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Quantum-statistical mechanics of extended objects. II. Breathers in the sine-Gordon system

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The analysis of the solitons in paper I is extended to breathers in the sine-Gordon system. The breather energy, the inertial mass of the breather, and the breather density at finite temperatures have been calculated. It was shown that the quantized breathers behave like elementary particles at low temperatures. At high temperatures ($E_s > T \gg m$, where E_s is the soliton energy and m is the mass of radiation), the total breather density becomes comparable to that of radiation, implying that the breather degree of freedom is essential in the thermodynamics of the sine-Gordon system.

I. INTRODUCTION

In the first part of this series,[1] which we shall hereafter refer to as I, we have extended the earlier quantum-field-theoretical treatments of the sine-Gordon system by Coleman[2] and by Dashen, Hasslacher, and Neveu[3] to finite temperatures. In particular, we have determined the thermodynamic properties of kinks of the sine-Gordon system. However, the question of breathers was deliberately left untouched. In this paper we shall analyze the thermodynamics of breathers in the sine-Gordon system. We shall again follow in great detail a work by Dashen, Hasslacher, and Neveu[4] (DHN II), where they discussed the quantization of the breathers.

The paper is organized as follows: In Sec. II we calculate the partition function for the system with a single breather. Also a moving breather is considered. From the partition function we obtain the breather energy, the breather inertial mass, and the breather density at finite temperatures, which are summarized in Sec. III. We show that the quantized breather contributes to the thermodynamics significantly. In particular, in the weak-coupling limit the total breather density becomes comparable to the radiation density at the high-temperature region ($E_s > T \gg m$). The present result contradicts the earlier transfer-matrix-technique (TMT) result[5,6] for the sine-Gordon system but is consistent with an heuristic ideal-gas model for the sine-Gordon system.[6]

II. PARTITION FUNCTION FOR BREATHER

As in I we shall consider the system described by the following Hamiltonian[1]:

$$H = \frac{1}{2}\int dx\left[\pi^2(x) + \left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{2(m^*)^2}{g^2}\cos g\phi - \frac{2m_0^2}{g^2}\right], \qquad (1)$$

where $\pi(x) = \partial\phi/\partial t$, and m^* and g are the bare mass of the Bose field $\phi(x)$ and the coupling constant, respectively. The last term with m_0, the physical mass of the ϕ field at zero temperature, is added for later convenience. The bare mass m^* in the coefficient of the cosine potential will be replaced by m, the temperature-dependent physical mass after renormalization of loop diagrams, which are logarithmically divergent.[1]

A. Static breather

As is well known, the sine-Gordon equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^2}{g}\sin(g\phi) = 0, \qquad (2)$$

which is obtained from Eq. (1), allows, besides kink solutions, breather solutions given by

$$\phi_B(w) = 4g^{-1}\tan^{-1}\left(\frac{(1-w^2)^{1/2}}{w}\frac{\sin(mwt)}{\cosh[m(1-w^2)^{1/2}t]}\right). \qquad (3)$$

The breather may be considered as a bound state of a soliton and an antisoliton. In the classical theory w is a continuous parameter $0 < w < 1$ and the energy of the breather is given by

$$E_B^{cl}(w) = \frac{16m}{g^2}(1-w^2)^{1/2}. \qquad (4)$$

As shown by DHN II in the quantum-field theory w cannot take continuous values but is limited to the discrete values. DHN have first shown that the Bohr-Sommerfeld (BS) quantization of the breathers[4]

$$\int_0^T dt \int_{-\infty}^{\infty} dx\, \pi_B(w)\left(\frac{\partial\phi_B(w)}{\partial t}\right) = 2\pi n, \qquad (5)$$

with $T = 2\pi/mw$, and n an integer yields

$$w_n^{BS} = \cos(\tfrac{1}{16}ng^2). \qquad (6)$$

$\pi(x)$ has been already defined after Eq. (1).

Substituting Eq. (6) into Eq. (4), DHN obtained a discrete energy spectrum for breathers

$$E_B^{\rm BS}(n)=\frac{16m}{g^2}\sin(\tfrac{1}{16}ng^2). \tag{7}$$

Furthermore, DHN were able to improve the Bohr-Sommerfeld result (7) by including the corrections arising from the quantum fluctuation around the Bohr-Sommerfeld solution at the absolute zero of temperature. These corrections modify Eq. (7) into

$$E_B^0(n)=\frac{16m_0}{g'^2}\sin(\tfrac{1}{16}ng'^2) \tag{8}$$

at $T=0$ K, where $g'^2=g^2/[1-(1/8\pi)g^2]$. They also speculated that Eq. (8) may be exact, although its derivation is approximate. Equation (8) is one of the basic results of DHN.

We are interested here in the breathers at finite temperatures. For this purpose we first substitute $\phi(x)$ in Eq. (1) by $\phi_B(w)+\phi$, with $w=w_n^{\rm BS}$ given by Eq. (6), and $\phi_B(w)$ the classical solution Eq. (3). Then expanding Eq. (1) in powers of ϕ, we can rewrite Eq. (1) as

$$H=E_B^{\rm cl}(w)+\tfrac{1}{2}\int dx\left[\pi(x)^2+\left(\frac{\partial\phi}{\partial x}\right)^2+m^2\cos[g\phi_B(w)]\phi^2 -m^2\cos[g\phi_B(w)]D-\frac{2m_0^2}{g^2}\right], \tag{9}$$

where we have kept up to the quadratic terms in ϕ. Here $E_B^{\rm cl}(w)$ is the energy of the classical solution given by Eq. (4) with $w_n^{\rm BS}$ [i.e., Eq. (7)], m is the physical mass of the ϕ field, and D is the equal space-time propagator of the ϕ field

$$D=T\sum_n\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\frac{1}{k^2+m^2}=\frac{1}{2\pi}\ln\left(\frac{2\Lambda}{m}\right)+f_0 \tag{10}$$

and

$$f_0(\beta m)=\frac{1}{\pi}\int_0^\infty dk\,\omega_k^{-1}N(\omega_k), \tag{11}$$

where $\omega_k=(k^2+m^2)^{1/2}$ and $N(E)=(e^{\beta E}-1)^{-1}$.

In deriving Eq. (9), we have first rewritten the cosine potential in terms of the "normal product" at finite temperatures introduced in I,

$$\frac{(m^*)^2}{g^2}\cos(g\phi)=\frac{m^2}{g^2}N(\cos g\phi),$$

with

$$m=(m^*)\exp(-\tfrac{1}{4}g^2D), \tag{12}$$

and then make use of the relation

$$N(\phi^2)=\phi^2-D. \tag{13}$$

Furthermore, as in the analysis of the solution, we have substituted for m and D, the ones corresponding to those in the absence of a breather.

In order to calculate the partition function associated with the breather, we have to solve the eigenvalue problem associated with the potential $V(w;x,t)=m^2\cos[g\phi_B(w)]$. In a sharp constrast to the case of the soliton, the potential depends now on time t as well as the space coordinate x. However, as in the case of the soliton, the potential is again reflectionless. Therefore, in spite of its time dependence, the incident radiation acquires asymptotically only a phase shift, which is independent of time. The phase shift of the radiation with incident wave vector k is given by[7]

$$\Delta(w,k)=4\tan^{-1}\left(\frac{mw}{|k|}\right). \tag{14}$$

This phase shift implies that the breather potential gives rise to two bound states with energy $w_{b1}=0$, and $w_{b2}=mw$. With this knowledge at hand, we can calculate the partition function associated with the breather as

$$Z_B(w)=\left\langle 0\right|\int\mathfrak{D}(\phi)\exp\left(-\int_0^\beta d\tau\,\mathcal{H}(\phi_B+\phi)\right)\left|0\right\rangle, \tag{15}$$

while earlier Z_0, the partition function of the $N=0$ sector in I, is given by

$$Z_0=\left\langle 0\right|\int\mathfrak{D}(\phi)\exp\left(-\int_0^\beta d\tau\,\mathcal{H}(\phi)\right)\left|0\right\rangle. \tag{16}$$

Then the ϕ-field contribution is evaluated within quadratic approximation as before, which should be valid at least in the weak-coupling limit.

Although Eq. (15) is the exact transcription of the DHN approach for finite temperatures, a few remarks will be useful. Since the breather belongs to the $N=0$ sector, we need here a finer classification of the Hilbert space (or the functional space) than that used in I. Here we assume with DHN that each nonlinear classical solution defined a new "sector" in the Hilbert space; we assume that the Hilbert space defined by a breather is different from the one with no breather. This finer partition of the Hilbert space is quite consistent with the separation of independent variables of the sine-Gordon system discussed by Korepin and Faddeev,[8] for example. However, contrary to the sectors discussed in I, there are many local operators which can connect two sectors discussed above. Therefore our basic assumption that the Hilbert space can be partitioned into disconnected parts by means of different classical solutions appears plausible, but its rigorous justification is difficult for the moment.

Then the thermodynamic potential $\Omega_B(w)$ ($\equiv -\beta^{-1}[\ln Z_B(w)-\ln Z_0]$) for the breather is given by

$$\Omega_B(w) = E_B^{\rm cl}(w) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} dk\, \Delta(w,k)\frac{\partial}{\partial k}\lambda(\omega k) + \lambda(mw) - 2\lambda(m) + \tfrac{1}{2}m^2\int dx\{1-\cos[g\phi_B(w)]\}D\,, \quad (17)$$

where

$$\lambda(y) = \beta^{-1}\ln[2\sinh(\tfrac{1}{2}\beta y)]\,,$$

and $\Omega_0(\equiv -\beta^{-1}\ln Z_0)$ has been calculated in I. The third and the fourth terms in Eq. (17) arise from the bound state associated with the breather and two states in the $\phi=0$ vacuum compensating for two bound states. The last term in Eq. (17) is time dependent and has to be replaced by its time average. Then, as in the case of the thermodynamic potential of the soliton, the logarithmic divergence of the second term is exactly canceled by the last term. Equation (17) is further simplified to

$$\Omega_B(w) = \frac{16m}{g'^2}(1-w^2)^{1/2} + 4m(1-w^2)^{1/2}f_0 + 4Tf_2(w) + T\ln(1-e^{-\beta m w})\,, \quad (18)$$

where f_0 has been already defined in Eq. (11), and

$$f_2(w) = -\frac{(1-w^2)^{1/2}m}{\pi}\int_0^\infty dk(\omega_k^2 - m^2w^2)^{-1}\ln(1-e^{-\beta\omega_k})\,, \quad (19)$$

and w hereafter takes only the discrete values w_n given by

$$w_n = \cos(\tfrac{1}{16}ng'^2)\,, \quad (20)$$

which is essentially Eq. (6) with quantum corrections ($g\to g'$).

The breather density [the probability of finding a static breather ($v=0$) with parameter $w=w_n$] is given by

$$n_B(w_n, v=0) = e^{-\beta\Omega_B(\omega_n)}\,. \quad (21)$$

Furthermore, we can extract the breather energy at finite temperatures from Eq. (18) by

$$E_B(w) \equiv \Omega_B(w) - T\frac{d\Omega_B(w)}{dT} = \left(\frac{16}{g'^2}+4f_0\right)m(1-w^2)^{1/2} + mwN(mw)\left(1+\frac{g'^2}{4}T\frac{\partial f_0}{\partial T}\right) - 4T^2\left(\frac{df_2}{dT}\right)\,, \quad (22)$$

where we have discarded the terms of the order of g'^4. At $T=0$ K, where f_0 and f_2 vanish exponentially, Eq. (22) reduces to Eq. (8) the DHN result. At high temperatures the term $\frac{1}{4}g'^2T(\partial f_0/\partial T)$ in Eq. (22) gives rise to a correction of order of T/E_s, and we shall neglect this term hereafter.

B. Moving breather

Now let us consider a breather moving with uniform velocity v. The corresponding classical solution is given by Eq. (3), where now t and x are replaced by $t'=(t+vx)(1-v^2)^{-1/2}$ and $x'=(x+vt)(1-v^2)^{-1/2}$, respectively. Again as in the analysis of a moving soliton in I, it is more convenient to analyze the ϕ-field corrections in the rest frame of the breather. Then we obtain (see Appendix A for details)

$$\Omega_B(w,v) = \left(\frac{16}{g'^2}+4f_0\right)\left(\frac{1-w^2}{1-v^2}\right)^{1/2} + 4Tf_2(w,v) + T\ln\{1-\exp[-\beta m w(1-v^2)^{-1/2}]\}\,, \quad (23)$$

where

$$f_2(w,v) = -\frac{(1-w^2)^{1/2}m}{2\pi}\int_{-\infty}^{\infty} dk(\omega_k^2 - m^2w^2)^{-1} \times \ln[1-\exp(-\beta\omega_k')]$$

and

$$\omega_k' = (\omega_k + v_k)(1-v^2)^{-1/2}\,. \quad (24)$$

The energy of the moving breather is then given by

$$E_B(w,v) = \Omega_B(w,v) - T\frac{d\Omega_B(w,v)}{dT}\,. \quad (25)$$

III. BREATHERS AT FINITE TEMPERATURES

A. Breather energy, inertial mass, and fugacity

The breather energy at finite temperatures is given by Eq. (22). As already mentioned, in the low-temperature region ($T<m$), Eq. (22) reduces to the DHN result [Eq. (8)]. However, in general $E_B(w)$ has a more complicated temperature dependence than, say, the soliton energy discussed in I. In particular, in the high-temperature region ($T\gg m$), we obtain

$$E_B(w_n) = 2(E_s+T)\sin(\tfrac{1}{16}ng'^2) - T + O(\beta m)^2\,, \quad (26)$$

where E_s is the temperature-dependent soliton energy calculated in I;

$$E_s = 8m(g')^{-2}\,. \quad (27)$$

Here we have made use of the asymptotic expansions of f_0 and $f_2(w)$; for $\beta m \ll 1$,

$$f_0(z) = \tfrac{1}{2}z^{-1} + \frac{1}{2\pi}\ln\left(\frac{\gamma z}{4\pi}\right) - \frac{\zeta(3)}{2(2\pi)^3}z^2\,, \quad (28)$$

$$f_2(w) = -\tfrac{1}{2}\ln\{z[1+(1-w^2)^{1/2}]\} + z\left[\left(\frac{1}{4}-\frac{1}{2\pi}\cos^{-1}w\right)w - \frac{1}{2\pi}(1-w^2)^{1/2}\ln\left(\frac{\gamma z}{4\pi}\right)\right]\,, \quad (29)$$

where $z = \beta m$.

Equation (28) has been already derived in I, while the derivation of Eq. (29) is given in Appendix B. In contrast to the soliton energy in I, the breather energy at finite temperatures is no longer a simple extension of the zero-temperature result. In particular, $E_B(w_n)$ appears to become negative at a finite temperature, although this has no significant consequences on the thermodynamics of the breather as $\Omega_B(w_n)$ is always positive.

The energy of a moving breather Eq. (25) becomes

$$E_B(w_n, v) = E_B^0(n)(1-v^2)^{-1/2} + O(e^{-\beta m w_n}, e^{-\beta m}) \quad (30)$$

at low temperatures ($T \ll m$), as is required from the Lorentz invariance. This implies

$$E_B^I(w_n) = E_B^0(n) \quad (31)$$

at low temperatures ($T \ll m$), where $E_B^I(w)$ is the inertial mass of the breather. At high temperatures ($T \gg m$), on the other hand, $E_B(w_n, v)$ is given by

$$\begin{aligned} E_B(w_n, v) &\cong E_B^I(w_n)(1-v^2)^{-1/2} - T \\ &\cong E_B(w_n) + \tfrac{1}{2}v^2 E_B^I(w_n), \end{aligned} \quad (32)$$

where $E_B^I(w_n)$ $[\equiv E_B^I(n)]$ is given by

$$\begin{aligned} E_B^I(w_n) &= E_B(w_n) + T \\ &= 2(E_s + T)\sin(\tfrac{1}{16}ng'^2), \end{aligned} \quad (33)$$

where $E_B(w_n)$ has already been given in Eq. (26). The difference between $E_B^I(n)$ and $E_B(w_n)$ may be interpreted as the inertia of the scattered radiation due to the moving breather.

Finally, the fugacity $\zeta_B(w, v)$ is given by

$$\zeta_B(w, v) = \exp[\beta\mu_B(w, v)] = \exp[S_B(w, v)], \quad (34)$$

where the entropy $S_B(w, v)$ is given by

$$\begin{aligned} S_B(w, v) &= -\frac{d\Omega_B(w, v)}{dT} \\ &= -4f_2(w, v) - 4T\frac{df_2(w, v)}{dT} \\ &\quad + \beta m w(1-v^2)^{-1/2} N[mw(1-v^2)^{-1/2}] \\ &\quad - \ln\{1 - \exp[-\beta m w(1-v^2)^{-1/2}]\}. \end{aligned} \quad (35)$$

In the low-temperature limit ($T < m$), Eq. (35) is proportional to $e^{-\beta m w}$ and we obtain

$$\zeta_B(w, v) = 1. \quad (36)$$

On the other hand, in the high-temperature region ($E_s \gg T \gg m$), we obtain

$$\begin{aligned} S_B(w, v) &\cong -1 + \ln\left(\frac{\beta m}{w}\frac{[1+(1-w^2)^{1/2}(1-v^2)^{1/2}]^2}{(1-v^2)^{1/2}}\right) \\ &\quad + O(\beta m, (\beta E_s)^{-1}) \end{aligned} \quad (37)$$

and

$$\zeta_B(w, v) \cong \frac{m[1+(1-w^2)^{1/2}(1-v^2)^{1/2}]^2}{eTw(1-v^2)^{1/2}} \quad (38)$$

$$\cong \frac{m[1+(1-w^2)^{1/2}]^2}{eTw}\left[1 + \left(\frac{1-(1-w^2)^{1/2}}{1+(1-w^2)^{1/2}}\right)\tfrac{1}{2}v^2 + O(v^4)\right]. \quad (39)$$

In particular, we have

$$\zeta_B(w_n) = \zeta_B(w_n, 0) = \frac{m[1+\sin(\tfrac{1}{16}ng'^2)]^2}{eT\cos(\tfrac{1}{16}ng'^2)} \quad (40)$$

in the high-temperature region ($E_s \gg T \gg m$).

B. Breather density

We have already seen that the probability of finding a breather with velocity $v = 0$ and $w = w_n$ is given by

$$n_B(w_n, 0) = \exp[-\beta\Omega_B(w_n)] \quad (41)$$

for $\beta\Omega_B(w_n) \gg 1$. Similarly, the probability of finding a breather with velocity v is given by

$$n_B(w_n, v) = \exp[-\beta\Omega_B(w_n, v)], \quad (42)$$

where $\Omega_B(w_n, v)$ has been given in Eq. (23). The total probability of finding one breather with $w = w_n$ (i.e., the breather density) is then given by

$$\bar{n}_B(w_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, n_B(w_n, v), \quad (43)$$

where the momentum p is determined by

$$p = \int_0^v v\, dE_B(w, v). \quad (44)$$

In evaluating Eq. (43), it is convenient to consider three temperature regions separately. First, at low temperatures ($T \ll m$), we obtain

$$\bar{n}_B(w_n) = \left(\frac{E_B^0(n)}{2\pi\beta}\right)^{1/2}\exp[-\beta E_B^0(n)], \quad (45)$$

where $E_B^0(n)$ is defined in Eq. (8). Here the breather behaves like an elementary Boltzmann particle. This disagrees with the expression derived by Trullinger,[9] previously based on a heuristic ideal-gas model for solitons and breathers of the sine-Gordon system. The origin of the discrepancy is that he has neglected the corrections due to the radiation scattered by the breather, which we have included in the present theory.

Second, in the intermediate-temperature region $[E_B(w_n) > T \gg m]$, we have

$$\begin{aligned} n_B(w_n) &\cong \frac{m}{Tw_n}[1+(1-w_n^2)^{1/2}]^2\left(\frac{E_B^I(n)}{2\pi\beta}\right)^{1/2} \\ &\quad \times\left[1 + \frac{1}{2}\left(\frac{1-(1-w_n^2)^{1/2}}{1+(1-w_n^2)^{1/2}}\right)\frac{T}{E_B^I(n)}\right] \\ &\quad \times \exp[-\beta E_B^I(n)]. \end{aligned} \quad (46)$$

In this temperature region only the breathers with small velocity (i.e., $|v| \ll 1$) are thermally excited. Finally, for low-lying breathers, the region $E_s > T \gg E_B(w_n)$ may be easily accessible. In this temperature region the thermally excited breathers are almost relativistic ($v \simeq 1$), and we have to take into account all the singular terms in $\Omega_B(w_n, v)$, which give rise to diverging contributions in the limit $v = 1$. In this limit we have

$$\bar{n}_B(w_n) = \frac{mE_B^I(n)}{2\pi T w_n}\{K_2[\beta E_B^I(n)] + 4(1-w_n^2)^{1/2}K_1[\beta E_B^I(n)] + (3-2w_n^2)K_0[\beta E_B^I(n)]\}$$

$$\simeq \frac{mT}{\pi w_n E_B^I(n)}\left[1 + 0\left(\frac{E_B^I(n)}{T}\right)\right], \quad (47)$$

where $K_n(z)$ is the modified Bessel function. Here we have made use of the relation

$$p = E_B^I(n) v(1-v^2)^{-1/2}. \quad (48)$$

At first sight Eq. (47) appears to predict ever-increasing $\bar{n}_B(w_n)$ with temperature. However, this tendency is only true for T, $E_s \gg T \gg E_B^I(n)$. As the temperature approaches the $T_{cr} \simeq e^{-1}E_s^0$ discussed in I, the ratio $T/E_B^I(n)$ becomes

$$T/E_B^I(n) \simeq T[2E_s^0 \sin(\tfrac{1}{16}ng'^2)]^{-1} \simeq [2e\sin(\tfrac{1}{16}ng'^2)]^{-1}. \quad (49)$$

Therefore in such high temperatures $\bar{n}_B(w_n)$ is saturated.

From Eq. (47) we can calculate the total breather density in the high-temperature region ($T_{cr} > T \gg m$):

$$\bar{n}_B = \sum_n \bar{n}_B(w_n). \quad (50)$$

If we substitute Eq. (47) into Eq. (50), the summation in n has to be cut off around $E_B^I(n) = T$:

$$n_B = \sum_{n=1}^{n_0}\left(\frac{mT}{\pi w_n E_B^I(n)}\right), \quad (51)$$

with

$$n_0 \cong \frac{16}{g'^2}\sin^{-1}\left(\frac{T}{2E_s^0}\right) \cong \frac{T}{m_0}.$$

Since the most important contributions come from the breather modes with small n, we can approximate $\sin(\frac{1}{16}ng'^2)$ by $(\frac{1}{16}ng'^2)$ for small g'^2 (i.e., $0 < g'^2 \ll 1$), and we obtain

$$\bar{n}_B = \frac{T}{\pi}\left(\frac{m}{m_0}\right)\ln\left(\frac{\gamma T}{m_0}\right) \quad (52)$$

for small g'^2. The total breather density in the small g'^2 limit is comparable to the radiation density given by

$$\bar{n}_{\rm rad} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, N(\omega_k) \cong \frac{T}{\pi}\ln\left(\frac{\gamma T}{m}\right). \quad (53)$$

Therefore, we conclude that the breathers are as important as the radiations in the thermodynamics of the sine-Gordon system.

The above conclusion confirms the heuristic ideal-gap model for the sine-Gordon system,[6] but contradicts the result of the transfer matrix technique (TMT).[5] The origin of this discrepancy may be ascribed to the deficiency of either the TMT or the quantum-statistical mechanics used in this paper. In TMT the Hamiltonian is first separated into two parts: the one with $\pi(x)$ and the other without $\pi(x)$. In an ordinary system this decoupling is equivalent to the classical approximation; the quantum corrections arising from the noncommutability of $\pi(x)$ and $\phi(x)$ are neglected. However, in the case of the sine-Gordon system with the infinite conservation laws, it is possible that this decoupling of the Hamiltonian into two parts may destroy these conservation laws even at the classical level. In this case the degree of freedom associated with the breathers, whose existence is guaranteed by these conservation laws, is likely dropped from the outset.

On the other hand, at the present moment we cannot exclude another possibility, i.e., that the fault lies in our present approach. In particular, it has been shown previously[4] that the radiation becomes the $n = 1$ breather, if it is properly renormalized. If this is the case, we are in fact overcounting the degrees of freedom associated with $\phi(x)$; the radiation contribution might have to be subtracted from our Ω.

If the radiation contribution is subtracted from our Ω, we can again recover the TMT result at high temperatures at least in the weak-coupling limit. Indeed, how to define the functional space properly is one of the fundamental problems in the modern field theories.[10]

C. Correlation function

As in the case of the soliton, the breathers contribute to the cos-cos correlation function. In particular, the breathers give rise to a series of δ-function contributions in the Fourier transform of the correlation function:

$$S_B(q,\omega) \propto \sum_n \left(1 + n_B\{w_n, q[E_B^I(n)]^{-1}\}\right) g_n^2 \times \delta(\omega - (\{[E_B^I(n)]^2 + q^2\}^{1/2} - T)) \quad (54)$$

and

$$g_n^2 = \left(\frac{2}{m}\frac{(\pi/2m)\omega}{\sin[(\pi/2m)\omega]}\right)^2, \quad (55)$$

where $n_B(w_n, v)$ has been defined in Eq. (42), and g_n^2 is obtained from the matrix element for the soliton scattering by analytical continuation. Details of this procedure will be published elsewhere. The summation over n has to be carried out over all the breather modes. The correlation function Eq. (54) describes the creation of breathers from the ground state. It is therefore of great interest to see if the quasi-one-dimensional ferromagnet $CsNiF_3$ in a strong magnetic field exhibits such fine structures associated with the breather in the neutron scattering experiment.[11,12]

In the classical limit (i.e., $g'^2 \to 0$) the above discrete spectra will degenerate into a broad continuous background, which may resemble that obtained in a molecular-dynamical analysis of the classical sine-Gordon system.[13] However, when the quantum corrections are included the breather peaks appear in a higher-energy region than the radiation peak.

IV. CONCLUDING REMARKS

We have extended the previous analysis of the soliton to the breathers in the sine-Gordon system. The breather energy, the breather inertial mass, and the breather density at finite temperatures are obtained. The quantized breathers behave like a new class of elementary excitations. In the high-temperature limit ($E_s > T \gg m$) and in the weak-coupling limit ($0 < g'^2 \ll 1$), the total breather density becomes comparable to the total radiation density. Therefore, the breather modes give significant contribution to the thermodynamics of the sine-Gordon system in the high-temperature region, which is consistent with the heuristic ideal-gas model but at variance with the TMT result. Thus the present calculation resolves one of the outstanding problems in classical statistical mechanics.

We predict also that the existence of breathers can be most directly seen in the cos-cos correlation function. More details of the dynamic response of the sine-Gordon system will be given in a future publication.

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APPENDIX A: PARTITION FUNCTION FOR MOVING BREATHER

As in I the partition function associated with a moving solution is evaluated in terms of H' and P' in the rest frame of the breather, where

$$H = (H' + vP')(1 - v^2)^{-1/2}, \tag{A1}$$

where v is the velocity of the moving solution. Then the partition function $\Omega_B(w,v)$ ($\equiv -\beta^{-1} \times [\ln Z_B(w,v) - \ln Z_0]$) is given by

$$\Omega_B(w,v) = \frac{E_B^{\rm cl}(w)}{(1-v^2)^{1/2}} - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} dk'\,\Delta(w,k')\frac{\partial}{\partial k'}\lambda(\omega_k) + \lambda[mw(1-v^2)^{-1/2}] - 2\lambda[m(1-v^2)^{-1/2}] + \frac{m^2}{2}(1-v^2)^{-1/2}\int dx'\{1-\cos[g\phi_B(w)]\}D, \tag{A2}$$

where $k' = (k + vw_k)(1 - v^2)^{-1/2}$.

As in Eq. (17) in the text, the last term in (A2) is understood as the time average. We can then evaluate the integral over k' and obtain Eq. (23).

APPENDIX B: ASYMPTOTIC BEHAVIOR OF $f_2(w,v)$

The function of $f_2(w,v)$ given in Eq. (24) is transformed as

$$f_2(w,v) = -\frac{(1-w^2)^{1/2}}{2\pi}\int_{-\infty}^{\infty} d\theta \cosh\theta(\cosh^2\theta - w^2)^{-1} \times \ln\{1 - \exp[-z(\cosh\theta + v\sinh\theta) \times (1-v^2)^{-1/2}]\}, \tag{B1}$$

with $z = \beta m$.

In the high-temperature region ($z \ll 1$), we transform (B1) as

$$f_2(w,v) = -\frac{(1-w^2)^{1/2}}{2\pi}\int_{-\infty}^{\infty} d\theta \cosh\theta(\cosh^2\theta - w^2)^{-1} \times \left[+ \tfrac{1}{2}z(\cosh\theta + v\sinh\theta)(1-v^2)^{-1/2} + \ln[z(\cosh\theta + v\sinh\theta)(1-v^2)^{-1/2}] + \sum_{\nu=1}^{\infty}\ln\left(1 + \frac{z^2(\cosh\theta + v\sinh\theta)^2(1-v^2)^{-1}}{(2\pi\nu)^2}\right)\right], \tag{B2}$$

where we made use of a relation

$$1-e^{-y}=e^{-y/2}y\prod_{\nu=1}^{\infty}\left[1+\left(\frac{y}{2\pi\nu}\right)^2\right]. \tag{B3}$$

Each term in Eq. (26) is integrated easily although it is tedious, and we obtain

$$\begin{aligned} f_2(w,v)=-\frac{(1-w^2)^{1/2}}{2\pi}\Bigg\{&-z(1-v^2)^{1/2}\left[\theta_0+\frac{w}{(1-w^2)^{1/2}}\tan^{-1}\left(\frac{w}{(1-w^2)^{1/2}}\right)\right]\\ &+\frac{\pi}{(1-w^2)^{1/2}}\ln\{z\,[(1-v^2)^{-1/2}+(1-w^2)^{1/2}]\}\\ &+\sum_{\nu=1}^{\nu_0}\frac{1}{(1-w^2)^{1/2}}\left(\int_0^{\pi/2}d\theta\,\ln[X_\nu(\theta)]+2\pi\ln 2\right)\Bigg\}, \end{aligned} \tag{B4}$$

where

$$X_\nu(\theta)=\left\{1+\left(\frac{z}{2\pi\nu}\right)^2-\left[1+\left(\frac{z}{2\pi\nu}\right)^2(v^2+w^2-v^2w^2)\right]\sin^2\theta\right\}^2+4\left(\frac{z}{2\pi\nu}\right)^2 v^2(1-w^2)\sin^2\theta\cos^2\theta \tag{B5}$$

and

$$\nu_0=(z/2\pi)\cosh\theta_0\,. \tag{B6}$$

In the last term of (B4), the cutoff in the integral is transferred to that of the summation as discussed in I. Finally, for $z\ll 1$, (B4) reduces to

$$f_2(w,v)=-\tfrac{1}{2}\ln\{z[(1-w^2)^{1/2}+(1-v^2)^{-1/2}]\}+z(1-v^2)^{1/2}\left[\frac{w}{2\pi}\sin^{-1}w-\frac{1}{2\pi}(1-w^2)^{1/2}\ln\left(\frac{\gamma z}{4\pi}\right)\right]+O(z^2), \tag{B7}$$

where the divergent term ($\sim\theta_0$) in the first term is exactly canceled by the last term. In the limit $v\to 0$, (B7) reduces to Eq. (29).

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