour is closed in the right half-s-plane an expression appropriate for large x is obtained. However evaluation of Eq. (C1) is more involved in this case than for small x, due to the existence of double poles of $[\Gamma(1-s)]^2$ at $s=p$, $p=1, 2, \ldots$ and the double pole of $\Gamma(1-s)\zeta(s-\alpha)$ at $s=1+\alpha$, $\alpha=0,1,\ldots$.

The occurrence of double poles in the above expression is a direct analytical consequence of the two dimensionality of the system being considered in this paper. For the three-dimensional charged Bose gas, as treated in HF, only simple poles are present in the expression analogous to Eq. (C1). The analysis of the expressions which occur in the description of the two-dimensional Bose gas is therefore rather more involved than that which is required when dealing with the corresponding three-dimensional expressions. This, along with the absence of a Bose-Einstein condensation in two dimensions, is why the results in this paper differ so from those in HF.

Nonetheless, the residues of the integrands of Eq. (C1) at these double poles can be calculated using standard analysis. For the residue of

$$\Gamma(s) \frac{[\Gamma(1-s)]^2}{\Gamma(\frac{3}{2}-s)} (-\ln z)^{s-1} x^{-s}$$

at $s=p$, $p=1,2,\ldots$, we obtain

$$\frac{(-\ln z)^{p-1}}{(p-1)!\Gamma(\frac{3}{2}-p)x^p} \{2[C-\chi(p-1)] + \psi(p)$$

$$+ \psi(\tfrac{3}{2}-p) + \ln(-\ln z) - \ln x\} ,$$

where

$$\psi(x) = \frac{d}{dx}[\ln\Gamma(x)]$$

is the digamma function,[19]

$$\chi(m) = \begin{cases} 0, & m=0 , \\ \sum_{k=1}^{m} \frac{1}{k}, & m=1,2,\ldots \end{cases}$$

and C is Euler's constant.[19]

The residue of

$$\Gamma(s) \frac{\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)} \zeta(s-\alpha) x^{-s}$$

at $s=1+\alpha$, $\alpha=0,1,\ldots,$ is

$$\frac{(-1)^{1+\alpha}}{\Gamma(\frac{1}{2}-\alpha)x^{1+\alpha}}[2C-\chi(\alpha)+\psi(1+\alpha)+\psi(\tfrac{1}{2}-\alpha)-\ln x] \ .$$

Then for large x Eq. (C2) provides the following asymptotic expansion:

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}\phi(ix^{1/2}j^{1/2})e^{-xj}=i\left[\sum_{\alpha=0}^{\infty}\sum_{\substack{p=0,\\ p\neq\alpha}}\frac{(-1)^{\alpha+p}\zeta(1+p-\alpha)(-\ln z)^{\alpha}}{\alpha!\Gamma(\frac{1}{2}-p)x^{1/2+p}}+\sum_{p=0}^{\infty}\frac{(-\ln z)^p}{p!\Gamma(\frac{1}{2}-p)x^{1/2+p}}[\chi(p)-\ln(-\ln z)]\right] \ . \tag{C3}$$

APPENDIX D

Using the Mellin integral representation of[10] e^{-x} it is easy to derive asymptotic expansions of

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}e^{-xj}$$

for small x in the limit as $z\to 0+$ and for both large and small x in the limit as $z\to 1-$.

The results are as follows: when $z\simeq 0$

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}e^{-xj}=\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}g_{1/2-p}(z)x^p \ , \tag{D1}$$

useful for $x \ll 1$; when $z\simeq 1$

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}e^{-xj}=\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}(-\ln z)^p g_{1/2-p}(e^{-x}) \ , \tag{D2}$$

good for $x\gg 1$ and

$$\sum_{j=1}^{\infty}\frac{z^j}{j^{1/2}}e^{-xj}=\sum_{p,\alpha=0}^{\infty}\frac{(-1)^{p+\alpha}}{\alpha!p!}\zeta(\tfrac{1}{2}-p-\alpha)(-\ln z)^{\alpha}x^p+\frac{1}{(-\ln z)^{1/2}}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\Gamma(\tfrac{1}{2}+p)\left[\frac{x}{-\ln z}\right]^p \ , \tag{D3}$$

which is an asymptotic expansion suitable for $x\ll 1$.

APPENDIX E

We wish to obtain expressions for $z=z(\rho,T)$ in the two limits $z\to 0+$ $(T\to\infty,\rho\to 0)$ and $z\to 1-$ $(T\to 0+,\rho\to\infty)$.

The number equation for ideal bosons in two dimensions can easily be shown to have the closed form

$$\rho\lambda_T^2=-\ln(1-z) \ . \tag{E1}$$

The identity

$$-\ln(1-x)=\sum_{j=1}^{\infty}\frac{x^j}{j}, \quad |x|<1 \tag{E2}$$

shows that the right-hand size of Eq. (E1) is a rapidly converging series for small z so that we need only directly invert Eq. (E1) for $z\simeq 0$, which then yields

$$z=\rho\lambda_T^2[1-\tfrac{1}{2}(\rho\lambda_T^2)+\tfrac{1}{6}(\rho\lambda_T^2)+O((\rho\lambda_T^2)^3)] \ , \tag{E3}$$

$\rho\lambda_T^2\ll 1$ (the high-temperature, low-density limit).

Using Eq. (E2) it is trivial to rewrite Eq. (E1) as

$$-\ln z=-\ln(1-e^{-\rho\lambda_T^2})=\sum_{j=1}^{\infty}\frac{e^{-j\rho\lambda_T^2}}{j} \ . \tag{E4}$$

This relation is immediately useful for $\ln z\simeq 0$ $(z\simeq 1)$, when $\rho\lambda_T^2\gg 1$ (the low-temperature, high-density limit).

We also require an expansion for $-\ln(-\ln z)$ when $z\simeq 1$. It is easily shown that

$$\sum_{j=1}^{\infty}\frac{z^j}{j}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\zeta(1+s)\,ds}{(-\ln z)^s}, \quad c>0 \ .$$

This integral may be evaluated by closing the contour in the left half-plane. The residue at the double pole $s=0$ is $-\ln(-\ln z)$. Cauchy's theorem then gives

$$-\ln(-\ln z)=\rho\lambda_T^2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\zeta(1-n)(-\ln z)^n \ , \tag{E5}$$

where Eqs. (E1) and (E2) have been used.

APPENDIX F

We here evaluate the integral

$$I(R)=\int_0^{\infty}dx\,\frac{x^3J_0(Rx)}{1+x^3} \ . \tag{F1}$$

It is more conveniently written

$$I(R)=\frac{1}{R}-\int_0^{\infty}dx\,\frac{J_0(Rx)}{1+x^3} \ , \tag{F2}$$

where the result[19]

$$\int_0^\infty J_0(ax)\,dx = \frac{1}{a}$$

has been used.

The factor $1/(1+x^3)$ can be split into a sum of partial fractions of the form

$$\sum_{n=1}^{3} \frac{U_n}{3(x+U_n)} \ ,$$

where the U_n are the three cube roots of unity. Then the three resultant integrals are standard[19] and we obtain

$$\int_0^\infty dx \frac{J_0(Rx)}{1+x^3} = \tfrac{1}{6}\pi \{H_0(R) - N_0(R) + e^{2\pi i/3}[H_0(Re^{2\pi i/3}) - N_0(\mathrm{Re}^{2\pi i/3}] + e^{-2\pi i/3}[H_0(Re^{-2\pi i/3}) - N_0(\mathrm{Re}^{-2\pi i/3})]\} \ , \tag{F3}$$

where $N_0(z)$ is the Bessel function of the second kind of order zero and $H_0(z)$ is the Struve function of order zero. Using the power-series expansions[19] of $N_0(z)$ and $H_0(z)$, Eq. (F3) can be written as follows:

$$\int_0^\infty \frac{J_0(Rx)}{1+x^3} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{64}R^6\right)^n}{[(3n)!]^2}\left[\frac{2\pi}{3(3)^{1/2}} + \frac{R^2}{4(3n+1)^2}\left(C + \ln(\tfrac{1}{2}R) - \chi(3n+1) - \frac{\pi R^2}{6(3)^{1/2}(3n+2)^2}\right)\right]$$

$$+ R^5 \sum_{n=0}^{\infty} \frac{(-R^6)^n}{[(6n+5)!!]^2} \ . \tag{F4}$$

The asymptotic form for this function as $R \to \infty$ is easily found by using the standard asymptotic result[19] for $H_0(z) - N_0(z)$, which, when substituted into Eq. (F3), gives

$$\int_0^\infty \frac{dx\, J_0(Rx)}{1+x^3} = \frac{1}{R} - \frac{225}{R^7} + \frac{693^2}{R^{13}} + O\left(\frac{1}{R^{19}}\right) \ ,$$

$$\text{as } R \to \infty \ . \tag{F5}$$

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