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# Venturi level differences in liquid-helium-II flow

Robert F. Lynch
*Physics Department, University of Petroleum and Minerals, Dhahran, Saudi Arabia*
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A calculation is given of the expected Venturi level difference in liquid-helium-II flow. It disagrees with a calculation by Putterman which finds no level difference, and agrees with Meservey that a level difference $\Delta z = (\rho_n/2\rho g)\Delta v_n^2$ entirely due to normal fluid will obtain. The experimental situation is surveyed, particularly the pioneering experiment of Pellam.

The late J. R. Pellam was the first investigator to study Venturi level differences in liquid-helium-II flow.[1] According to the "generalized Bernoulli equation" derived by Pellam and Morse,[2] the normal fluid and the superfluid were each expected to give rise to a level difference due to the flow of that component. However, Meservey[3] concluded that the requirement $\vec{\nabla} \times \vec{v}_s = 0$ would prevent any level difference for (subcritical) flow of superfluid. This has been confirmed in clamped ($v_n = 0$) geometries by Van Alphen *et al.*[4]

Meservey also predicted that for arbitrary flow of both superfluid and normal-fluid components there will be a level difference of $(\rho_n/2\rho g)\Delta v_n^2$, although he did not furnish an explicit proof. On the other hand a recent calculation by Putterman[5] leads him to conclude that *no* level difference will be observed in an arbitrary flow.

Pellam's early measurements[1] of the level difference caused by a resonant second-sound wave bear on this disagreement, since in this arrangement there is both a normal flow and a superflow. In fact he did report a small level difference ("a fraction of a millimeter") of the right sign at an unspecified temperature and heat input. Unfortunately his observations must be considered too qualitative to constitute a definitive test of the conflicting theories.

Putterman's result seems to contradict one's physical intuition in this problem. One purpose of this paper is to show that, in fact, Putterman's calculation is incorrect, and that a normal-fluid level difference should be obtained, as predicted by Meservey. A calculation is also made of the expected magnitude of the level difference in a Pellam-type resonator under optimal conditions.

Consider first the standard two-fluid equation (including the external gravitational field $\Omega = gz$)

$$\frac{D_s \vec{v}_s}{Dt} \equiv \frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\vec{\nabla}\mu - \vec{\nabla}\Omega \ . \quad (1)$$

Here $\mu$ is the chemical potential which satisfies

$$d\mu = (1/\rho)\, dp - s\, dT - \tfrac{1}{2}(\rho_n/\rho)\, d(\vec{v}_n - \vec{v}_s)^2 \ , \quad (2)$$

where $s$ is the specific entropy and the other symbols have their usual meanings.

Assuming $\vec{\nabla} \times \vec{v}_s = 0$ and steady flow, $\partial \vec{v}_s/\partial t = 0$, with the help of a vector identity one easily finds

$$\vec{\nabla}(\mu + \tfrac{1}{2}v_s^2 + gz) = 0 \ . \quad (3)$$

Now add and subtract from Eq. (3) the quantity $s\vec{\nabla}(\vec{v}_n \cdot \vec{A})$, where $\vec{A} \equiv (\rho_n/\rho s)(\vec{v}_n - \vec{v}_s)$. Rearranging Eq. (3) one finds

$$(1/\rho)\vec{\nabla}p - s\vec{\nabla}(T + \vec{v}_n \cdot \vec{A}) + s\vec{\nabla}(\vec{v}_n \cdot \vec{A}) - \tfrac{1}{2}(\rho_n/\rho)\vec{\nabla}(\vec{v}_n - \vec{v}_s)^2 + \tfrac{1}{2}\vec{\nabla}v_s^2 + g\vec{\nabla}z = 0 \ . \quad (4)$$

Ignore the spatial derivatives of the thermodynamic quantities $\rho$, $\rho_n$, and $s$. If we next assume that $\vec{\nabla} \times \vec{v}_n = 0$, then we have from the definition of $\vec{A}$, $\vec{\nabla} \times \vec{A} = 0$. Putterman (Ref. 5, p. 31) has shown that (for steady flow conditions) this implies that the quantity $\vec{\nabla}(T + \vec{v}_n \cdot \vec{A}) = 0$.

Using this in Eq. (4) we finally find[6]

$$\vec{\nabla}(p + \tfrac{1}{2}\rho_n v_n^2 + \tfrac{1}{2}\rho_s v_s^2 + \rho g z) = 0 \ . \quad (5)$$

Let us now calculate the level difference expected in an apparatus such as that depicted in Fig. 1. Before applying Eq. (5), however, we must critically examine the physical situation which obtains. For sufficiently narrow standpipes[7] viscosity will keep the normal fluid at rest. There will be a vortex sheet at the entrance to the standpipes, with $v_n = 0$ in the standpipe. Assume that we have one-dimensional flow in the channel, i.e., that $\vec{v}_n$, $\vec{v}_s$ do not vary across the channel (except very near the walls in the case of $\vec{v}_n$ due to viscosity). At the vortex sheet $\vec{\nabla} \times \vec{v}_n$ and $\vec{\nabla} \times \vec{A} \neq 0$. But in the standpipe, at $A$, $B$, and in the body of the channel we may take these quantities equal to zero. Hence Eq. (5) will hold separately in the standpipes and in the channel, and the quantity in the bracket will be a constant, the constant differing across the vortex sheet.

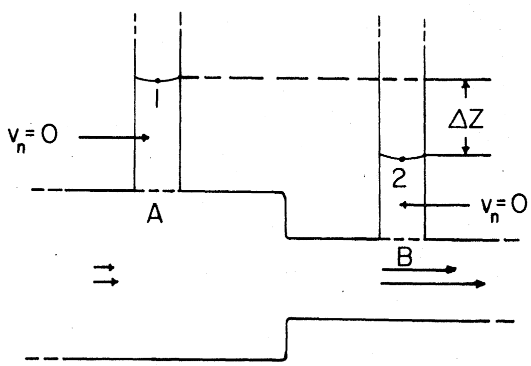


FIG. 1. Flow apparatus. Dashed lines at entrances to standpipes indicate normal-fluid vortex sheets.

Now consider the level difference between points 1 and 2 in Fig. 1. At the top of the standpipes, $p = p_0$, the vapor pressure. Meservey's calculations show that although the superfluid flow penetrates the standpipe, the velocity falls off exponentially with height, and for all practical purposes we can take $v_s = 0$ at the tops. Thus from Eq. (5),

$$p_0 + \rho g z_1 = C_1 \ ,$$
$$p_0 + \rho g z_2 = C_2 \ , \qquad (6)$$

so that

$$\rho g (z_1 - z_2) = \rho g \Delta z = C_1 - C_2$$
$$= (C_1 - C) + (C - C_2) \ , \qquad (7)$$

where $C$, $C_1$, and $C_2$ are the values of the constant quantity in parentheses in Eq. (5) in the channel and the two standpipes, respectively. Since the pressure, superfluid velocity, and gravitational potential are continuous through the normal-fluid vortex sheet, it follows that the change in the constant is just the change in the quantity $\frac{1}{2} \rho_n v_n^2$ through the sheet. Therefore,

$$\Delta z = (\rho_n / 2\rho g) \Delta v_n^2 = (\rho_n / 2\rho g)(v_{nB}^2 - v_{nA}^2) , \qquad (8)$$

i.e., the result which Merservey stated. Putterman failed to get this result because he neglected to include the influence of the vortex sheet in his calculations.

Next consider the expected magnitude of the level difference in a Pellam-type second-sound resonator (see Fig. 2). Even though this is not a case of steady flow the theory derived is completely applicable since in this case we are interested in the mean-squared average of $v_n^2$, $v_s^2$; the time-dependent terms are first order in the velocity and average to zero.

For this theory to be valid for a resonator of characteristic diameter $d$, the superfluid velocity must be limited to the critical velocity $v_{s,c}$, where $v_{s,c}$ satisfies the relationship[8] $v_{s,c} \sim d^{-1/4}\ \mathrm{cm}^{5/4}\mathrm{sec}^{-1}$. Employing the condition for zero mass flow, $\rho_n \overline{v}_n + \rho_s \overline{v}_s = 0$, one has

$$\Delta z_{\max} = (\rho_s^2 / 2 g \rho \rho_n) \langle v_{s,c}^2 \rangle$$
$$= (\rho_s^2 / 2 g \rho \rho_n)(0.5 / d^{1/2})\ \mathrm{cm}^{5/2}\ \mathrm{sec}^{-2} \ . \qquad (9)$$

Figure 3 is a plot of $\Delta z_{\max}$ vs $T$ for a channel of dimension $\sim 1$ cm — presumably approximately that of

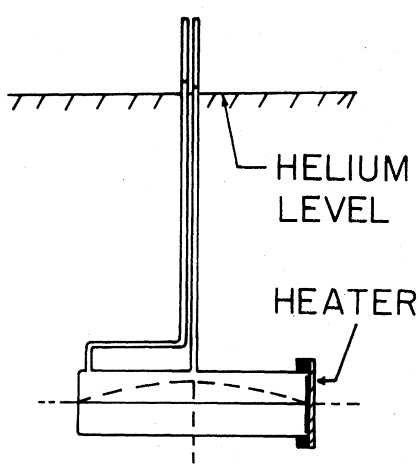


FIG. 2. Schematic of Pellam's apparatus. Dashed line indicates particle velocity amplitude for half-wave resonance (after Pellam, Ref. 1).

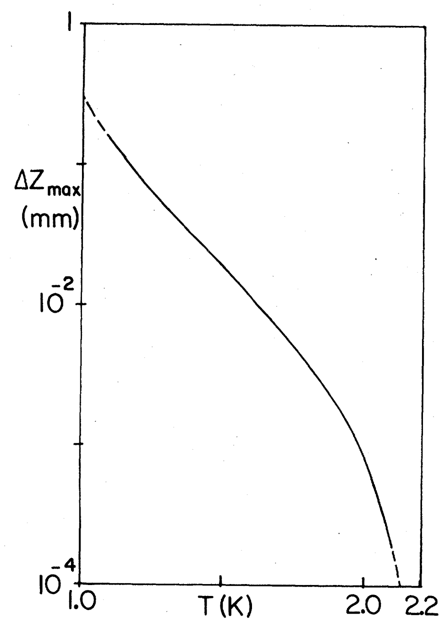


FIG. 3. Maximum level difference $\Delta z_{\max}$ vs temperature $T$ for a channel of dimension $\sim 1$ cm, for which $v_{s,c} = 1$ cm/sec.

Pellam's apparatus. It is seen that for $T \sim 1$ K a level difference of approximately the magnitude observed by Pellam will obtain.

It appears the effect is only marginally detectable by visual observation. It would be interesting indeed to see the results of an experiment in which small level differences could be detected with greater sensitivity.

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[1]J. R. Pellam, Phys. Rev. 78, 818 (1950).

[2]J. R. Pellam and P. M. Morse, Phys. Rev. 78, 474 (1950).

[3]R. Meservey, Phys. Fluids 8, 1209 (1965).

[4]W. M. Van Alphen, R. De Bruyn Ouboter, K. W. Taconis, and E. van Spronsen, Physica (Utrecht) 39, 109 (1968).

[5]S. J. Putterman, *Superfluid Hydrodynamics* (North-Holland, Amsterdam, 1974), p. 158.

[6]A complete derivation of this equation is given so that the assumptions underlying it can be critically examined. It was first given, however, by Landau, J. Phys. (Moscow) 5, 71 (1941) as pointed out by J. L. Olsen, Physica (Utrecht) 69, 136 (1973). It is interesting to note that the latter author has verified at least the partial validity of Eq. (5)—somewhat qualitatively—by means of his observations of level differences at a free surface, caused by resonant second sound waves.

[7]We may estimate the needed standpipe diameter $t$ as follows: in the Poiseuille flow formula $\dot{V} = (\pi\rho_n/128\eta) \times (\Delta p/l)t^4$ set the mass flow rate $\dot{V} \sim (\rho_n v_n)[\frac{1}{4}(\pi t^2)]$, $\Delta p \sim \frac{1}{2}\rho_n v_n^2$, and $l \sim 1$ cm. Then $t = (64\eta/\rho_n v_n)^{1/2}$. For $\eta \sim 10^{-5}$ poise, $\rho_n \sim \rho$, $v_n \sim 1$ cm/sec, $t \sim 1$ mm.

[8]R. De Bruyn Ouboter, K. W. Taconis, and W. M. Van Alphen, *Progress in Low Temperature Physics* (North-Holland, Amsterdam, 1967), Vol. V, p. 74.