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Generalization of the random-walk process

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A generalization of the random-walk process is given. To this end, two simple models for "memory functions" are proposed. They are a step-dependent memory (model I) and a site-step-dependent memory (model II). Recursion relations representing the processes are converted to generalized Fokker-Planck equations. Specifically, it is to be noted that the basic equation for model II is the same as the nonlinear equation describing turbulence, or solitons. Based on these model processes, modifications due to memory effects on traveling waves are studied. A method of determining the memory function for model II in concise form is shown.

I. INTRODUCTION

Recently, the dynamical macrosystems have been extensively studied by using the stochastic equations, due to the fact that successful phenomenological descriptions have been obtained by this approach. In particular, it is well known that the diffusion equation is easily derived from the random-walk process.[1] In general, however, the basic equation which governs the time evolution of the macro variable $B(x,t)$ at position x and time t is expressed in nonlinear form

$$\frac{\partial B(x,t)}{\partial t}=f\left(B,\frac{\partial B}{\partial x},t\right) . \tag{1.1}$$

To derive Eq. (1.1) from the stochastic point of view, some memory effects must be included, as shown later. The problem of non-Markov processes is very difficult and requires specific considerations.

In this paper, to clarify the memory effects in the stochastic equation, two simple models for "memory functions" are utilized and the random walks are extended in more general form; generalized random walks (GRW).[2,3] The GRW's are different from the generalized random walks as given by Montroll *et al.*[4] The present two models for memory functions are a step-dependent memory function (model I) and a site-step-dependent memory function (model II). The basic equation for model II is the same as the nonlinear equation, the Burgers equation[5] describing a freely decaying, homogeneous turbulence. This fact suggests that the non-Markov processes generated by model II are reduced to the random-walk processes after making the Hopf-Cole transformation.[6,7] This result may be readily understood from the fact that the turbulence described by the Burgers equation is, in its broadest sense, a random motion of a continuous medium.[8,9]

In Sec. II, a basic expression which describes a special "walk" process with memory is given. Next, it is shown that a projection of motions in the above processes onto an axis yields "random"-walk processes with memory: generalized random walks (GRW). In Sec. III, two models for jumping probabilities with memory are introduced. Recursion relations for these processes are converted to generalized Fokker-Planck (GFP) equations. In Sec. IV, based on truncated GFP equations for the two model processes, modifications due to memory effects on traveling waves are studied. In Sec. V, a method of determining the memory in model II is shown.

II. SPECIAL WALK PROCESSES

First, consider a stochastic process with no memory, which is characterized by a set of positions x_i and the distances of each step, a_i [see Fig. 1(a)]. In this process, the probability density that the walker starting at x_i arrives at x after $N-1$ steps, irrespective of intermediate positions, $W_{N-1}(x|a_{N-1}, a_{N-2}, \dots, a_1,x_1)$, is expressed by $W_{N-2}(x_{N-1}|a_{N-2}, \cdots ,a_1,x_1)$ and jumping probabilities $P_{a_{N-1}}(x|x_{N-1})$

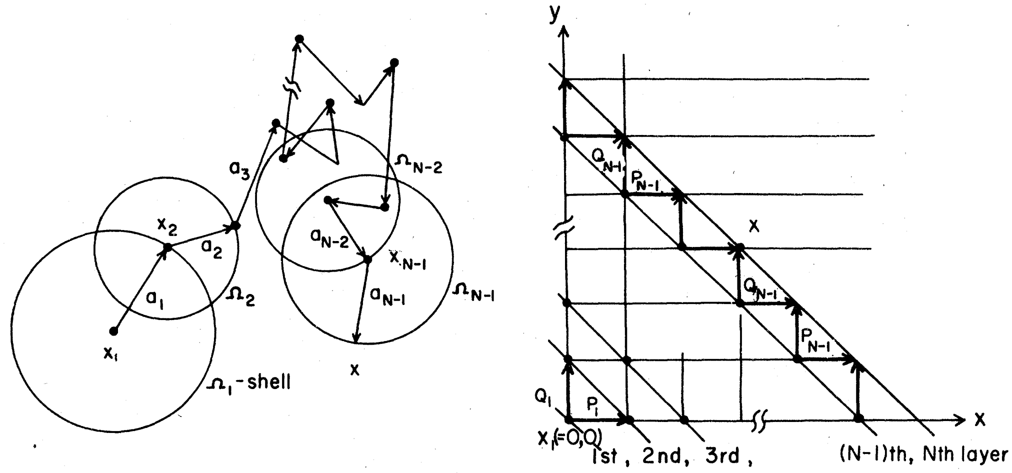
$$W_{N-1}(x|a_{N-1},a_{N-2},\dots,a_1,x_1) = \int_{\Omega_{N-2}} P_{a_{N-1}}(x|x_{N-1}) \times W_{N-2}(x_{N-1}|a_{N-2},\dots,a_1,x_1)\, dx_{N-1} , \tag{2.1a}$$

$$W_1(x|a_1,x_1)=P_{a_1}(x|x_1)\, W_0(x_1) , \tag{2.1b}$$

$$\int P_{a_{N-1}}(x|x_{N-1})\, dx=1 , \tag{2.1c}$$

where $W_0(x_1)=1$ and x_{N-1} is a position on the shell Ω_{N-2}.

For a special case in which positions x and x_i are



(a) (b)

FIG. 1. (a) Stochastic processes specified by Eqs. (2.1a), (2.1b), and (2.1c). (b) Stochastic processes represented by Eqs. (2.5) and (2.6). Positions of sites on $(N-1)$th layer, x_{N-1}, are expressed by $(k-1,N-1-k)$, where k is integer, $1 \leqslant k \leqslant N-1$.

restricted to a set of lattice points in the first quadrant [see Fig. 1(b)], the set of jumping probabilities $P_{a_{N-1}}(x\,|\,x_{N-1})$ is reduced to a pair of jumping probabilities P_{N-1} and Q_{N-1}

$$P_a((x,y)\,|\,k-1,N-1-k)) = P_{N-1}\delta_{x,1+(k-1)} + Q_{N-1}\delta_{y,1+(N-1-k)} , \quad (2.2)$$

where k is integer $(1 \leqslant k \leqslant N-1)$,

$$(x,y) = x, \quad (k-1,N-1-k) = x_{N-1} \quad (2.3)$$

and the distance of each step is a. The sum of P_{N-1} and Q_{N-1} is normalized as

$$P_{N-1} + Q_{N-1} = 1 . \quad (2.4)$$

The recursion formula for this process is written

$$W_N((x,y)\,|\,a,a,\ldots,a,(0,0)) = \sum_{k=1}^{N-1} P_a((x,y)\,|\,(k-1,N-1-k))\, W_{N-1}((k-1,N-1-k)\,|\,a,a,\ldots,a,(0,0)) , \quad (2.5)$$

$$W_2((x,y)\,|\,a,(0,0)) = P_a((x,y)\,|\,(0,0))\, W_1((0,0)) = P_1\delta_{x,1} + Q_1\delta_{y,1} . \quad (2.6)$$

Here it is noted that the suffixes of the layers are changed such that on the Nth layer there are N sites.

A method of including memory effects in the process under consideration is a rather difficult problem. For some processes, however, the memory effects may be expressed by replacing P_{N-1} and Q_{N-1} by $\tilde{P}_{N-1}$ and $\tilde{Q}_{N-1}$, respectively, which are related to a set of jumping probabilities at the previous step as follows:

$$\tilde{P}_{N-1} = f(\tilde{P}_{N-2},\tilde{Q}_{N-2}), \quad \tilde{Q}_{N-1} = g(\tilde{P}_{N-2},\tilde{Q}_{N-2}) , \quad (2.7)$$

where functions f and g specify the memory. The sum of $\tilde{P}_N$ and $\tilde{Q}_N$ is normalized as

$$\tilde{P}_N + \tilde{Q}_N = 1 \quad (2.8)$$

for each N steps.

Substituting relation (2.3), with P_{N-1} and Q_{N-1} replaced by $\tilde{P}_{N-1}$ and $\tilde{Q}_{N-1}$, respectively, into Eq. (2.5) gives

$$W_N((x,y)\,|\,a,a,\ldots,a,(0,0)) = \tilde{P}_{N-1}{}'\, W_{N-1}((x-1,y)\,|\,a,a,\ldots,a,(0,0)) + \tilde{Q}_{N-1}{}'\, W_{N-1}((x,y-1)\,|\,a,a,\ldots,a,(0,0)) , \quad (2.9)$$

where

$$\tilde{P}_{N-1}{}' \equiv \tilde{P}_{N-1}(1-\delta_{x,0}), \quad \tilde{Q}_{N-1}{}' \equiv \tilde{Q}_{N-1}(1-\delta_{y,0}) , \quad (2.10)$$

and $W_1(0,0) = 1$ and

$$W_N((x,y)\,|\,a,a,\ldots,a,(0,0)) = 0$$

when (x,y) is not on the Nth layer.

As is seen in Fig. 1(b), the processes represented by Eqs. (2.9) and (2.10) may be understood as a kind of "self-avoiding" walk. Moreover, it is noted that the walker has the memory specified by Eqs. (2.7) and (2.8). For this reason, this process will be called a special walk process.

III. MEMORY-FUNCTION MODELS

In Sec. II, a special process with "memory function" was derived [see Eqs. (2.7), (2.9), and (2.10)]. Here, by projecting the motions generated by Eq. (2.9) onto a single axis, the GRW axis as shown in Fig. 2, they can be reduced to a generalized-random-walk (GRW) process in the one-dimensional case. Similarly, two models for memory functions are introduced in the GRW.

In order to represent positions projected onto the GRW axis, a different notation is required as follows:

$$m=(x,y),\quad m+1=(x,y-1),\quad m-1=(x-1,y) \quad (3.1)$$

(see Fig. 2). Moreover, it is assumed that $\tilde{P}_{N-1}$ and $\tilde{Q}_{N-1}$ depend on the positions as well as on the steps: $\tilde{P}_{N-1}(m\,|\,m-1)$ and $\tilde{Q}_{N-1}(m\,|\,m+1)$.

Then the recursion formula (2.9) can be rewritten in the form

$$W(m,N)=\tilde{P}_{N-1}(m\,|\,m-1)\,W(m-1,N-1)+\tilde{Q}_{N-1}(m\,|\,m+1)\,W(m+1,N-1)\ , \quad (3.2)$$

where, to avoid the complexity of a subscript on W, an abbreviated notation is used, as follows:

$$W(m,N)\ [=W((x,y),N)] = W_N((x,y)\,|\,a,a,\ldots,a,(0,0))\ , \quad (3.3)$$

and primes on $\tilde{P}$ and $\tilde{Q}$ have been omitted. The expression (3.2) has a form identical to the random-walks processes, except for the memory effects denoted by $\tilde{P}_{N-1}(m\,|\,m-1)$ and $\tilde{Q}_{N-1}(m\,|\,m+1)$. Similarly, the relation (2.7) is expressed by

$$\tilde{P}_{N-1}(m\,|\,m-1)=f(\tilde{P}_{N-2}(m-1\,|\,m-2),\ \tilde{Q}_{N-2}(m-1\,|\,m))\ ,$$
$$\tilde{Q}_{N-1}(m\,|\,m+1)=g(\tilde{P}_{N-2}(m+1\,|\,m),\ \tilde{Q}_{N-2}(m+1\,|\,m+2))\ , \quad (3.4)$$

with

$$\tilde{P}_{N-1}(m+1\,|\,m)+\tilde{Q}_{N-1}(m-1\,|\,m)=1\ . \quad (3.5)$$

As simple models for memory function, consider the following two cases:

Model I (site-dependent case)

$$\tilde{P}_{N-1}=\tilde{P}_{N-2}(1-\tilde{Q}_{N-2}M_{N-2})\ ,$$
$$\tilde{Q}_{N-1}=\tilde{Q}_{N-2}(1+\tilde{P}_{N-2}M_{N-2})\ , \quad (3.6)$$

for $N\geqslant 3$. Both first terms in Eqs. (3.6) represent the constant part of each jumping probability and the second terms express the memory specified by M_{N-2}, which depends on the step.

Model II (step-site-dependent case)

$$\tilde{P}_{N-1}(m\,|\,m-1)=\tfrac{1}{2}[1+bW(m-1,N-2)]\ ,$$
$$\tilde{Q}_{N-1}(m\,|\,m+1)=\tfrac{1}{2}[1-bW(m+1,N-2)]\ . \quad (3.7)$$

The second terms in Eqs. (3.7) represent the memory specified by a parameter b. Namely, it is assumed that the memory in model II depends on the probability density at the previous step, $W(m+1,N-2)$.

To convert Eq. (3.2) using Eq. (3.4), that is Eq. (3.6) or Eq. (3.7), into continuous form, introduce a set of continuous variables z and t and put

$$z\,(=x-y)=am,\quad t=N/\nu \quad (3.8)$$

(a: unit step; $1/\nu$: unit time) when x and y are on the Nth layer. Expanding the functions concerned with these variables around $1/\nu=0$ and $a=0$, rewrite Eq. (3.2) in the differential form

$$\frac{\partial w}{\partial t}=a\nu\frac{\partial}{\partial z}(\tilde{q}-\tilde{p})w+\frac{a^2\nu}{2}\frac{\partial^2}{\partial z^2}(\tilde{p}+\tilde{q})w \quad (3.9)$$

($a^2\nu$ finite) where $\tilde{p}$, $\tilde{q}$, and w represent continuous functions which correspond to $\tilde{P}$, $\tilde{Q}$, and W, respectively.

Similarily, Eq. (3.6) can be rearranged to give[10]

$$\frac{\partial\tilde{p}(t)}{\partial t}=-\tilde{p}(t)\tilde{q}(t)m(t)\ , \quad (3.10a)$$

$$\frac{\partial\tilde{q}(t)}{\partial t}=\tilde{q}(t)\tilde{p}(t)m(t)\ , \quad (3.10b)$$

with

$$\tilde{p}(t)+\tilde{q}(t)=1\ . \quad (3.11)$$

The condition (3.11) is derived from the normaliza-

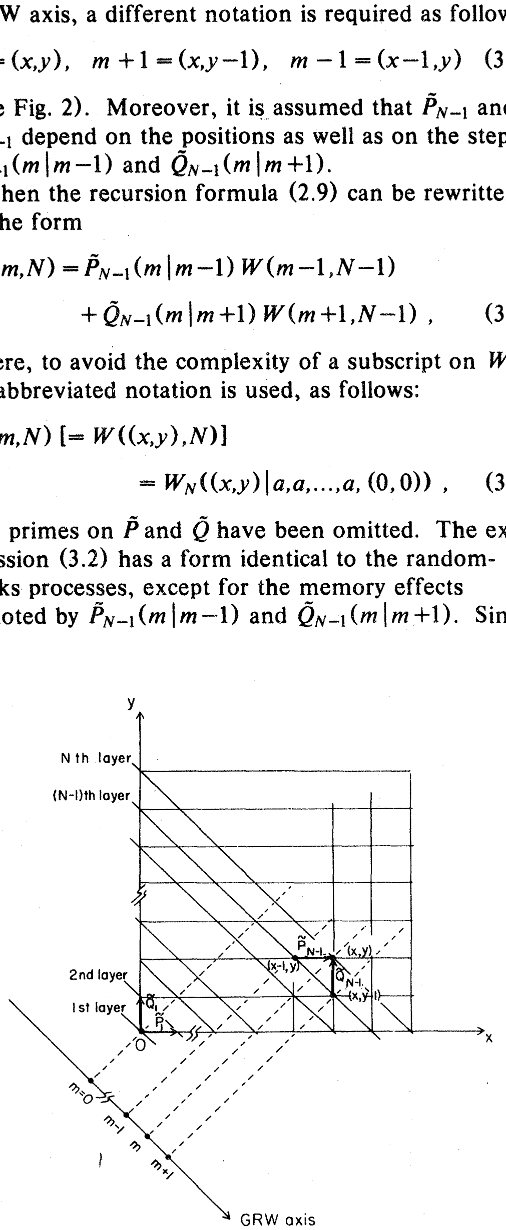


FIG. 2. Relation between a special walk process specified by Eqs. (2.7), (2.9), and (2.10) and a generalized-random-walk process given by Eqs. (3.2), (3.3), and (3.4).

tion of Eq. (2.8). For model II, Eq. (3.7) is rewritten in the form

$$\tilde{p}(z,t:w)=\tfrac{1}{2}[1+bw(z,t)]\ , \qquad (3.12a)$$

$$\tilde{q}(z,t:w)=\tfrac{1}{2}[1-bw(z,t)]\ . \qquad (3.12b)$$

Consequently the basic equation for model II is given by

$$\frac{\partial w}{\partial t}+C_1 w\frac{\partial w}{\partial z}-D_0\frac{\partial^2 w}{\partial z^2}=0\ , \qquad (3.13)$$

where

$$C_1=2a\nu b,\quad D_0=a^2\nu/2\ . \qquad (3.14)$$

This equation is a nonlinear equation defined by Eq. (1.1) and it is the same as the so-called Burgers equation.[5]

IV. MEMORY EFFECTS ON TRAVELING WAVES

In this section, a study of memory effects on traveling waves is made. Two simple models for memory functions are used to determine their effects on the traveling waves which are obtained by truncating the differential equation (3.9). Namely, for simplicity, it is assumed that $w(z,t)$ is a slowly varying function such that $\partial^2 W/\partial z^2$ can be neglected.

A. Time-dependent memory function (model I)

First consider the case in which the memory effects on the process are expressed by Eq. (3.10a). Under the initial conditions $\tilde{p}(t=0)=p_1$ and $\tilde{q}(t=0)=q_1$, the solution is

$$\tilde{p}(t)=\frac{M'}{1+M'}\ , \qquad (4.1)$$

$$M'(t)=C_0\exp\left(-\int_0^t m(t')\,dt'\right)\ ,$$

$$C_0\equiv p_1/(1-p_1)\ , \qquad (4.2)$$

and the memory effect is expressed as

$$\tilde{p}(t)-\tilde{q}(t)=\frac{M'-1}{M'+1}=U(t)\ . \qquad (4.3)$$

In Fig. 3(a), the behavior of $\tilde{p}(t)$ is shown by taking a suitable form; $m(t)=m_0\,(1-e^{-bt})$, where m_0 and b are parameters.[11] Substituting Eq. (4.3) into Eq. (3.9) gives

$$\frac{\partial w}{\partial t}+a\nu U(t)\frac{\partial w}{\partial z}=0\ , \qquad (4.4)$$

where the term $\partial^2 w/\partial z^2$ has been neglected. It is well known that the solution to Eq. (4.4) denotes a traveling wave with a phase velocity $a\nu U(t)$. The

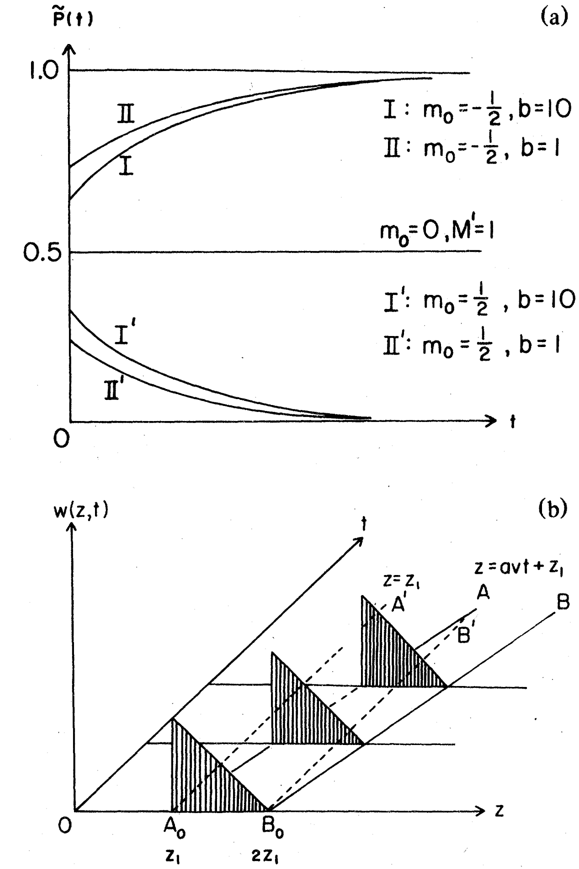


FIG. 3. (a) Here put $p_1=\frac{1}{2}$, $m(t')=m_0(1-e^{-bt'})$, where m_0 is a "residual memory" and b is a parameter which specifies the memory effect. (b) As an initial condition for w, put $w(z,0)=0$ $(0\leq z<z_1)$, $2-z/z_1$ $(z_1\leq z<2z_1)$, 0 $(z\geq 2z_1)$. A profile of $w(z,t)$ is preserved along a characteristic curve $z-a\nu\int_0^t U(t')\,dt'=z_0$, where z_0 belongs to the interval $[z_1,2z_1]$. When $M'=1$, a set of characteristic curves is located in the strip $A_0B_0A'B'$. A strip A_0B_0AB denotes a memory effect in which M' is a very large constant, that is for $m_0=-\frac{1}{2}$ and $t\to\infty$.

memory effect on the traveling wave of Eq. (4.4) is shown graphically in Fig. 3(b). The memory effect is related to changing the trajectory of $w(z,t)$.

B. Space (site) and time (step) dependent memory function (model II)

Here the memory effect is expressed by

$$\tilde{p}(z,t,w)-\tilde{q}(z,t;w)=bw(z,t)+O\left(\frac{1}{\nu}\frac{\partial w}{\partial z},a\frac{\partial w}{\partial z}\right)\ , \qquad (4.5)$$

and the basic equation is given by Eq. (3.13). The

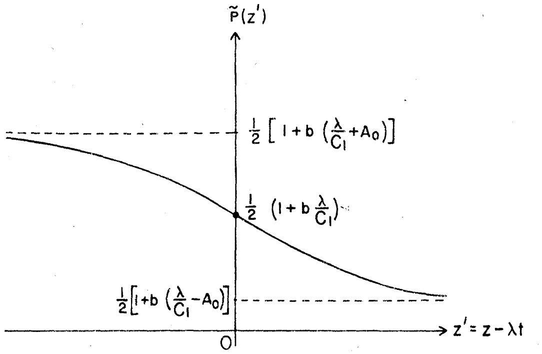


FIG. 4. Behavior of the jumping probability density $\tilde{p}(z')$ in non-Markov processes.

nonlinear equation (3.13) is solved for a steady state[12] as

$$w(z')-\frac{\lambda}{C_1}=A_0\tanh\left[-\frac{C_1A_0}{2D_0}z'\right], \tag{4.6}$$

with

$$z'=z-\lambda t,\quad A_0=\left[\frac{\lambda}{C_1}-w(z')\right]_{z'\to\infty}, \tag{4.7}$$

where λ is constant and $C_1w(z'=0)-\lambda=0$. Replacing w in Eq. (4.6) by a velocity field $u(z,t)$, the steady-state solution shows a "shock wave".[13,14] By substituting Eq. (4.6) into Eq. (3.12a), the behavior of $\tilde{p}(z,t)$ or $\tilde{q}(z,t)$ can be plotted. Figure 4 shows the behavior of the jumping probability density $\tilde{p}(z')[=1-\tilde{q}(z')]$. To study the memory effects explicity, truncate Eq. (3.13) as follows:

$$\frac{\partial w}{\partial t}+C_1w\frac{\partial w}{\partial z}=0\ . \tag{4.8}$$

Note that Eq. (4.8) is the same as the basic equation

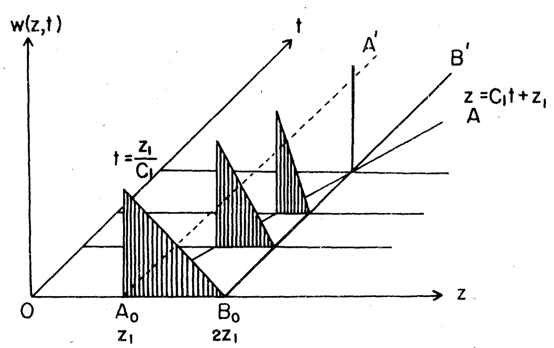


FIG. 5. For the sake of simplicity, assume that $p_1=q_1$ in the Markov process and regard a trajectory deviation from A_0A' to A_0A to be a memory effect due to the non-Markov process.

which describes a "compressive" or "divergent" wave, according to its boundary condition.[14] For the purpose of comparing the solution of Eq. (4.8) with the solution of Eq. (4.4), choose the same initial condition for w

$$w(z,0)=\begin{cases}0 & (0\le z<z_1)\\ 2-\dfrac{z}{z_1} & (z_1\le z<2z_1)\ .\\ 0 & (z\ge 2z_1)\ .\end{cases} \tag{4.9}$$

The behavior of this solution is shown in Fig. 5. In this model, the memory effect is related to changing a profile of $w(z,t)$.

V. HOPF-COLE TRANSFORMATION OF MEMORY FUNCTION

In Sec. IV, the memory effect has been investigated by using two simple models, characterized by Eqs. (4.4) and (4.8), respectively. For model II, using the Hopf-Cole transformation[6,7], one can obtain the memory function in a more concise form.

From Eq. (3.13) and model II as defined by Eq. (3.12a), a basic equation for $m(z,t)[=p(z,t)-q(z,t)]$ is then given by

$$\frac{\partial m}{\partial t}+C_2m\frac{\partial m}{\partial z}-D_0\frac{\partial^2 m}{\partial z^2}=0\ , \tag{5.1}$$

where $C_2=2a\nu=C_1/b$.

Now define a function $M(z,t)$ generated by $m(z,t)$

$$M(z,t)=\exp\left[-\frac{1}{2D_0^1}\int_0^z m(z',t)\,dz'\right],\quad \left[D_0^1=\frac{D_0}{C_2}\right] \tag{5.2}$$

[cf. Eq. (4.2)]. By substituting Eq. (5.2) into Eq. (5.1), Eq. (5.1) can be rewritten as the diffusion equation

$$\frac{\partial M(z,t)}{\partial t}=D_0\frac{\partial^2 M(z,t)}{\partial z^2}\ , \tag{5.3}$$

and hence the solution of Eq. (5.1) is expressed as[15]

$$m(z,t)=-\frac{2D_0^1}{M}\frac{\partial M}{\partial z}=\frac{\displaystyle\int_{-\infty}^{\infty}\exp\left[-\frac{(z-y)^2}{4D_0t}\right]M(y,0)\,m(y,0)\,dy}{\displaystyle\int_{-\infty}^{\infty}\exp\left[-\frac{(z-y)^2}{4D_0t}\right]M(y,0)\,dy} \tag{5.4}$$

Note that Eq. (5.3) can be reinterpreted by introducing a suitable transition probability $\rho(y,t\,|\,y-\Delta y,t-\Delta t)$ between the two quantities

$M(y,t)$ and $M(y-\Delta y, t-\Delta t)$

$$M(y,t)=\int \rho(y,t\,|\,y-\Delta y,t-\Delta t)\times M(y-\Delta y,t-\Delta t)\,d(\Delta y)\ , \qquad (5.5)$$

where

$$\rho(y,t\,|\,y-\Delta y,t-\Delta t)=\frac{1}{(4\pi D_0\Delta t)^{1/2}}\exp\left[-\frac{(\Delta y)^2}{4D_0\Delta t}\right]\ . \qquad (5.6)$$

The memory specified by Eq. (5.1) is determined in terms of the initial values $m(y,0)$ and $M(y,0)$ and the transition probability $\rho(z,t\,|\,y,0)$.

Finally, with the aid of the relations (5.2) and (5.4), $m(z,t)$ is obtained as follows:

$$m(z,t)=\frac{\int_{-\infty}^{\infty}\frac{z-y}{C_2t}\exp\left[-\frac{1}{2D_0^1}F(z,y,t)\right]dy}{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2D_0^1}F(z,y,t)\right]dy}\ , \qquad (5.7)$$

where

$$F(z,y,t)=\frac{(z-y)^2}{2tC_2}+\int_0^y m(y',0)\,dy'\ . \qquad (5.8)$$

This form suggests that in the present treatment the memory $m(z,t)$ may be interpreted as an averaged quantity for $(z-y)/C_2t$ in a "memory field" specified by $\exp[-F(y,z,t)2D_0^1]$.

VI. CONCLUDING REMARKS

In this paper, it has been shown what happens when memory is included in generalized random walks (GRW). The basic equation which governs the GRW's is obtained by taking the continuous limit in the recursion formula for the GRW's. To clarify the memory effects in the GRW, two simplified models were used.

The most important assumption of the present arguments is Eq. (2.7) [or Eq. (3.4)]. By allowing this assumption, the equation which specifies the memory of model II, $m(z,t)[=\tilde{p}(z,t)-\tilde{q}(z,t)]$, is reduced to Eq. (3.13) or Eq. (5.1); i.e., the Burgers equation. Next, it is noted that by considering Eqs. (3.13) or (5.1) as a continuity equation for w, a nonlinear flux J can be expressed as follows:

$$J(w)=a\nu bw^2-D_0\frac{\partial w}{\partial z}\qquad (D_0=\tfrac{1}{2}a^2\nu)\ . \qquad (6.1)$$

This paper has been confined to the case in which D_0 is constant (from the condition $p+q=1$), but extending the present procedure to the more general case in which the jumping probabilities p,q, and r appear, D_0 can be modified to the form $D(z)$ or $D(w)$, as shown in Ref. 3. Moreover, in the present paper, terms up to the second derivatives with respect to x have been included in the expansion of the recursion formula for the GRW's [e.g., Eq. (3.13)]. Including higher-order terms gives a considerably more complicated nonlinear equation. i.e.,

$$\frac{\partial w}{\partial t}=-2a\nu bw\frac{\partial w}{\partial z}+\left[\frac{a^2\nu}{2}-a^3\nu b\left(\frac{\partial w}{\partial z}\right)\right]\frac{\partial^2 w}{\partial z^2}-\frac{a^3\nu b}{3}w\frac{\partial^3 w}{\partial z^3}\ , \qquad (6.2)$$

with the corresponding flux expressed as

$$J(w)=a\nu bw^2-\frac{\partial}{\partial z}(Dw)+\frac{a^3\nu}{3!}\frac{\partial^2}{\partial z^2}(w^2)\ , \qquad (6.3)$$

where $D=\frac{1}{2}[a^2\nu(p+q)]$. Further studies related to the works in Refs. 16-19 would be interesting.

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[12]Here it is assumed that $w(z,t) = w(z - \lambda t)$, where λ is constant. Then Eq. (3.13) is reduced to an ordinary differential equation with respect to $z' = z - \lambda t$. The steady-state solution is the solution of the reduced differential equation.

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[15]To derive Eq. (5.4), use a representation for the solution of Eq. (5.3): i.e.,

$$M(z,t) = (4\pi D_0 t)^{-1/2} \int_{-\infty}^{\infty} \exp\left[-\frac{(z-y)^2}{4D_0 t}\right] M(y,0)\, dy$$

[see Eq. (5.5)]. Then from the relation (5.2) and the identity

$$\frac{\partial}{\partial z} \exp\left[-\frac{(z-y)^2}{4D_0 t}\right] = -\frac{\partial}{\partial y} \exp\left[-\frac{(z-y)^2}{4D_0 t}\right],$$

the expression (5.4) is readily obtained after an integration by parts.

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