Instability in the Potts model

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It is shown that for $p > 2$ the transverse susceptibility of a p-dimensional Potts model with uniaxial order is negative, indicating that the ordered state is unstable, even though a stable critical fixed point is obtained for all $p < \frac{10}{3}$.

INTRODUCTION

Recent renormalization-group calculations[1,2] as applied to the Edwards-Anderson theory[3] of spin-glasses led to a very unusual result. Starting from the disordered phase an accessible stable critical fixed point was obtained,[1] which usually indicates a second-order phase transition. However, in the ordered phase,[2] the state described by the assumed order parameter was found to be unstable. It is the purpose of this note to point out that a similar instability is present in the Potts model although only in a limited range of p values.

Renormalization-group calculations have been carried out for the disordered phase of the Potts model by Harris *et al.*[4] and by Priest and Lubensky[5] in $d = 6 - \epsilon$ dimensions. The fundamental variables are p-dimensional diagonal traceless tensors. A stable fixed point was obtained (to lowest order in ϵ) for $p < \frac{10}{3}$. In this paper these calculations are extended to the ordered phase. An instability is obtained for $p > 2$. As in the spin-glass case[2] the instability manifests itself by a negative value of a susceptibility, χ_T, which by definition is positive. Thus in the range $2 < p < \frac{10}{3}$ one finds a stable fixed point starting from the disordered phase and an instability in the ordered phase. For $p < 2$ there is no instability. This includes the physically interesting case $p = 1$ which describes the percolation transition. In the spin-glass case the instability is obtained for all values of the spin components m except in the limit $m \rightarrow \infty$.

The lack of a stable fixed point for $p > \frac{10}{3}$ is believed due to the fact that the transition in this case is first order.

I. DISORDERED PHASE

Following Priest and Lubensky[5] the effective reduced Hamiltonian is written

$$\mathcal{H} = -\frac{1}{4} \int (r + k^2) \sum Q_{ij}(k) Q_{ij}(-k) + \omega \int \sum Q_{ij}(k) Q_{jk}(k') Q_{kl}(-k-k') \ , \tag{1}$$

where Q_{ij} are symmetric traceless tensors and in the case of the Potts model also diagonal. The corresponding propagator has the form

$$\langle Q_{ii}(k) Q_{jj}(-k) \rangle = \frac{2}{r + k^2} \left(\delta_{ij} - \frac{1}{p} \right) \ . \tag{2}$$

The tensor components Q_{ii} are related to the components A_α of the p-state Potts model,

$$\mathcal{H} = -J \sum_{\langle xx' \rangle} \vec{A}(x) \cdot \vec{A}(x') \ , \tag{3}$$

by

$$Q_{ii} = \sum_{\alpha=1}^{p} A_\alpha a_{ii}^\alpha \ , \tag{4}$$

where

$$a_{ii}^\alpha = \left(\frac{p-\alpha}{p-\alpha+1} \right)^{1/2} \times \begin{cases} 0, & \text{if } i < \alpha \\ 1, & \text{if } i = \alpha \\ -1/(p-\alpha), & \text{if } i > \alpha \ . \end{cases} \tag{5}$$

When written in differential form the recursion relations are given by

$$\frac{dr}{dl}(l) = [2 - \eta(l)] r(l) - 144 K_6 \omega^2(l) \left(1 - \frac{2}{p} \right) \frac{1}{[1 + r(l)]^2} \ , \tag{6}$$

$$\frac{d\omega}{dl}(l) = (\tfrac{1}{2}\epsilon - \tfrac{3}{2}\eta(l)) \omega(l) + 288 K_6 \left(1 - \frac{3}{p} \right) \frac{\omega^3(l)}{[1 + r(l)]^3} \ ,$$

where $\epsilon = 6 - d$ and

$$\eta(l) = 48 K_6 \left(1 - \frac{2}{p} \right) \omega^2(l) \ . \tag{7}$$

These may be integrated up in the usual way[2,6] to give

$$r(l) = t(l) + 72 K_6 \omega^2(l) \left(1 - \frac{2}{p} \right) \times \left(1 - 2t(l) \ln[1 + t(l)] - \frac{t^2(l)}{1 + t(l)} \right) \ , \tag{8}$$

$$\omega^2(l) = \omega^2 e^{\epsilon l} / W(l) \ , \tag{9}$$

where

$$t(l) = te^{2l}W(l)^{-(5/3)(2-p)/(10-3p)} , \quad (10)$$

$$W(l) = 1 + 144K_6\left(\frac{10}{p} - 3\right)\frac{\omega^2}{\epsilon}(e^{\epsilon l} - 1) , \quad (11)$$

and

$$t = t(l=0), \quad \omega(l=0) .$$

The critical behavior is determined by relating properties in the critical region to those far from criticality be means of the recursion relations. The free energy is given by[2,6]

$$F = \tfrac{1}{2}K_6(p-1)\int_0^{l^*} \ln[1 + r(l')]e^{-dl'}\,dl'$$

$$\sim \frac{t^3}{12^3\omega^2}(p-1)[W(l^*)^{2p/(10-3p)} - 1] , \quad (12)$$

where l^* is determined by setting $t(l^*) = 1$. Similarly the susceptibility is given by

$$\chi^{-1} = \exp\left[-2l^* + \int_0^{l^*} \eta(l')\,dl'\right]\chi^{-1}(l^*) \quad (13)$$

$$\sim tW(l^*)^{-2(2-p)/(10-3p)} . \quad (14)$$

Equations (7), (12), and (14) yield the exponents

$$\alpha = -1 + [p/(10-3p)]\epsilon ,$$
$$\gamma = 1 + [(2-p)/(10-3p)]\epsilon , \quad (15)$$
$$\eta = -\tfrac{1}{3}\epsilon(2-p)/(10-3p) .$$

These are just the exponents obtained previously[4,5] from the recursion relations and the scaling assumption. For $p=2$, which corresponds to the Ising model, $\mathrm{Tr}Q^3 = 0$, and we obtain exponents appropriate to the Gaussian model. If quartic terms were included, this would give rise to an expansion about $d = 4-\epsilon$ and the usual Ising exponents would be obtained.

II. ORDERED PHASE

To describe the ordered phase we set

$$A_\alpha(x) = \langle A_\alpha\rangle + \mathcal{L}(x) , \quad (16)$$

where the brackets denote the thermal average and $\mathcal{L}_\alpha$ is the fluctuating part. Following Priest and Lubensky[5] only uniaxial order will be considered. Thus

$$\langle A_\alpha\rangle = \begin{cases} Q, & \alpha = 1 \\ 0, & \text{otherwise} .\end{cases} \quad (17)$$

The corresponding expression for Q_{ii} is given by Eq. (4)

$$Q_{ii} = \left(\frac{p-1}{p}\right)^{1/2}(Q + \mathcal{L}) ,$$
$$(18)$$
$$Q_{ii} = \frac{-1}{p-1}\left(\frac{p-1}{p}\right)^{1/2}(Q+\mathcal{L}) + q_{ii} \text{ for } i \neq 1 ,$$

where q_{ii} is a traceless diagonal tensor of dimension $p-1$ and where the subscript on $\mathcal{L}_1$ has been dropped.

We add a fictitious field $-A_1(x)h(x)$ to the Hamiltonian and separate it into its fluctuating part,

$$\mathcal{H} = -\tfrac{1}{4}\int (r_L + k^2)\mathcal{L}(k)\mathcal{L}(-k) - \tfrac{1}{4}\int (r_T + k^2)\sum_{i\neq 1} q_{ii}q_{ii} - \int \tilde{h}(k)\mathcal{L}(-k)$$
$$+ (p-2)c\omega\int \mathcal{L}(k)\mathcal{L}(k')\mathcal{L}(-k-k') - 3\omega c\int \mathcal{L}(k)\sum_{i\neq 1} q_{ii}(k')q_{ii}(-k-k')$$
$$+ \omega\int \sum_{i\neq 1} q_{ii}(k)q_{ii}(k')q_{ii}(-k-k') , \quad (19)$$

and its fluctuation-independent (mean-field) part,

$$\mathcal{H}_{\mathrm{MF}} = -\tfrac{1}{4}rQ^2 + (p-2)c\omega Q^3 + hQ . \quad (20)$$

In these equations

$$r_L = r - 12(p-2)\omega c Q ,$$
$$r_T = r + 12\omega c Q , \quad (21)$$
$$\tilde{h} = h - \tfrac{1}{2}rQ + 3\omega c(p-2)Q^2 ,$$
$$c = [p(p-1)]^{-1/2} .$$

The propagators are now given by

$$\langle \mathcal{L}(k)\mathcal{L}(-k)\rangle = G_L(k) ,$$
$$(22)$$
$$\langle q_{ii}(k)q_{jj}(-k)\rangle = \left(\delta_{ij} - \frac{1}{p-1}\right)G_T(k) ,$$

where $G_{L,T}(k) = 2/(r_{L,T} + k^2)$. The derivation of the recursion relations is again straightforward,

$$\frac{dr_L}{dl}(l) = [2-\eta(l)]r_L(l) - 144K_6(p-2)c^2\omega^2(l)\left[\frac{p-2}{[1+r_L(l)]^2} + \frac{1}{[1+r_T(l)]^2}\right] ,$$

$$\frac{dr_T}{dl}(l) = [2-\eta(l)]r_T(l) - \frac{288K_6c^2\omega^2(l)}{[1+r_T(l)][1+r_L(l)]} - 144K_6\frac{p-3}{p-1}\omega^2(l)\frac{1}{[1+r_T(l)]^2} , \qquad (23)$$

$$\frac{d\tilde{h}}{dl} = \left[4 - \frac{\epsilon}{2} - \frac{\eta(l)}{2}\right]\tilde{h}(l) + 6K_6(p-2)c\,\omega(l)\left[\frac{1}{1+r_L(l)} - \frac{1}{1+r_T(l)}\right] .$$

The leading order solutions are

$$r_L(l) = t(l) - 12(p-2)c\,\omega(l)Q(l) + O(\omega^2) ,$$

$$r_T(l) = t(l) + 12c\,\omega(l)Q(l) + O(\omega^2) , \qquad (24)$$

$$\tilde{h}(l) = h(l) - \tfrac{1}{2}t(l)Q(l) + 3\omega(l)(p-2)cQ^2(l)$$
$$+3\omega(l)(p-2)cK_6(r_L(l)\{1-r_L(l)\ln[1+r_L(l)]\} - r_T(l)\{1-r_T(l)\ln[1+r_T(l)]\}) ,$$

where $t(l)$ and $\omega(l)$ are given by Eqs. (9) and (10) and where

$$h(l) = h\exp\left[(4-\tfrac{1}{2}\epsilon)l - \tfrac{1}{2}\int_0^l \eta(l')\,dl'\right] ,$$
$$Q(l) = Q\exp\left[(2-\tfrac{1}{2}\epsilon)l + \tfrac{1}{2}\int_0^l \eta(l')\,dl'\right] . \qquad (25)$$

The trajectory integral is evaluated with the help of Eqs. (6) and (7)

$$\exp\int_0^{l^*} \eta(l)\,dl = W(l^*)^{-(1/3)(2-p)/(10-3p)} . \qquad (26)$$

The fluctuation-corrected noncritical order parameter is given implicitly by the condition that $\langle \mathcal{L}(k)\rangle = 0$. With $l = l^*$ we obtain, to leading order in ω,

$$\tilde{h}(l^*) + 6(p-2)\omega(l^*)K_6 \times \int_0^1 k^5\,dk\left[\frac{1}{r_L+k^2} - \frac{1}{r_T+k^2}\right] = 0 . \qquad (27)$$

Performing the integral, substituting Eq. (24) for $\tilde{h}(l^*)$, and taking the limit $h(l^*)/Q(l^*) \to 0$ then gives, to leading order,

$$Q(l^*) = \frac{t(l^*)}{6(p-2)c\,\omega(l^*)} . \qquad (28)$$

From Eqs. (9), (10) and (25) the order parameter in the critical region is given by

$$Q = \frac{t}{6(p-2)c\,\omega} W(l^*)^{2/(10-3p)} . \qquad (29)$$

For $p = 2$, Q diverges. For the Ising model the quartic terms are required to give a finite value for the order parameter.

The susceptibilities are given by

$$\chi_{T,L}^{-1} = \exp\left[-2l^* + \int_0^{l^*} \eta(l)\,d(l)\right]^{-1} \chi_{T,L}^{-1}(l^*) , \qquad (30)$$

where, to leading order,

$$\chi_{T,L}^{-1}(l^*) = r_{T,L}(l^*) . \qquad (31)$$

The free energy now takes the form

$$F = \tfrac{1}{4}rQ^2 - (p-2)c\,\omega Q^3 + \tfrac{1}{2}K_6\int_0^{l^*} e^{-l'd}dl'\{\ln[1+r_L(l')] + (p-2)\ln[1+r_T(l')]\} . \qquad (32)$$

To leading order the trajectory integral is given by

$$\tfrac{1}{6}K_6\int_0^{l^*} [r_L^3(l') + (p-2)r_T^3(l')]e^{-l'd}dl' . \qquad (33)$$

The contribution from the lower limit cancels the mean-field terms in Eq. (32) and we obtain simply

$$F = -\frac{p-1}{(p-2)^2}\frac{|t|^3}{12^3\omega^2}[(p^2+4)W(l^*)^{2p/(10-3p)} - (p-2)^2] \qquad (34)$$

In the ordered phase l^* is determined by the condition

$$r_L(l^*) = 1 = t(l^*) - 12(p-2)c\,\omega(l^*)Q(l^*) , \qquad (35)$$

which by Eq. (28) is equivalent to

$$-t(l^*) = 1 \qquad (36)$$

such that, to leading order,

$$e^{2l^*} = |t|^{-1} . \qquad (37)$$

For $r_L(l^*) = 1$, it further follows from Eqs. (24) and (28) that

$$r_T(l^*) = \frac{-p}{p-2} \qquad (38)$$

and the susceptibilities take the form

$$\chi_{L,T}^{-1} = \begin{Bmatrix} 1 \\ -p/(p-2) \end{Bmatrix} \times |t| \, W(l^*)^{-2(2-p)/(10-3p)} \quad . \tag{39}$$

From Eqs. (29), (30), and (39) we obtain

$$Q \propto |t|^{\beta} \;,$$
$$c_v \propto |t|^{-\alpha} \;,$$
$$\chi_{L,T} \propto |t|^{-\gamma} \;,$$

with α and γ given by Eqs. (15) and β given by

$$\beta = 1 - \epsilon/(10-3p) \;, \tag{40}$$

in agreement with scaling. However, from Eqs. (39) it follows that $\chi_T < 0$ for $p > 2$. Priest and Lubensky[5] suggested that the fixed point for $2 < p < \frac{10}{3}$ might correspond to a metastable state. The present calculations show that the state is, in fact, unstable for this range of p values.

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