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Electron multiplication in solids

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Treating collisions as Markov processes, a theory is developed for the calculation of electron multiplication in solids. The electron mean free paths for acoustic-phonon scattering, optical-phonon emission, and pair production processes are assumed to be independent of the energy of the electron. The number of ionization events per mean free path is a universal function of dimensionless energy ratios. Several types of phonon scatterings are studied. In the case of isotropic scattering, comparison of the present work with the results obtained by numerically solving the Boltzmann equation indicates that good agreement is found only in fast-multiplication cases. The discrepancy noted in slow-multiplication cases is believed to be due to either the less stringent requirements or the mild singularity of the transition probabilities in the numerical approach to solving the Boltzmann equation.

I. INTRODUCTION

Under the stress of a very strong electric field, semiconductors and dielectrics tend to exhibit avalanche breakdown. A quantitative measurement of the charge multiplication in p-n junctions of Si and Ge was carried out and its implication on the ionization rate per unit length, as a function of the applied electric field, was analyzed two decades ago.[1] The fundamental processes for charge multiplication in solids were assumed to be analogous to that (Townsend's β mechanism[2]) of gas discharge. The ionization rate is an important parameter which brings out the details of the microscopic solid-state properties from the macroscopic characterestics of breakdown measurements. Methods have been developed to calculate this parameter, most of them involving the solution of the Boltzmann equation in a high electric field. Thus, neglecting the then-unknown band structure of silicon, Wolff[3] expanded the electron distribution function in terms of Legendre polynomials, kept the first two terms, and solved the Boltzmann equation in a steady state. Baraff[4] employed the concept of collision density and derived an integral equation by Laplace-transforming the Boltzmann equation which he then solved numerically. To solve the Boltzmann equation at high fields is generally difficult, as pointed out by Wannier.[5] On the other hand, Shockley[6] considered the collision processes as probabilistic processes with exponential probability distribution and, treating only the electrons which survive any collision, directly obtained the ionization rates in the low-field and high-field limits. Although the validity of some assumptions made by Shockley was questioned at that time, the concept of exponential probability is particularly attractive in its simplicity. It appears that for the calculation of ionization rates, one may avoid solving the Boltzmann equation by treating all processes as Markov processes and following the motion of the electron in a completely stochastic way.

In this paper, we will address the calculation of ionization rates in semiconductors or dielectrics where band-structure details can be neglected. A strong electric field $\vec{\mathcal{E}}$ is applied to a thick slab of material. The electron whose track we will follow starts with zero energy in the conduction band. We assume that there are three energy ranges in each of which only one single-electron process is possible. Thus from zero to $\hbar\omega$ (the energy of the only optical phonon), only acoustic-phonon scattering is possible; between $\hbar\omega$ and E_i (the ionization threshold to produce an electron-hole pair), the electron can only emit the optical phonon, assuming no optical phonons to be present so that the absorption of optical phonons is negligible; above E_i, electron excitation from the valence band is the only interaction the electron has with its environment. We also assume that the mean free paths of these three fundamental processes are independent of the electron energy. Furthermore, these three mean free paths are assumed for simplicity to have the same value λ. These assumptions are in line with previous works.[3,4,6] Most of them can be relaxed if one so wishes: the involved modifications of the equations, to be derived below, either are trivial or can be carried out with minor effort.

The main quantity we are interested in is the mean ionization distance for an electron starting with zero kinetic energy at $z=0$ (z is the depth coordinate). An electron, emitting n optical phonons, will reach E_i at location $z=(E_i+n\hbar\omega)/e\mathcal{E}$. Since z and n are two random variables, the mean distance $\bar{z}$ per mean free path is related to the mean number of optical-phonon emissions, $\bar{n}$, by

$$\bar{z}/\lambda = E_i/e\mathcal{E}\lambda + \bar{n}(\hbar\omega/e\mathcal{E}\lambda). \quad (1)$$

Setting $X \equiv E_i/e\mathcal{E}\lambda$ and $R \equiv \hbar\omega/E_i$, the mean number of electron-hole pairs produced by a single electron in traveling a mean free path λ is given by

$$\bar{N}_{\text{pair}} = (X + \bar{n}RX + 1)^{-1}. \quad (2)$$

The 1 on the right-hand side takes care of the extra mean distance λ_i the electron has to travel, after reaching E_i, in order to produce a pair. The exact value of this extra mean distance is not important,[3,4] however. Here we assume $\lambda_i = \lambda$.

Equation (2) indicates that $\bar{N}_{\text{pair}}$ is a universal function of X and R since $\bar{n}$, as will be shown later, depends only on X and R.

The concept described here could be applied to the calculation of the average energy of hot electrons emerging from a slab of given thickness z_0. The proper equation reads

$$\bar{E} = e\mathcal{E}z_0 - \bar{n}\hbar\omega \quad (3)$$

and $\bar{n}$ could be obtained exactly in the same way, as described in Sec. II. The only change is to replace $y - \Delta y$ in the arguments by y. Though Monte Carlo results[7] do exist, the suggested approach here gives an exact answer.

II. THEORY

Let λ be the mean free path of a collision (a Markov process). The no-collision probability for an electron in traveling a distance z is $e^{-z/\lambda}$ and the probability of one collision in dz is dz/λ. The mean collision distance is given by

$$\langle z \rangle = \int_0^\infty \frac{z e^{-z/\lambda} dz}{\lambda} = \lambda, \quad (4)$$

the mean free path of this process.

At some reference level, an electron is released with both kinetic and potential energies zero. Under the influence of a constant electric field $\mathcal{E}$, this electron behaves like a particle of some inertial mass in a gravitational field. Now, if this electron experiences an acoustic (elastic) collision, it simply adjusts the direction of its velocity with magnitude unchanged. If an optical-phonon emission occurs, the electron not only changes the velocity direction but also loses some quantum kinetic energy (the quantum energy $\hbar\omega$ of the optical phonon). Since in a conservative field, there is a one-to-one correspondence between kinetic energy and potential energy, once the reference level of zero potential is chosen, the fact that the electron loses $\hbar\omega$ kinetic energy will be equivalent to pushing this electron a distance $\hbar\omega/e\mathcal{E}$ closer to the reference level, as shown in Fig. 1. The probability of having a collision is controlled

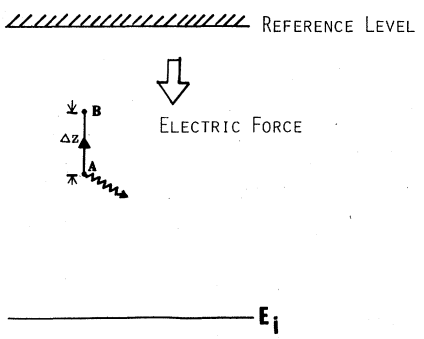

FIG. 1. Electron at A emits on optical phonon and its location is consequently put at B, $\Delta z = \hbar\omega/e\epsilon\lambda$.

by the exponential function of the length of the trajectory this electron travels after the latest collision.

For a Markov process, the history of the electron before a collision is completely wiped out when a collision occurs. In other words, the angular distribution of the cross section of a collision is independent of the direction of this incident electron. A few examples are in order:

(a) Forward scattering. The electron can only travel in the direction of the electric force. There is no elastic collision in this case because the direction is fixed.

(b) Two-direction scattering. The electron can move either along or against the electric field. Movement in either direction can cause collision with the consequence of, say, half chance of the electron going along and half chance of going against the electric field.

(c) Isotropic scattering. The cross section is isotropic and thus of Markov type.

In general, we can assume the angular distribution of the probability to be $P(\theta)$ no matter what direction the electron is moving in immediately before a collision. θ is the angle the velocity vector makes with the horizontal line as shown in Fig. 2. The electron is at a location y_1 ($y = z/\lambda$) below the reference level. Let $T(y_2, y_1)dy_2$ be the probability distribution that the electron, suffering a collision at y_1, will not have any collision before reaching the height y_2 below the reference level and having a collision in dy_2 at y_2. The collision here could be acoustic or optical. This probability is computable from classical mechanics, as can be done

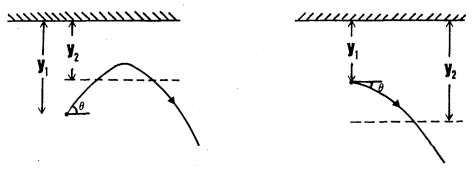

FIG. 2. Electron as a projectile.

following the results of the Appendix. It is easy to prove that $\int_0^\infty T(y_2,y_1)dy_2 = 1$ if one realizes that, for a given projectile angle, the probability of having a collision regardless of how far the electron travels is unity, i.e., $\int_0^\infty e^{-s/\lambda}ds/\lambda = 1$. Once $P(\theta)$ is assigned and $T(y_2,y_1)$ is computed, the ionization rate can be obtained as follows.

Let

$$N(y_1) = 1 - \int_0^{y_0} T(y_2,y_1)dy_2 = \int_{y_0}^\infty T(y_2,y_1)dy_2$$

be the probability than an electron suffering a collision at y_1 will not have any collision before reaching y_0 ($\equiv X$), the ionization threshold for producing an electron-hole pair. Let Δy be the width of the acoustic-scattering region, i.e.,

$$\Delta y = e\mathcal{E}\Delta z/e\mathcal{E}\lambda = \hbar\omega/e\mathcal{E}\lambda = RX.$$

We can then compute, by the technique of regrouping, the probability of having n optical-phonon emissions, regardless of the number of scatterings due to acoustic phonons.

The electron released at zero kinetic energy at the potential reference level reaches E_t either without any collision or with at least one collision (acoustic or optical). In other words,

$$1 = N(0) + \int_0^{y_0} 1 \times T(y_1,0)dy_1 . \tag{5}$$

After breaking up the integration limits into the acoustic and optical regions, the 1 on the right-hand side can be replaced by

$$N(y_1) + \int_0^{y_0} T(y_2,y_1)dy_2$$

when y_1 is in the acoustic scattering region, or by

$$N(y_1-\Delta y) + \int_0^{y_0} T(y_2,y_1-\Delta y)dy_2$$

when y_1 is in the optical emission region:

$$1 = N(0) + \int_0^{\Delta y} N(y_1)T(y_1,0)dy_1 + \int_0^{\Delta y} dy_1 \int_0^{y_0} dy_2 T(y_2,y_1)T(y_1,0) + \int_{\Delta y}^{y_0} dy_1 N(y_1-\Delta y)T(y_1,0)$$
$$+ \int_{\Delta y}^{y_0} dy_1 \int_0^{y_0} dy_2 T(y_2,y_1-\Delta y)T(y_1,0). \tag{6}$$

If (n_a,n_0) denotes the numbers of acoustic and optical scatterings, then the first term in Eq. (6) is (0, 0), the second (1, 0), and the fourth (0, 1). For the third and the last, we need to break up the integration $\int_0^{y_0}$ and then substitute the appropriate equation for 1 as before. This procedure could be repeated indefinitely. Collecting the terms with $(n_a,0)$ for $n_a = 0,1,2,3,\ldots,\infty$ gives

$$N(0) + \int_0^{\Delta y} dy_1 N(y_1)T(y_1,0) + \int_0^{\Delta y} dy_1 \int_0^{\Delta y} dy_2 N(y_2)T(y_2,y_1)T(y_1,0) + \cdots$$
$$= N(0) + \int_0^{\Delta y} dy_1 \tilde{N}(y_1)T(y_1,0) = \tilde{N}(0), \tag{7}$$

where $\tilde{N}(y_1)$ satisfies the following inhomogeneous integral equation of the second kind:

$$\tilde{N}(y_1) = N(y_1) + \int_0^{\Delta y} dy_2 \tilde{N}(y_2)T(y_2,y_1). \tag{8}$$

We can interprete $\tilde{N}(y)$ as the renormalized (by interaction with acoustic phonons) version of the bare quantity $N(y)$.

For one optical-phonon emission, all the terms of the type $(n_a, 1)$, $n_a = 0, 1, 2, \ldots$ need to be collected and regrouped. After two regroupings and one relabeling, the probability of one optical emission is given by

$$\int_{\Delta y}^{y_0} dy_1 \tilde{N}(y_1-\Delta y)\tilde{T}(y_1,0), \tag{9}$$

where $\tilde{T}(y_2,y_1)$ also satisfies an integral equation,

$$\tilde{T}(y_2,y_1) = T(y_2,y_1) + \int_0^{\Delta y} dy_3\, T(y_2,y_3)\tilde{T}(y_3,y_1). \tag{10}$$

Again, due to the presence of the acoustic-scattering region, the bare transition probability density $T(y_2,y_1)$ is renormalized to $\tilde{T}(y_2,y_1)$.

A similar procedure leads to the following probability for two optical-phonon emissions,

$$\int_{\Delta y}^{y_0} dy_1 \int_{\Delta y}^{y_0} dy_2 \tilde{N}(y_2-\Delta y)\tilde{T}(y_2,y_1-\Delta y)\tilde{T}(y_1,0). \tag{11}$$

The rule for writing the probability of any num-

ber of optical-phonon emissions can be obtained by examining Eq. (11). $\tilde{T}(y_1, 0)$ indicates that the electron is released at $y = 0$, making all possible interactions with the acoustic phonons, and reaches y_1 in the optical emission region. Emitting an optical phonon at y_1 and thereby changing its location to $y_1 - \Delta y$, the electron encounters all possible, including zero, numbers of acoustic-phonon scatterings before the next optical-phonon emission at y_2 in the optical emission region, as described by $\tilde{T}(y_2, y_1 - \Delta y)$. The optical-phonon emission at y_2 pushes the location of the electron to $y_2 - \Delta y$. Between this new position and the pair-production threshold no optical emission occurs while any number of acoustic scatterings may take place. This is represented by $\tilde{N}(y_2 - \Delta y)$. Reading equations like Eq. (11) is completely equivalent to describing the physical processes they represent.

It is gauranteed, from the way the above probabilities are derived, that the sum of all probabilities equals unity.

The average number of optical-phonon emissions is given by the sum of the products of the number of optical phonons emitted and the corresponding probability,

$$\bar{n} = 0\tilde{N}(0) + 1\int_{\Delta y}^{y_0} dy_1 \tilde{N}(y_1 - \Delta y)\tilde{T}(y_1, 0) + 2\int_{\Delta y}^{y_0} dy_1 \tilde{N}(y_1 - \Delta y)\int_{\Delta y}^{y_0} dy_2 \tilde{T}(y_1, y_2 - \Delta y)\tilde{T}(y_2, 0)$$
$$+ 3\int_{\Delta y}^{y_0} dy_1 \tilde{N}(y_1 - \Delta y)\int_{\Delta y}^{y_0} dy_2 \tilde{T}(y_1, y_2 - \Delta y)\int_{\Delta y}^{y_0} dy_3 \tilde{T}(y_2, y_3 - \Delta y)\tilde{T}(y_3, 0) + \cdots . \quad (12)$$

Laying down the probabilities with weights (the number of phonons emitted) as in Fig. 3, and summing vertically for each column, one obtains a function G satisfying

$$G(y_1) = \tilde{T}(y_1, 0) + \int_{\Delta y}^{y_0} dy_2 \tilde{T}(y_1, y_2 - \Delta y)G(y_2) \quad (13)$$

for each column. When all the columns are summed, the final expression for the average number of optical-phonon emissions is given by

$$\bar{n} = \int_{\Delta y}^{y_0} dy \tilde{N}(y - \Delta y)H(y), \quad (14)$$

where

$$H(y_1) = G(y_1) + \int_{\Delta y}^{y_0} dy_2 \tilde{T}(y_1, y_2 - \Delta y)H(y_2). \quad (15)$$

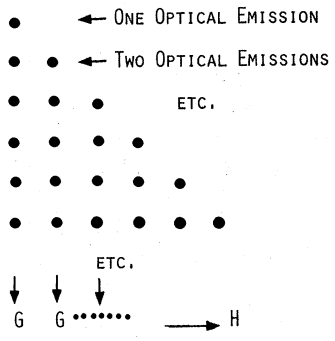

FIG. 3. Summing a column gives G. Summing all G's gives H.

Equation (14) indicates $\bar{n}$ is a function of y_0, which is equivalent to the energy ratio $X \equiv E_i/e\mathcal{E}\lambda$, and Δy, another energy ratio, $\hbar\omega/e\mathcal{E}\lambda$ ($\equiv RX$). Given R and X, $\bar{N}_{\text{pair}}$ is uniquely determined by Eq. (2).

III. RESULTS AND DISCUSSION

As mentioned in Sec. II, there are three interesting cases of scatterings: forward scattering, two-direction scattering, and isotropic scattering. The first case provides a check of the formulation described above against an algebraic approach to be published elsewhere.[8] The second case gives a hint of how the backward scattering (against the electric force) would change the mean ionization distance, while the complication of calculating the trajectory length is kept to a minumum. The third case is that of the real physical situation which has been discussed by others in the literature.

Case (i): forward scattering.

$$T(y_2, y_1) = \begin{cases} 0 & \text{for } y_2 < y_1 \text{ or } y_1 < y_2 < \Delta y \\ \exp - [y_2 - \max(y_1, \Delta y)] & \text{otherwise.} \end{cases} \quad (16)$$

Backward scattering ($y_2 < y_1$) is forbidden in this case. The $N(y)$ corresponding to this transition probability is given by $\exp[\max(y, \Delta y) - y_0]$. Using the formulation of Sec. II, one obtains $e^{RX} - 1$, and $e^{2RX} + (1 - RX)e^{RX} - 2$ for the average number of optical-phonon emissions in the respective cases of $R = \frac{1}{2}$ and $\frac{1}{3}$. These results agree with that of the simple algebraic approach.[8] For smaller R's

of the type $1/(m+1)$ with integer m, numerical evaluations in a self-consistent way of the integral equations in Sec. II yield the same $\bar{n}$ as that of the algebraic solution.[8] For the analytic confirmation of the equivalence of these two different approaches, a conjecture, $h_m=[K(m+1)-m]e^{m\Delta y}$ for $R=1/(m+1)$, needs to be proven: the $h(y)$ is defined as $H(y)=h(y)/e^{y-\Delta y}$ and h_m is $h(y)$ for $m\Delta y<y<y_0$. The definition of $K(m)$ can be found in Ref. 8. That this conjecture is true for $m=1$ and 2 has been checked.

Case (ii): two-direction scattering with equal probabilities.

$$T(y_2,y_1)=\tfrac{1}{2}\exp(-|y_2-y_1|)+\tfrac{1}{2}\exp[-(y_1+y_2)]. \quad (17)$$

$N(y)=e^{-y_0}\cosh(y)$ in this case. Shown in Fig. 4 are the transition probabilities for various y_1. Note that an electron, after having a collision at a given location, may move against the external electric force (backward scattering) and have another collision somewhere upstream in the electric field, further delaying the time at which it will reach the ionization threshold. Consequently, the mean ionization distance for this case is somewhat longer than for case (i) above. In Fig. 5, the ionization events per mean free path ($\bar{N}_{\text{pair}}$) of these two cases for $R=0.05$ are compared in a semilog scale. The one-half chance to scatter backward does change $\bar{N}_{\text{pair}}$ substantially for low electric fields (large $E_i/e\mathcal{E}\lambda$).

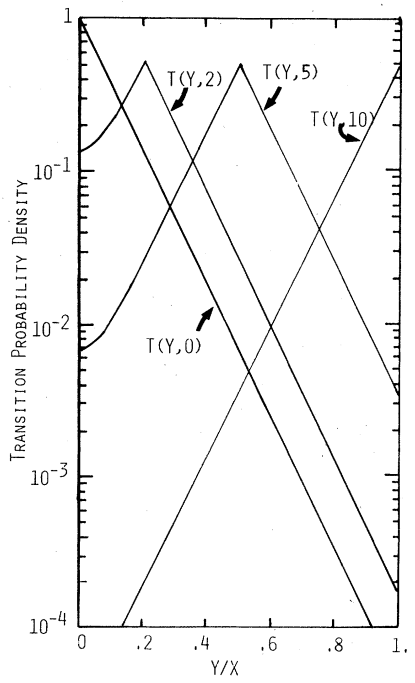

FIG. 4. $T(y_2,y_1)$ for two-direction scattering with equal probabilities. Here $X=E_i/e\mathcal{E}\lambda=10$.

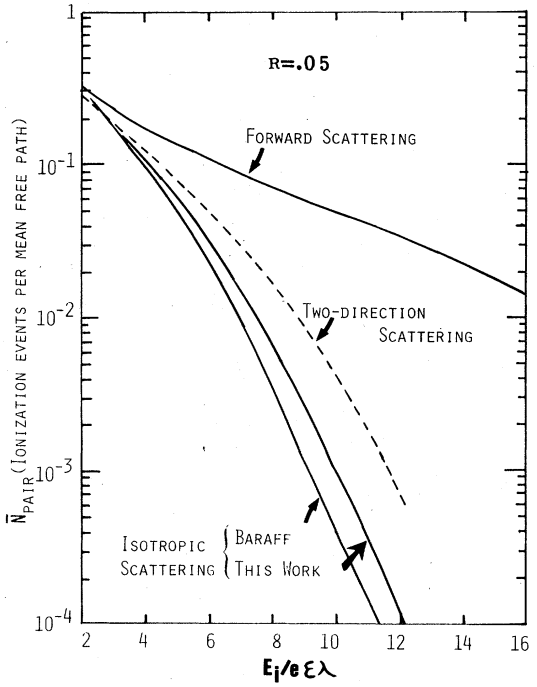

FIG. 5. Ionization events per mean free path for various types of scattering $\hbar\omega/E_i=0.05$.

Case (iii): general scattering.

$$T(y_2,y_1)dy_2=\tfrac{1}{2}\sum_{\theta=0}^{\pi/2}P(\theta)d(e^{-S_1(y_2,y_1,\theta)/\lambda}+e^{-S_2(y_2,y_1,\theta)/\lambda}). \quad (18)$$

The functions $S_1(y_2,y_1,\theta)$ and $S_2(y_2,y_1,\theta)$, as can be found in the Appendix, are the arc lengths of the electron from y_1 to the points on the trajectory at which the height is y_2. θ is the projectile angle and $P(\theta)$ the corresponding probability [$P(\theta)=$ const for isotropic scattering]. Trivially,

$$N(y_1)=\tfrac{1}{2}\sum_{\theta=0}^{\pi/2}P(\theta)(e^{-S_1(y_0,y_1,\theta)}+e^{-S_2(y_0,y_1,\theta)}). \quad (19)$$

There are three important observations concerning Eq. (18). The first is that the angular integration appears as a multiplicative factor whereas in Baraff's formulation[4] it shows up in the exponent. The second is about the behavior of $T(y_2,y_1)$ at some seemingly singular points. For example, at $\cos^2\theta=y_2/y_1$, both terms on the right-hand side of Eq. (18) are divergent. They sum up to a finite value because $S_1=S_2$ when $\cos^2\theta=y_2/y_1$. Another example is when $\cos\theta=0$ so that the denominator in the logarithmic function of $S_2(y_2,y_1,\theta)$ in Eq. (A2) becomes zero. The $\cos^2\theta$ factor in front of this logarithmic function eliminates this

singular behavior. The good behavior of $T(y_2,y_1)$ is sort of physically clear from the definition of $T(y_2,y_1)$ itself: given a scattering at y_1, the probability of the next scattering within a strip dy_2 at y_2 should be finite because the segment of the trajectory arc inside this strip shrinks to zero at least as fast as dy_2 does. Baraff's approximate expression for T does contain a singularity at $y_2=y_1$. The third observation involves the normalization of $T(y_2,y_1)$. Equation (18) automatically satisfies the requirement of normalization because it is formulated in a purely probabilistic way.

An equivalent way of carrying out the calculations indicated in Sec. II is to actually compute the probability of m optical-phonon emissions for $m=0,1,2,\ldots$. Once $\tilde{N}$ and $\tilde{T}$ are known, these probabilities are just simple integrals. The important advantage of this is to check and see if the sum of all probabilities equals unity, a very stringent constraint on the accuracy of both the most fundamental function $T(y_2,y_1)$ itself and all the numerical procedures involved. In other words, once $T(y_2,y_1)$ is set, a probability distribution is obtained which has to be summed up to unity to be correct. This requirement has been obeyed to guarantee the correctness of the computed results.

In actual calculations, a steady state will be reached, as correctly observed by Baraff,[4] in which the ratio of adjacent probabilities of emitting m and $m+1$ optical phonons approaches a constant. This constant, called γ, is strictly controlled by $T(y_2,y_1)$ itself and may be computed by using either the eigenvalue method or iterative procedures. Using this constant, the probability at the onset of the steady state, and the sum of the probabilities before reaching the steady state, one can compute the total probability which should be unity if everything is right.

In Fig. 6 we show the probability distributions of case (ii) and case (iii) for $R=0.05$ and $X=6$. The steady state is determined when the fractional change of the probability ratio γ is less than 10^{-6} (in some cases 10^{-7}).

The result of the present work for the isotropic-scattering case is presented in Fig. 5 for $R=0.05$. Also shown in the figure is a curve of Baraff for identical assumptions. The difference is about a factor of 2 at $X=10$ where, on the average, an ionization event occurs every thousand mean free paths (slow charge multiplication). The discrepancy gradually disappears as X is decreased. Comparison of Baraff's work and the present approach for $R=0.01$ shows excellent agreement for X from 2 to 14: the mean ionization distance in this case is generally less than 100 mean free paths (fast charge multiplication). It is noted that in the slow-multiplication cases the steady states always possess probability ratios very close to 1, i.e., $\gamma\approx 1$. Since the equation for the mean ioni-

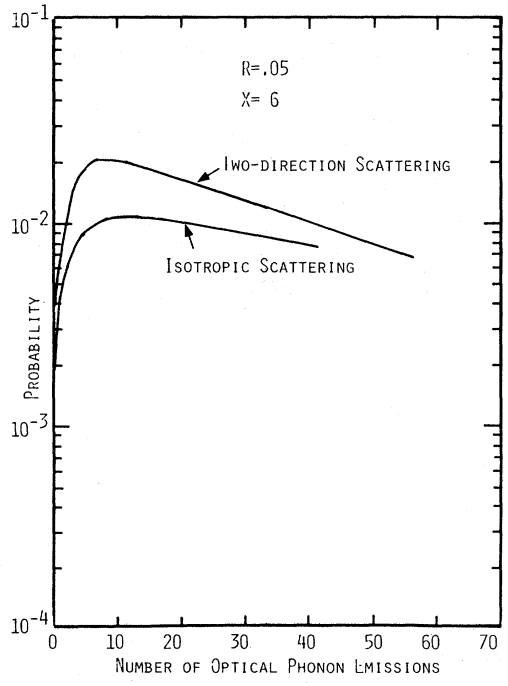

FIG. 6. Probability distributions of two different scattering functions for same R and X. The sum of each probability distribution equals unity.

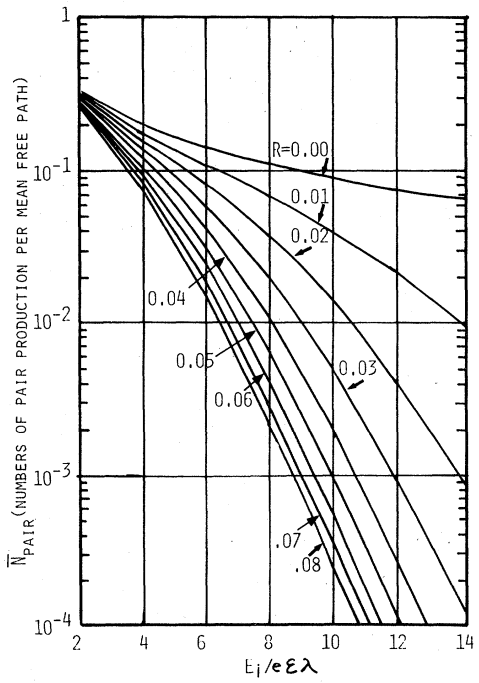

FIG. 7. Number of pairs produced per mean free path as a function of $E_i/e\mathcal{E}\lambda$ for various values of $R\equiv\hbar\omega/E_i$.

TABLE I. $\bar{N}_{pair}$ for $R=0.01-0.08$. For $R=0$, $N_{pair}=(1+X)^{-1}$. $a(-b)$ stands for $a\times10^{-b}$.

R \ X	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0.01	4.96(−1)	3.23(−1)	2.34(−1)	1.76(−1)	1.37(−1)	1.08(−1)	8.52(−2)	6.66(−2)	5.14(−2)	3.90(−2)	2.90(−2)	2.10(−2)	1.47(−2)	1.00(−2)
0.02	4.92(−1)	3.14(−1)	2.18(−1)	1.54(−1)	1.12(−1)	7.95(−2)	5.49(−2)	3.63(−2)	2.29(−2)	1.37(−2)	7.73(−3)	4.11(−3)	2.07(−3)	9.90(−4)
0.03	4.88(−1)	3.05(−1)	2.04(−1)	1.35(−1)	9.02(−2)	5.75(−2)	3.45(−2)	1.93(−2)	1.00(−2)	4.86(−3)	2.20(−3)	9.40(−4)	3.82(−4)	1.49(−4)
0.04	4.85(−1)	2.96(−1)	1.90(−1)	1.18(−1)	7.27(−2)	4.16(−2)	2.20(−2)	1.06(−2)	4.76(−3)	1.99(−3)	7.87(−4)	2.98(−4)	1.09(−4)	3.86(−5)
0.05	4.81(−1)	2.88(−1)	1.77(−1)	1.03(−1)	5.89(−2)	3.06(−2)	1.45(−2)	6.30(−3)	2.55(−3)	9.74(−4)	3.57(−4)	1.27(−4)	4.42(−5)	1.52(−5)
0.06	4.78(−1)	2.80(−1)	1.66(−1)	9.35(−2)	4.82(−2)	2.31(−2)	1.00(−2)	4.03(−3)	1.53(−3)	5.56(−4)	1.96(−4)	6.78(−5)	2.32(−5)	7.86(−6)
0.07	4.74(−1)	2.72(−1)	1.55(−1)	8.28(−2)	4.00(−2)	1.79(−2)	7.31(−3)	2.79(−3)	1.02(−3)	3.59(−4)	1.24(−4)	4.26(−5)	1.45(−5)	4.92(−6)
0.08	4.72(−1)	2.64(−1)	1.45(−1)	7.37(−2)	3.36(−2)	1.43(−2)	5.57(−3)	2.05(−3)	7.31(−4)	2.54(−4)	8.73(−5)	2.98(−5)	1.02(−5)	3.47(−6)

zation distance involves terms proportional to $1/(1-\gamma)$ and $1/(1-\gamma)^2$, the result is very sensitive to $1-\gamma$ when γ is near 1. The γ by the present calculation for $R=0.05$ and $X=10$, for example, is 0.99951 and the sum of all probabilities is a satisfactory 1.0012.

The results of the calculations of $\bar{N}_{pair}$ are shown in Fig. 7. Table I gives the computed values. The agreement between these results and Baraff's work is generally better for large $\bar{N}_{pair}$ (fast multiplication) than for small $\bar{N}_{pair}$ (slow multiplication). As described above, this may be due to the different requirements in getting γ's or the mild singularity in Baraff's expression of the approximate transition probability density $T(E_2, E_1)$. The overall agreement is considered good in viewing the very different approaches used.

Shockley's $\bar{N}_{pair}$ for the high-field limit conceptually corresponds to our $\bar{n}=0$. In his simple model, the high-field limit corresponds to the case of very hot electrons whose energies are always greater than E_i. In our model, those hot electrons cannot produce any optical phonon, thus $\bar{n}=0$. The argument for obtaining low-field limit in his model is not applicable to the special situation of $\lambda_i/\lambda=0$. Even if we choose $\lambda_i/\lambda=1$, the formula for $\bar{N}_{pair}$ is this simple model reads $\bar{N}_{pair}=e^{-X}/RX$, which does not correspond to any of our curves.

IV. CONCLUSION

A theory is developed for the calculation of the mean ionization distance in solids. The foundation of this formalism is the same as that of the Boltzmann equation, namely, concepts of probability. This shortcut approach clearly relates the physics to the equations and has versatility in applications. A simple application to the isotropic-scattering case produces results that generally agree with that obtained by numerically solving the Boltzmann equation.

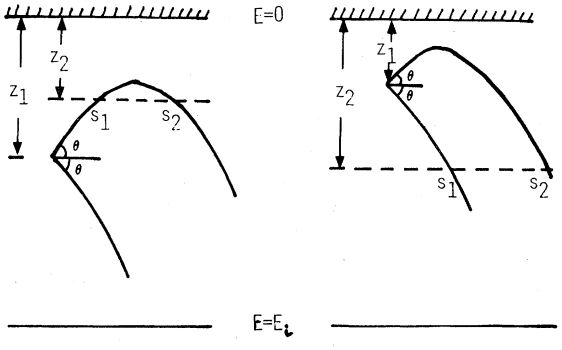

FIG. 8. Geometry of the projectile.

APPENDIX

This appendix gives the formula for the length of the electron trajectory under the action of a constant external electric force. The ceiling where the kinetic energy of the electron is zero is defined as $z=0$. The coordinate z is measured downward (along the direction of the force) and therefore $e\mathcal{E}z=\frac{1}{2}mv^2$. The electron at z_1 is projected at angles $\pm\theta$ (θ positive) with respect to the horizon, as shown in Fig. 8. By classical mechanics, the trajectory lengths from the point z_1 to the intersections with a horizontal line at height z_2 are given by

$$S_1(z_2,z_1,\theta)=z_1\left|\sin\theta+\cos^2\theta\ln\left(\frac{1+\sin\theta}{(z_2/z_1)^{1/2}+(z_2/z_1-\cos^2\theta)^{1/2}}\right)-\left(\frac{z_2}{z_1}\right)^{1/2}\left(\frac{z_2}{z_1}-\cos^2\theta\right)^{1/2}\right|, \tag{A1}$$

$$S_2(z_2,z_1,\theta)=z_1\left[\sin\theta+\cos^2\theta\ln\left(\frac{1+\sin\theta}{(z_2/z_1)^{1/2}-(z_2/z_1-\cos^2\theta)^{1/2}}\right)+\left(\frac{z_2}{z_1}\right)^{1/2}\left(\frac{z_2}{z_1}-\cos^2\theta\right)^{1/2}\right]. \tag{A2}$$

If these lengths and the z coordinate are scaled by λ and y is defined as z/λ, the dimensionless trajectory lengths $S_1(y_2,y_1,\theta)$ and $S_2(y_2,y_1,\theta)$ are obtained. They are used in Eq. (18).

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