Direct generation of ultrasound by electromagnetic radiation in metals: Effect of surface scattering

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The usual theory of direct ultrasonic generation by electromagnetic radiation in metals is based on the assumption that the conduction electrons experience specular scattering at the boundary of the metal. The present study generalizes the theory of Reuter and Sondheimer for the anomalous skin effect in metals to take into account the motion of the lattice. We develop a procedure for the calculation of the conversion efficiency of electromagnetic to acoustic energy when the electrons are scattered diffusely by the surface of the metal. The results show that this efficiency is comparable in the two cases of specular and diffuse scattering. For example, for metallic potassium under the conditions of the experiments by Chimenti *et al.* (incident radiation of frequency equal to 9 MHz and electron mean free path of the order of 1.4×10^{-2} cm) we find that the ratio of acoustic to electromagnetic fluxes is 4.5×10^{-12} for specular scattering and 4.7×10^{-12} for diffuse scattering.

I. INTRODUCTION

In the theory of the skin effect in metals it is generally assumed that the atoms remain undisturbed by the action of the radio-frequency (rf) electromagnetic field which acts only on the conduction electrons. If the frequency of the incident electromagnetic wave lies in the microwave region, at low temperatures in high-purity metals, the penetration of the wave is anomalously large compared with what is expected according to the standard theory.[1] The reason for this effect is to be found in the fact that, when the mean free path of the electrons is larger than the skin depth, only those electrons whose mean free path is traversed for the most part within the skin depth contribute effectively to the screening of the interior of the metal. This concept was developed by Pippard[2] to account for the experimental facts. It was subsequently studied mathematically by Reuter and Sondheimer[3] who provided detailed justification for Pippard's approach.

If we consider that not only the conduction electrons but also the positive ions experience the action of the rf field we conclude that acoustic oscillations of the same frequency as the incident wave are excited within the metal. This is caused by the fact that, when the electron mean free path is long, the screening of the motion of the ions is not locally exact.

This phenomenon has been observed by several investigators. In the experiments by Abeles[4] short microwave pulses were made to impinge on an indium film on a substrate formed by a slab of germanium. The generation of acoustic waves was demonstrated by the echoes of these pulses having a delay equal to the transit time of an acoustic wave through the germanium substrate which was thick compared to the indium film. Other investigators have studied this phenomenon by observing transmission of energy across a metallic slab under conditions in which direct electromagnetic coupling across it is not possible.[5–10] Recently, Perlow *et al.*[11] have studied a similar effect in copper foils doped with ^{57}Co and observed the Mössbauer resonance in a suitable absorber. In this way, the presence of an electromagnetically generated acoustic wave may be detected as frequency modulated sidebands in the Mössbauer spectrum. The results of this experiment, however, appear to be related to a different phenomenon.[11]

Theoretical investigations of ultrasonic generation in metals by electromagnetic radiation have been given by Quinn,[12] Southgate,[13] and Alig.[14]

In any study of this effect it is necessary to consider the nature of the scattering of the conduction electrons by the surface of the metal. In the investigations by Quinn[12] and Alig[14] it was assumed that an electron reaching the surface of the metal from its interior experiences a change in momentum only along the normal to the surface. We call this specular scattering. It is also possible that the component of the momentum of an electron in the plane of the surface is random after the collision. This is expected to be the case if the surface is not perfectly flat within atomic dimensions. There is no rigorous theory of this effect. Approximate discussions have been given by Abeles,[4] Southgate,[13] and others.[15,16]

There exist studies[17–24] of the ultrasonic generation in metals in a strong dc magnetic field normal to the surface and thus parallel to the direction of propagation of the wave. In this case the efficiency of generation is enhanced because the Lorentz force provides a force on the metal that is not balanced as in the case previously discussed. This force gives rise to an acoustic excitation whose amplitude is proportional to the magnetic field. While there does not exist, at present, quantitative agreement between the experimental measurements and the theory in the case of zero applied magnetic field,[25,26] in the presence of a strong magnetic field the agreement is satisfactory. At low fields, there exists a nonmonotonic behavior of the acoustic amplitude as a function of magnetic field which is yet to receive an adequate explanation.

The purpose of this paper is to give a theory of the electromagnetic generation of acoustic waves in metals assuming that the electrons are scattered diffusely at the surface. Section II gives a description of the anomalous skin effect in metals in a manner which is useful for our subsequent development. The results are those already obtained by Reuter and Sondheimer.[3] Section III gives a calculation of the surface force arising from the transfer of transverse momentum by the electrons to the surface of the metal. We then study the motion of the positive ion background when the acoustic wavelength is long compared to the atomic spacings. We obtain the amplitude of the traveling wave by introducing the surface force as a boundary condition and requiring that there be no reflected acoustic wave since we take the metal to be semi-infinite. This result is obtained neglecting the attenuation of the acoustic wave. We call the amplitude of such a wave $\xi(\infty)$. The actual amplitude measured must be corrected for the attenuation as indicated in Sec. III. Section IV gives the calculation of the insertion rate defined as the ratio of the energy fluxes of the acoustic and electromagnetic waves. This ratio is a measure of the efficiency of the transformation of electromagnetic to acoustic energy. The method used in the determination of the acoustic amplitude follows that employed in the calculation of acoustic generation in superconducting surfaces given by Kartheuser and Rodriguez.[27] However, in that study, it was supposed that the electrons were scattered specularly at the surface of the material.

The result of our investigation indicates that the detailed scattering mechanism does not significantly alter the efficiency of the conversion of electromagnetic to acoustic energy in the absence of a magnetic field. In the presence of a dc magnetic field the effect may give a significant difference[28] and we shall return to it in a future publication.

For metallic potassium, taking an electron mean free path of the order of that in the samples used by Chimenti *et al.*[23] and for the frequency used in their measurements we find an insertion rate

$$(\eta_1)_S = 4.45\times 10^{-12} \tag{1.1}$$

for specular scattering and

$$(\eta_1)_D = 4.67\times 10^{-12} \tag{1.2}$$

for diffuse scattering. The difference is attributed to the surface force. This force is in the opposite direction from that of the electric field of the wave because the average electron momentum is directed antiparallel to the electric field. We conclude, therefore, that the detailed mechanism of scattering of the electron at the surface of the metal does not profoundly affect the conversion efficiency of electromagnetic to acoustic energy in the absence of a dc magnetic field at these frequencies.

II. ANOMALOUS SKIN EFFECT

We consider a metal bounded by a plane surface which we take to be the $z=0$ plane of a Cartesian coordinate system. We suppose that the metal occupies the $z>0$ region of space and that a linearly polarized plane electromagnetic wave of angular frequency ω is incident normally on this surface from the empty region $z<0$ and propagates into the metal. The x axis is taken to be parallel to the direction of the electric field of the wave.

The electric and magnetic fields are described by the complex quantities $E(z)e^{-i\omega t}$ and $B(z)e^{-i\omega t}$, where $\vec{B}$ is directed along the y axis. The Maxwell equations give the condition (we use Gaussian units)

$$E''(z)+\left(\frac{\omega^2}{c^2}\right)E(z)=-\left(\frac{4\pi i\omega}{c^2}\right)J(z)\ ; \tag{2.1}$$

the quantity $J(z)e^{-i\omega t}$ is the total current density which we take to be parallel to the electric field. The current density $J(z)$ has ionic and electronic components. We calculate the electronic component using the procedure of Reuter and Sondheimer.[3] A solution of the Boltzmann transport equation is derived for the semiclassical distribution function $f(z,\vec{k},t)$ of the conduction electrons. The distribution function is expressed as $f_0+f_1(z,\vec{k})e^{-i\omega t}$, where f_0 is the equilibrium Fermi-Dirac distribution function. The deviation f_1 of the distribution function from its equilibrium value depends on the position z and the wave vector $\vec{k}$. We work in the free-electron approximation in which the electron velocity $\vec{v}=\hbar\vec{k}/m$, m being the mass of the electron. We also suppose that the distribution relaxes to equilibrium with a characteristic constant relaxation time τ so that the rate of change of f due to collisions is $-(f-f_0)/\tau$. Let p be the fraction of electrons experiencing specular scattering at $z=0$. We obtain $f_1=f_1^{(1)}$ if $v_z<0$, and $f_1=f_1^{(2)}$ if $v_z>0$, where

$$f_1^{(1)}=-\left(\frac{ev_x}{v_z}\right)f_0'\int_z^\infty d\zeta\,E(\zeta)\exp(u) \tag{2.2}$$

and

$$f_1^{(2)}=\left(\frac{ev_x}{v_z}\right)f_0'\left[p\int_{-\infty}^z d\zeta\,E(\zeta)\exp(u)+(1-p)\int_0^z d\zeta\,E(\zeta)\exp(u)\right], \tag{2.3}$$

with u being a function of the component v_z of the electron velocity and of $(\zeta-z)$ given by

$$u=(1-i\omega\tau)(\zeta-z)/\tau v_z\ . \tag{2.4}$$

The charge on the electron is $-e$. The quantity f_0' is the derivative of the Fermi distribution function with respect to the energy ϵ of the electron. This is a highly peaked function which to a high degree of approximation equals $-\delta(\epsilon-\epsilon_F)$, where ϵ_F is the Fermi energy. In deriving Eq. (2.3) which is only valid for $z>0$ we have arbitrarily redefined $E(\zeta)=E(-\zeta)$ for $\zeta<0$. The electronic current density is deduced from the equation

$$J_e(z)=-(e/4\pi^3)\int d\vec{k}\,v_x f_1\ , \tag{2.5}$$

so that

$$J_e(z)=p\int_{-\infty}^\infty d\zeta\,K(z-\zeta)E(\zeta)+(1-p)\int_0^\infty d\zeta\,K(z-\zeta)E(\zeta)\ , \tag{2.6}$$

where the kernel function K is

$$K(z)\equiv\sigma_0G(z)=\frac{3\sigma_0}{4l}\int_1^\infty d\lambda\left(\frac{\lambda^2-1}{\lambda^3}\right)\exp[-(1-i\omega\tau)\lambda|z|/l]\ . \tag{2.7}$$

In Eq. (2.7), $\sigma_0=ne^2\tau/m$ is the dc conductivity, n is the electron density, and $l=v_F\tau$ is the electron mean free path; v_F is the Fermi velocity of the electrons.

We now describe how one obtains the profile of the electric field and the surface impedance of the metal for the two cases $p=1$ (specular scattering) and $p=0$ (diffuse scattering) in a manner which will be convenient for our further discussion in Sec. IV.

A. Specular scattering ($p=1$)

In this case

$$J_e(z)=\int_{-\infty}^\infty K(z-\zeta)E(\zeta)\,d\zeta\ . \tag{2.8}$$

We introduce the Fourier transform $E(k)$ of $E(z)$ by the equation

$$E(z)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikz}E(k)\,dk\ . \tag{2.9}$$

Substitution of Eqs. (2.8) and (2.9) into Eq. (2.1) yields, after an integration by parts and using $E(z)=E(-z)$,

$$E(k)=2E'(z=+0)[-k^2+(\omega/c)^2+(4\pi i\omega/c^2)K(k)]^{-1}\ . \tag{2.10}$$

Here $K(k)$ is the Fourier transform of Eq. (2.7) and is given by

$$K(k)=-\frac{3i\sigma_0}{4lk}\left[-\frac{2b}{k}+\frac{k^2-b^2}{k^2}\ln\left(\frac{b-k}{b+k}\right)\right], \tag{2.11}$$

where

$$bl=\omega\tau+i\equiv\alpha+i\ . \tag{2.12}$$

It is convenient to define the function $\psi(k)$ by

$$\psi(k)=k^2-(4\pi i\omega/c^2)K(k)\ . \tag{2.13}$$

Care must be exercised in selecting the principal sheet of the Riemann surface for $\psi(k)$. This study is described in Appendix A. Then, the Fourier transform $E(k)$ of the electric field is

$$E(k)=\frac{-2E'(z=+0)}{\psi(k)-(\omega/c)^2}\ . \tag{2.14}$$

The surface impedance of the metal in this case is

$$Z=R-iX=\left(\frac{4\pi}{c}\right)\left(\frac{E(+0)}{B(+0)}\right)=\frac{4\pi i\omega}{c^2}\frac{E(+0)}{E'(+0)}\ . \tag{2.15}$$

Use of Eq. (2.14) yields

$$Z=-\frac{4i\omega}{c^2}\int_{-\infty}^\infty\frac{dk}{\psi(k)-(\omega/c)^2}\ . \tag{2.16}$$

For the purpose of evaluating the integral (2.16) we study the behavior of the function

$$\psi(k)=b^2\left\{w^2+\frac{\alpha^3\Lambda}{(\alpha+i)^3}\left[\frac{1}{w^2}+\frac{w^2-1}{2w^3}\ln\left(\frac{1+w}{1-w}\right)\right]\right\}$$

in Appendix A. Here

$$w=k/b$$

and

$$\Lambda=\frac{3}{2}\left(\frac{v_F\omega_p}{c\omega}\right)^2,$$

with $\omega_p=(4\pi ne^2/m)^{1/2}$ the plasma frequency of the metal. The numerical values of Λ and $\alpha=\omega\tau$ in the region of interest for metallic potassium at low temperature are obtained using $\omega_p = 6.7 \times 10^{15}$ sec^{-1}, $v_F = 8.6 \times 10^7$ cm/sec, and the purity to be such that $l = v_F \tau = 1.4 \times 10^{-2}$ cm at liquid-helium temperature. We take the frequency[23] of the incident wave to be 9 MHz ($\omega = 5.6 \times 10^7$ sec^{-1}). We have $\Lambda = 1.7 \times 10^{11}$ and $\alpha = 9.3 \times 10^{-3}$. The quantity $\alpha^3 \Lambda$ is of the order of 1.4×10^5. Under such conditions, the denominator of the integral in Eq. (2.16) has zeros for $|w| \gg 1$ only. With the parameters given above the zeros occur at $w = \pm(30.6 - i51.4)$ and at $w = \pm(60.2 + i0.13)$. The value of the integral in Eq. (2.16) is approximately that obtained by taking $\psi(k)$ for $kl \gg 1$ with the value appropriate for $\mathrm{Im} k > 0$ because in calculating Z one must consider solutions in which E tends to zero as z approaches infinity. This gives[3]

$$Z = -\frac{4i\omega}{c^2} \int_{-\infty}^{\infty} \frac{dk}{k^2 - (3i\pi^2 \omega \sigma_0 / c^2 l |k|)} . \quad (2.17)$$

Evaluation of the integral gives

$$Z = \tfrac{8}{9} (\sqrt{3} \pi l \omega^2 / c^4 \sigma_0)^{1/3} (1 - i\sqrt{3}) . \quad (2.18)$$

B. Diffuse scattering ($p = 0$)

In this case we have

$$J_e(z) = \int_0^\infty K(z - \zeta)\, d\zeta , \quad (2.19)$$

and

$$E''(z) + \frac{\omega^2}{c^2} E(z) = -\frac{4\pi i \omega}{c^2} \int_0^\infty K(z - \zeta) E(\zeta)\, d\zeta . \quad (2.20)$$

This equation is valid for $z > 0$. Without loss of generality we can set $E(z) = 0$ for $z < 0$. We then encounter the difficulty that even though the left-hand side of Eq. (2.20) vanishes for $z < 0$, the right-hand side does not. We can extend the validity of Eq. (2.20) without altering it for $z > 0$ by rewriting it

$$E''(z) + \frac{\omega^2}{c^2} E(z) = -\frac{4\pi i \omega}{c^2} H(z) \int_0^\infty K(z - \zeta) E(\zeta)\, d\zeta , \quad (2.21)$$

where $H(z)$ is the Heaviside function defined by $H(z) = 1$ if $z \geqslant 0$ and $H(z) = 0$ if $z < 0$. We use the representation

$$H(z) = \frac{1}{2\pi i} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{e^{i\kappa z}}{\kappa}\, d\kappa \quad (2.22)$$

for $H(z)$ where ϵ is a positive number which can be taken as small as we wish. We can now carry out the Fourier analysis as before to obtain

$$-E'(z = +0) - ikE(z = +0) - \left(k^2 - \frac{\omega^2}{c^2} \right) E(k) = \frac{4\pi i \omega}{c^2} \frac{1}{2\pi i} \int_\Gamma \frac{K(k') E^{(-)}(k')}{k' - k}\, dk' . \quad (2.23)$$

The contour Γ is a straight line extending from $-\infty$ to ∞ just above the line $\mathrm{Im} k' = \mathrm{Im} k$. We can select k to be real so that Γ is parallel to the real axis and just above it. $E^{(-)}(k')$ is the analytic continuation of the transform of $E(k)$ holomorphic in the lower half plane as required by the condition that $E(z)$ be zero for large values of z and zero for $z < 0$.

From Eq. (2.23), for $|k| \to \infty$, we have the asymptotic behavior

$$E(k) = -ik^{-1} E(z = +0) - k^{-2} E'(z = +0) + O(k^{-3}) . \quad (2.24)$$

We now study the auxiliary function

$$\frac{1}{2\pi i} \int_\Gamma \frac{K(k') E^{(-)}(k')\, dk'}{k' - k} . \quad (2.25)$$

this function has a discontinuity of $+K(k) E^{(-)}(k)$ upon crossing Γ from below to above. From Eq. (2.23) we obtain, using the Plemelj[29] relations, for k on Γ,

$$\left(k^2 - \frac{\omega^2}{c^2} \right) E^{(+)}(k) = \left(k^2 - \frac{\omega^2}{c^2} - \frac{4\pi i \omega}{c^2} K(k) \right) E^{(-)}(k) . \quad (2.26)$$

Here $E^{(+)}(k)$ is holomorphic in the upper half plane. Remembering that ω has a positive infinitesimal imaginary part, we rewrite Eq. (2.26)

$$\left(k - \frac{\omega}{c} - i\epsilon \right) E^{(+)}(k) = \frac{k^2 - \omega^2/c^2 - (4\pi i \omega / c^2) K(k)}{k^2 - \omega^2/c^2} \times \left(k - \frac{\omega}{c} - i\epsilon \right) E^{(-)}(k) . \quad (2.27)$$

We follow the Wiener-Hopf-Carleman method and let[30,31]

$$(k - \omega/c - i\epsilon) E(k) = \Phi(k) . \quad (2.28)$$

The procedure consists in defining[32,33]

$$\frac{k^2 - \omega^2/c^2 - (4\pi i \omega / c^2) K(k)}{k^2 - \omega^2/c^2} = \frac{\psi(k) - \omega^2/c^2}{k^2 - \omega^2/c^2} = \frac{X^{(+)}(k)}{X^{(-)}(k)} , \quad (2.29)$$

where the functions $X^{(+)}(k)$ and $X^{(-)}(k)$ are holomorphic in the upper-half and lower-half planes, respectively.

Thus we have

$$\Phi^{(+)}(k) / X^{(+)}(k) = \Phi^{(-)}(k) / X^{(-)}(k) , \quad (2.30)$$

showing that $\Phi(k)/X(k)$ is analytic throughout the complex k plane.

As $k \to \infty$, the asymptotic value of $\Phi(k)$ is

$-iE(z=+0)$. The function $X(k)$ is defined as one whose logarithm has the discontinuity

$$\ln\left[\frac{\psi(k)-\omega^2/c^2}{k^2-\omega^2/c^2}\right]$$

across the path Γ and at the same time approaches unity as $|k|\to\infty$. These requirements are satisfied by

$$X(k)=\exp\left[\frac{1}{2\pi i}\int_\Gamma \frac{dk'}{k'-k}\ln\left[\frac{\psi(k')-\omega^2/c^2}{k'^2-\omega^2/c^2}\right]\right]. \tag{2.31}$$

It is necessary that the argument of the logarithm possess no zeros along Γ. Since the asymptotic value of $X(k)$ is unity as $|k|\to\infty$, we find that $\Phi(k)/X(k)$ equals $-iE(z=+0)$ everywhere, and we have

$$E^{(-)}(k)=\frac{-iE(z=+0)X^{(-)}(k)}{k-\omega/c-i\epsilon}. \tag{2.32}$$

From Eq. (2.23) for large k we have

$$-E'(z=+0)-ikE(z=+0)-k^2E^{(-)}(k)=0, \tag{2.33}$$

so that using Eq. (2.32) we have

$$\frac{E'(z=+0)}{E(z=+0)}=\lim_{k\to\infty}\left[-ik+\frac{i(k^2-\omega^2/c^2)X^{(-)}(k)}{k-\omega/c-i\epsilon}\right]$$
$$=-\frac{1}{2\pi}\int_\Gamma \ln\left[\frac{\psi(k)-\omega^2/c^2}{k^2-\omega^2/c^2}\right]dk. \tag{2.34}$$

Direct numerical evaluation of the integral for the parameters given above gives

$$\frac{E'(z=+0)}{E(z=+0)}=(-4.09+i2.43)\times10^3\ \text{cm}^{-1}. \tag{2.35}$$

On the other hand, neglecting the displacement current, i.e., dropping the term $(\omega/c)^2$ and using the form of $\psi(k)$ appropriate for $|k|l>>1$, we have[34]

$$\frac{E'(z=+0)}{E(z=+0)}=-\frac{1}{\pi}\int_0^\infty dk\ln\left[1-\frac{i\pi\Lambda\alpha^3}{2k^3l^3}\right]. \tag{2.36}$$

Substituting the variable k by

$$\rho=-\frac{i\pi\Lambda\alpha^3}{2k^3l^3}, \tag{2.37}$$

we have, after performing a suitable contour integration,

$$\frac{E'(z=+0)}{E(z=+0)}=-\frac{1}{3\pi l}\left(-\tfrac{1}{2}i\pi\Lambda\alpha^3\right)^{1/3}\times\int_0^\infty d\rho\,\rho^{-4/3}\ln(1+\rho),$$

so that

$$\frac{E(z=+0)}{E'(z=+0)}=-\frac{i\sqrt{3}l}{4\alpha}\left(\tfrac{1}{2}\pi\Lambda\right)^{-1/3}(1-i\sqrt{3}). \tag{2.38}$$

Thus the surface impedance in the diffuse scattering case is $\frac{9}{8}$ the value for specular scattering. The numerical value, corresponding to that in Eq. (2.35) in the approximation (2.38), is

$$\frac{E'(z=+0)}{E(z=+0)}=(-4.24+i2.45)\times10^3\ \text{cm}^{-1}, \tag{2.39}$$

in good agreement with Eq. (2.35).

III. CALCULATION OF THE SURFACE FORCE AND EQUATION OF MOTION OF THE LATTICE

An electron gas in equilibrium exerts a constant pressure on its surroundings. If the gas is nonuniform some layers move relative to adjacent layers giving rise to shear stresses. This is precisely what happens in the case under investigation since the electric field of the wave is not uniform. Considering the rate of i component of momentum transported parallel to the j axis ($i,j=1,2,3$) we obtain the stress tensor of the electrons on their environment. This is given by

$$T_{ij}(z)=\frac{m}{4\pi^3}\int v_iv_jf\,d\vec{k}. \tag{3.1}$$

In equilibrium we set $f=f_0$ and find T_{ij} to be diagonal and equal to $P\delta_{ij}$, where P is the pressure of the gas. If the scattering at the surface $z=0$ is specular $T_{xz}(0)=0$. However, for diffuse scattering, each electron approaching the $z=0$ plane with $v_z<0$ and actually reaching it before being scattered will, on the average, surrender to the atoms on the surface the momentum $m\vec{v}_\perp$, where $\vec{v}_\perp$ is the transverse velocity of the electron. Since the electric field is taken parallel to the x axis, the only shear components of $T_{ij}(z)$ which do not vanish are $T_{xz}(z)=T_{zx}(z)$. We calculate this quantity using $f=f_0+f_1$ and Eqs. (2.2) and (2.3). We obtain

$$T_{xz}(z)=\tfrac{3}{4}ne\int_{\pi/2}^{\pi}\sin^3\theta\,d\theta\int_z^\infty E(\zeta)\exp(u)\,d\zeta$$
$$-\tfrac{3}{4}ne\int_0^{\pi/2}\sin^3\theta\,d\theta\left[p\int_{-\infty}^{z}E(\zeta)\exp(u)\,d\zeta+(1-p)\int_0^zE(\zeta)\exp(u)\,d\zeta\right], \tag{3.2}$$

where u is identical to the value given in Eq. (2.4). The symbol θ stands for the angle between the velocity $\vec{v}$ of the electron and the positive z axis. To calculate the stress at the boundary of the metal we set $z=0$ in Eq. (3.2) and obtain

$$T_{xz}(0)=\tfrac{3}{4}ne(1-p)\int_{-1}^{0}(1-\mu^2)\,d\mu\int_0^{\infty}E(\zeta)\exp[(1-i\omega\tau)\zeta/l\mu]\,d\zeta\ , \tag{3.3}$$

where we have made several transformations and used the substitution $\mu=\cos\theta$.

Before going further we establish the equation of motion of the positive ions under the action of the electromagnetic field. We first recall that in Eq. (2.1), $J(z)$ stands for the total current density. In our analysis of the penetration of the electric field given in Sec. II we approximated $J(z)$ by the electronic current density. There is, however, an ionic current density. This contribution is

$$J_i(z)=-(\gamma e\rho/M)i\omega\xi(z)\ , \tag{3.4}$$

where γe is the charge and M is the mass of each positive ion, ρ is the mass density of the material, and $\xi(z)e^{-i\omega t}$ is the displacement at time t of an ion whose equilibrium position is at z. We take this displacement along the x axis. For this reason a single component is sufficient to specify it. If we call s the speed of transverse-acoustic waves, the motion of the positive ions is governed by the equation

$$-M\omega^2\xi(z)=Ms^2\xi''(z)+\gamma eE(z)+F_c(z)\ . \tag{3.5}$$

The left-hand side of this equation is simply the mass of the ion times its acceleration. The right-hand side is the sum of three forces. The first is the elastic force per atom, the second describes the action of the electric field of the wave, and the third is a collision force arising from the transfer of momentum from the electrons to the lattice. It is assumed that, on the average and in the frame of reference in which the lattice is instantaneously at rest, the momentum of the electron after the collision is zero. Thus in the frame of reference of the experimenter the electrons move with the velocity of the positive ions immediately after a collision. The rate of momentum transfer from the electrons to the lattice per unit volume of material is

$$(nm/\tau)(\langle\vec{v}\rangle-\dot{\vec{u}})\ ,$$

where $\langle\vec{v}\rangle$ is the average electron velocity and $\vec{u}(\vec{r},t)$ the displacement field of the lattice. Since $\vec{u}(\vec{r},t)=\hat{x}\xi(z)e^{-i\omega t}$, where $\hat{x}$ is a unit vector along the x axis, we obtain the expression

$$F_c(z)e^{-i\omega t}=(Mnm/\rho\tau)[\langle v_x\rangle+i\omega\xi(z)e^{-i\omega t}]\ ,$$

for the average collision force on a positive ion.

Substitution for the electron and ion velocities in terms of their currents yields

$$F_c(z)=-(Mm/\rho e\tau)J(z)\ . \tag{3.6}$$

From Eqs. (2.1) and (3.4)–(3.6) we obtain the coupled equations

$$E''(z)+\frac{\omega^2}{c^2}E(z)+\frac{4\pi\gamma e\rho\omega^2}{Mc^2}\xi(z)+\frac{4\pi i\omega}{c^2}J_e(z)=0 \tag{3.7}$$

and

$$\xi''(z)+\frac{\omega^2}{s^2}\xi(z)+\frac{i\omega\gamma m}{M\tau s^2}\xi(z)+\frac{\gamma e}{Ms^2}E(z)-\frac{m}{\rho e\tau s^2}J_e(z)=0\ . \tag{3.8}$$

In order to be precise $J_e(z)$ in this case differs from that in which the lattice is at rest. In fact, the distribution function f relaxes to a value where the electrons retain the velocity of the positive ions. This is equivalent to having an additional electric field in the x direction, equal to $i\omega m\xi/e\tau$. This contribution is negligible as we see below. In solving the coupled equations (3.7) and (3.8) we must recognize that the terms containing $\xi(z)$ in Eq. (3.7) are small and can be neglected. These include the third term in Eq. (3.7) and the contribution of $J_e(z)$ arising from the effective field which we just discussed. We can, if we wish, established an iteration procedure where $E(z)$ and $\xi(z)$ are expanded in powers of the parameter m/M. The expansion of $E(z)$ contains powers of m/M starting from zero while $\xi(z)$ contains first and higher powers of this parameter. Keeping only the lowest terms in $E(z)$ and $\xi(z)$ leads to neglecting the contributions discussed above and the third term in Eq. (3.8). Thus, Eq. (3.8) reduces to

$$\xi''(z)+\frac{\omega^2}{s^2}\xi(z)=-\frac{\gamma e}{Ms^2}\mathcal{E}(z)\ , \tag{3.9}$$

where

$$\mathcal{E}(z)=E(z)-\sigma_0^{-1}J_e(z)\ . \tag{3.10}$$

The general solution of the linear differential equation (3.9) is

$$\xi(z)=A\exp\frac{i\omega z}{s}+B\exp\frac{-i\omega z}{s}+\frac{\gamma e}{Ms\omega}\int_0^z\mathcal{E}(\zeta)\sin\frac{\omega(\zeta-z)}{s}\,d\zeta\ . \tag{3.11}$$

The xz component of the strain at the surface $z=0$ of the metal is given by

$$\xi'(z=+0)=(i\omega/s)(A-B)\ . \tag{3.12}$$

For large positive values of z, physical requirements impose the restrictions that $\mathcal{E}(z)$ approach zero and that $\xi(z)$ behave as $\xi(\infty)\exp(i\omega z/s)$, i.e., that it be a wave propagating in the positive z direction. The asymptotic behavior of $\xi(z)$ for large z is

$$\xi(z)=\left[A+\frac{i\gamma e}{2Ms\omega}\int_0^\infty \mathcal{E}(\zeta)\exp\frac{-i\omega\zeta}{s}\,d\zeta\right]\exp\frac{i\omega z}{s}+\left[B-\frac{i\gamma e}{2Ms\omega}\int_0^\infty \mathcal{E}(\zeta)\exp\frac{i\omega\zeta}{s}\,d\zeta\right]\times\exp\frac{-i\omega z}{s}\,. \tag{3.13}$$

The condition just outlined that there be no reflected wave determines the coefficient B in the form

$$B=\frac{i\gamma e}{2Ms\omega}\int_0^\infty \mathcal{E}(\zeta)\exp\frac{i\omega\zeta}{s}\,d\zeta\,. \tag{3.14}$$

If one studies the generation of ultrasound in thin slabs this condition is not satisfied since, certainly, a reflected wave exists. We shall return to this problem in the future. We determine A by the requirement (3.12) together with the relation[35]

$$T_{xz}(0)=\rho s^2\xi'(z=+0)\,. \tag{3.15}$$

If the scattering of the electrons at the surface were specular, $T_{xz}(0)=0$, and the surface $z=0$ would be regarded as strain free $[\xi'(+0)=0]$. For diffuse scattering this is not the case. Combining Eqs. (3.3), (3.12), (3.14), and (3.15) we get

$$A=\frac{i\gamma e}{2Ms\omega}\int_0^\infty \mathcal{E}(\zeta)\exp\frac{i\omega\zeta}{s}\,d\zeta-(1-p)\frac{3ine}{4\rho s\omega}\int_{-1}^0(1-\mu^2)\,d\mu\int_0^\infty E(\zeta)\exp\frac{(1-i\omega\tau)\zeta}{l\mu}\,d\zeta\,. \tag{3.16}$$

This allows us to find the amplitude $\xi(\infty)$ of the acoustic wave in the bulk of the metal. Combining Eqs. (3.16) and (3.13) gives

$$\xi(z)=\xi(\infty)\exp(i\omega z/s)\,, \tag{3.17}$$

with

$$\xi(\infty)=\frac{i\gamma e}{Ms\omega}\left[\int_0^\infty \mathcal{E}(\zeta)\cos\frac{\omega\zeta}{s}\,d\zeta-\tfrac{3}{4}(1-p)\int_{-1}^0(1-\mu^2)\,d\mu\int_0^\infty E(\zeta)\exp\frac{(1-i\omega\tau)\zeta}{l\mu}\,d\zeta\right]. \tag{3.18}$$

We give a detailed study of these results in Sec. IV. It should be mentioned that $\xi(\infty)$ does not represent the true amplitude of the wave in the bulk. One must correct for the fact that it is attenuated by the factor

$$e^{-k_I z}\,, \tag{3.19}$$

where k_I, the imaginary part of the propagation vector, is given by

$$k_I=\frac{\gamma m}{2M\tau s}\,\frac{1-G}{G^2+a^2}(G+a^2)\,, \tag{3.20}$$

where

$$G=G(\omega/s) \tag{3.21}$$

stands for the Fourier component after setting $\alpha=0$ with wave number ω/s of the function defined in Eq. (2.7) and studied in Appendix A, and

$$a=c^2\omega/s^2\omega_p^2\tau\,. \tag{3.22}$$

When $\omega l/s \ll 1$ this is

$$k_I=\frac{\gamma m v_F^2\tau\omega^2}{10Ms^3}\,, \tag{3.23}$$

while if $\omega l/s \gg 1$ we have

$$k_I=\frac{2\gamma m v_F\omega}{3\pi Ms^2}\,. \tag{3.24}$$

IV. CALCULATION OF THE AMPLITUDE OF THE ACOUSTIC WAVE

The results of Sec. III can now be applied to calculate $\xi(\infty)$. As before we consider specular and diffuse scattering separately. For specular scattering the results have already been given by Alig.[14] We reproduce them here by a method that allows a rapid derivation. For diffuse scattering the calculations are more involved and require some developments which are relegated to Appendix B.

A. Specular scattering

Here, use of Eq. (3.18) gives

$$\xi(\infty)=\frac{i\gamma e}{Ms\omega}\int_0^\infty \mathcal{E}(\zeta)\cos\frac{\omega\zeta}{s}\,d\zeta=\frac{i\gamma e}{2Ms\omega}\mathcal{E}\left(\frac{\omega}{s}\right), \tag{4.1}$$

where $\mathcal{E}(\omega/s)$ is the Fourier component of $\mathcal{E}$ for $k=\omega/s$ and we have taken $\mathcal{E}(\zeta)=\mathcal{E}(-\zeta)$ for $\zeta<0$ as before. We now remember that

$$\mathcal{E}(k)=E(k)-\sigma_0^{-1}J_e(k)=[1-G(k)]E(k)\,, \tag{4.2}$$

which reduces to

$$\mathcal{E}(k)=-2E'(z=+0)[1-G(k)][\psi(k)-(\omega/c)^2]^{-1}\,, \tag{4.3}$$

after making use of Eq. (2.10). Since

$$\psi(k)=k^2-(4\pi i\omega\sigma_0/c^2)G(k)\ , \tag{4.4}$$

we can neglect $(\omega/c)^2$ compared to $(\omega/s)^2$ in the expression for $\xi(\infty)$. We have

$$\xi(\infty)=-\frac{i\gamma e}{Ms\omega}\left[1-G\left[\frac{\omega}{s}\right]\right]\psi^{-1}\left[\frac{\omega}{s}\right]E'(z=+0)\ . \tag{4.5}$$

It is customary to express these results in terms of the insertion rate η_1. This is defined as the ratio of the acoustic flux in the bulk of the metal to the incident electromagnetic flux. It is, therefore, a measure of the efficiency with which electromagnetic energy is converted into acoustic energy. We have

$$\eta_1=P_a/P_{\rm em}\ , \tag{4.6}$$

where

$$P_a=\tfrac{1}{2}\rho\omega^2|\xi(\infty)|^2 s \tag{4.7}$$

and

$$P_{\rm em}=\frac{c}{8\pi}|B(0)|^2=\frac{c^3}{8\pi\omega^2}|E'(z=+0)|^2\ . \tag{4.8}$$

Therefore,

$$\eta_1=\frac{\gamma m}{M}\left[\frac{\omega_p}{\omega}\right]^2\left[\frac{s}{c}\right]^3\frac{(1-G)^2}{1+(G/a)^2}\ . \tag{4.9}$$

In this expression for the insertion rate it is understood that G is to be evaluated at $k=\omega/s$. We recall that, when $\alpha=\omega\tau\ll 1$ which is certainly appropriate to our case,

$$G(k)=\frac{3}{2\beta^2}\left[\frac{1+\beta^2}{\beta}\arctan\beta-1\right]\ , \tag{4.10}$$

where

$$\beta=kl=\omega l/s\ . \tag{4.11}$$

Taking $\omega l/s=4.5$, $\omega=5.64\times10^7\ \sec^{-1}$, and the remaining parameters appropriate for potassium (see Sec. II) we obtain

$$(\eta_1)_s=4.45\times10^{-12}\ .$$

B. Diffuse scattering

Setting $p=0$ in Eq. (3.18) we obtain

$$\xi(\infty)=\frac{i\gamma e}{Ms\omega}\int_0^\infty \mathcal{E}(\zeta)\cos\frac{\omega\zeta}{s}\,d\zeta-\frac{3i\gamma e}{4Ms\omega}\int_{-1}^{0}(1-\mu^2)\,d\mu\int_0^\infty E(\zeta)\exp\frac{\zeta}{l\mu}\,d\zeta\ , \tag{4.12}$$

where we have made the approximation $\alpha=\omega\tau\ll 1$.

From Eq. (2.1) setting $J(z)=J_e(z)$ we can rewrite $\mathcal{E}(\zeta)$ entirely in terms of $E(\zeta)$ and its derivatives. After substitution in Eq. (4.12) and an integration by parts we obtain

$$\xi(\infty)=E'(z=+0)(\gamma e/Ms\omega)(D_1+D_2+D_3)\ , \tag{4.13}$$

where

$$D_1=-as^2/\omega^2\ , \tag{4.14}$$

$$D_2=-\frac{1+ia}{2\pi E'(z=+0)}\int_{-\infty}^{\infty}\frac{kE^{(-)}(k)\,dk}{k^2-(\omega/s)^2}\ , \tag{4.15}$$

and

$$D_3=-\frac{3}{4E'(z=+0)}\frac{1}{2\pi}\int_{-\infty}^{\infty}E^{(-)}(k)S(k)\,dk\ . \tag{4.16}$$

The function $S(k)$ is defined by

$$S(k)=(il/6x^4)[-4x^3+3x^2+6x-6(x^2-1)\ln(1-x)]\ , \tag{4.17}$$

with

$$x=ikl\ . \tag{4.18}$$

The function $S(k)$ has the following expansions valid for $|x|\ll 1$ and $|x|\gg 1$. If $|x|\ll 1$

$$S(k)=il(\tfrac{1}{4}+\tfrac{2}{15}x+\tfrac{1}{12}x^2+\tfrac{2}{35}x^3+\cdots)\ , \tag{4.19}$$

and, if $|x|\gg 1$,

$$S(k)=-\frac{i2l}{3x}+\cdots\ . \tag{4.20}$$

We now substitute $E^{(-)}(k)$ from Eqs. (2.29) and (2.32) in the form

$$E^{(-)}(k)=\frac{-iE(z=+0)X^{(+)}(k)(k^2-\omega^2/c^2)}{[\psi(k)-\omega^2/c^2](k-\omega/c-i\epsilon)}\ , \tag{4.21}$$

into Eqs. (4.15) and (4.16).

The function $X^{(+)}(k)$ is evaluated in Appendix B and we have, on neglecting ω^2/c^2 and using Eq. (B8),

$$E^{(-)}(k)=\frac{-iE(z=+0)ke^{I(k)}}{(k-k_1)(k-k_2)}\ . \tag{4.22}$$

The integrals

$$I_2\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk\,k^2e^{I(k)}}{(k-k_1)(k-k_2)(k^2-\omega^2/s^2)}\ , \tag{4.23}$$

and

$$I_3\equiv\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\frac{kS(k)e^{I(k)}}{(k-k_1)(k-k_2)}\ , \tag{4.24}$$

now appearing in D_2 and D_3 are evaluated numerically by integration along Rek. As shown in Fig. 1, the path of integration for I_2 is indented at $\pm\omega/s$ in a manner so as to account for the positive imaginary

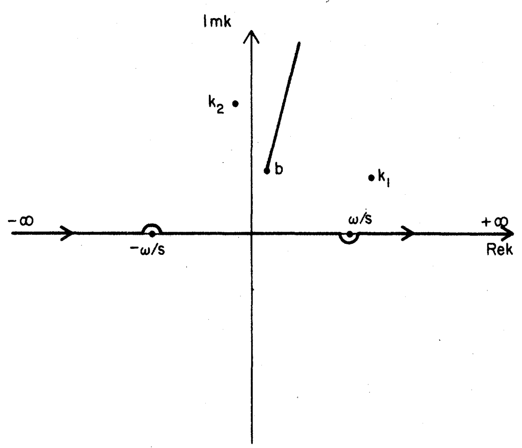

FIG. 1. Contour of integration for the integrals in Eqs. (4.15) and (4.16). The contour has been indented at $\pm\omega/s$ in order to include the damping associated with the frequency dependence.

part associated with ω. We obtain

$$I_2=(0.821-i2.91)\times 10^{-6}\text{ cm} \tag{4.25}$$

and

$$I_3=(-2.95+i4.23)\times 10^{-6}\text{ cm} . \tag{4.26}$$

With $E'(z=+0)/E(z=+0)$ given by Eq. (2.35) we evaluate the amplitude $\xi(\infty)$ of the acoustic excitation, and using Eqs. (4.6)–(4.8) we obtain the insertion rate for diffuse scattering

$$(\eta_1)_D=4.67\times 10^{-12} .$$

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APPENDIX A

1. Fourier transform of the conductivity kernel

The conductivity kernel $K(z)$ given by

$$K(z)=\frac{3\sigma_0}{4l}\int_1^\infty d\lambda\,\lambda^{-3}(\lambda^2-1)\exp\frac{-\lambda|z|(1-i\omega\tau)}{l}$$

has the Fourier transform

$$K(k)=\int_{-\infty}^{\infty}dz\,e^{-ikz}K(z)=i\frac{3\sigma_0}{4lk}\left[\frac{2b}{k}+\frac{k^2-b^2}{k^2}\ln\left[\frac{b+k}{b-n}\right]\right], \tag{A1}$$

where $b=(\omega\tau+i)/l$. We note that $K(k)$ is also expressible in the form given by Kjeldaas[36] since

$$\begin{aligned}K(k)&=\frac{3\sigma_0}{4l}\int_0^{\pi/2}\sin^3\theta\sec\theta\,d\theta\int_{-\infty}^{\infty}dz\,e^{-ikz}\exp\frac{-(1-i\omega\tau)|z|}{l\cos\theta}\\&=\frac{3\sigma_0}{4}\int_0^{\pi}\frac{\sin^3\theta\,d\theta}{1-i\omega\tau+ikl\cos\theta} .\end{aligned} \tag{A2}$$

The substitution $\cos\theta=\mu$, $\alpha=\omega\tau$, and $\beta=kl$ leads to

$$K(k)=\sigma_0\left[-\frac{3}{2}\frac{(1-i\alpha)}{\beta^2}+\frac{3}{4\beta^3}[\beta^2+(1-i\alpha)^2][\arctan(\beta-\alpha)+\arctan(\beta+\alpha)]-\tfrac{1}{2}i\ln\left[\frac{1+(\beta-\alpha)^2}{1+(\beta+\alpha)^2}\right]\right] . \tag{A3}$$

2. Phase of $\psi(k)$

The function $\psi(k)$ defined by

$$\psi(k)=k^2-\frac{4\pi i\omega}{c^2}K(k)$$

is analytically continued in the complex k plane from its value on the real k axis. For this purpose the form of $K(k)$ given in Eq. (A1) is suitable and it is then necessary to specify the principal branch of the logarithm appearing in Eq. (A1). The phases of the logarithm for $|k|>b$ on either side of the two branch cuts going from $\pm b$ to $\pm b\infty$, respectively, are determined by comparing Eq. (A1) with Eq. (A3) for $\alpha=\omega\tau\ll 1$, say. The cuts are then aligned along the imaginary axis, and in Eq. (A1) for Re$k>0$ we require $\ln[(b+k)/(b-k)]$ for $|k|>b$ to be $\ln[(k+b)/(k-b)]-i\pi$, while for Re$k<0$ we need $\ln[(k+b)/(k-b)]+i\pi$. For arbitrary α we then have, with $w=k/b$,

$$\psi(k)=b^2\left\{w^2+\frac{\Lambda\alpha^3}{(\alpha+i)^3}\left\{\frac{1}{w^2}+\frac{w^2-1}{2w^3}\left[\ln\left(\frac{w+1}{w-1}\right)\pm i\pi\right]\right\}\right\}, \quad \text{for } \mathrm{Im}\,w \gtrless 0\,,\quad |w|>1\,. \tag{A4}$$

3. Zeros of $\psi(k)$

The quantity $\Lambda=\frac{3}{2}(v_F\omega_p/c\omega)^2$, for $\omega\simeq 5.6\times 10^7$ Hz and with parameters appropriate for potassium, is 1.74×10^{11}. Also $\alpha=\omega\tau\simeq 9.3\times 10^{-3}$. Thus $\psi(k)$ has no roots for $|w|\leq 1$ because of the large value of $\Lambda\alpha^3$ ($\Lambda\alpha^3\simeq 1.4\times 10^5$). For large k (or equivalently, large $w=k/b$) the $\psi(k)$ is given by

$$\psi(k)=b^2\left\{w^2+\frac{\Lambda\alpha^3}{(\alpha+i)^3}\left[\pm i\pi\left(\frac{w^2-1}{2w^3}\right)+\frac{1}{w^2}+\frac{w^2-1}{w^3}\left(\frac{1}{w}+\frac{1}{3w^3}+\frac{1}{5w^5}+\cdots\right)\right]\right\}, \quad \text{for } \mathrm{Im}\,w\gtrless 0\,. \tag{A5}$$

At the frequency under consideraton $\alpha << 1$ so that the branch cuts in $\psi(k)$ are aligned with the imaginary k axis. Retaining only the leading term in $1/w$ we have

$$\psi(k)=b^2\left\{w^2\pm\frac{i\pi\Lambda\alpha^3}{2(\alpha+i)^3}\frac{1}{w}\right\},$$

so that with

$$k_0=(\tfrac{1}{2}\pi\Lambda)^{1/3}\left(\frac{\alpha}{l}\right),$$

the zeros of $\psi(k)$ are given by[3,37]

$$k_0\{e^{-i\pi/6}\,,\ e^{i\pi/2}\,,\ e^{-i5\pi/6}\}\,,\quad \text{for } \mathrm{Re}k<0$$

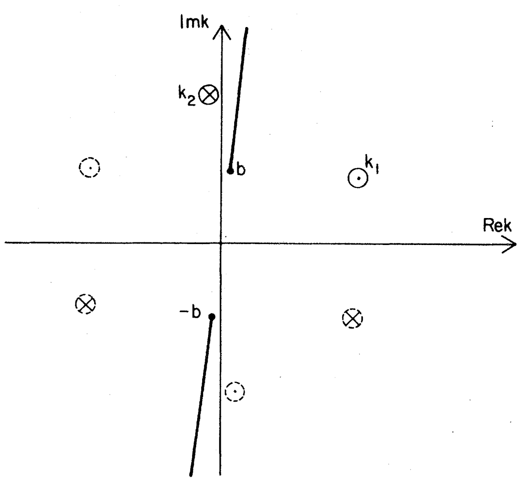

FIG. 2. Zeros of $\psi(k)$ and the branch cuts originating at $\pm b$ are shown. The zeros for $\mathrm{Re}k>0$ are indicated by circled dots, while the zeros for $\mathrm{Re}k<0$ are shown by circled crosses. Only the zeros at k_1 and k_2 are relevant to the discussion in this work.

and by

$$k_0\{e^{i\pi/6}\,,\ e^{-i\pi/2}\,,\ e^{i5\pi/6}\}\,,\quad \text{for } \mathrm{Re}k>0\,.$$

Only the roots in the upper-half k plane are needed and these are indicated in Fig. 2. They are

$$k_1=k_0e^{i\pi/6}=w_1b$$

and

$$k_2=k_0e^{i\pi/2}=w_2b\,.$$

In the numerical analysis the terms of order w^{-4} were retained and the values of w_1 and w_2 were determined in this approximation. The actual values of w_1,w_2 have been given in Sec. II.

APPENDIX B: EVALUATION OF THE FUNCTION $X(k)$

The function $X(k)$ appears in the expression for the electric field $E(k)$ in the diffuse scattering case. In the upper-half complex k plane we have

$$X^{(+)}(k)=\exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-k}\ln\left(\frac{\psi(k')}{k'^2}\right)\right]$$

$$\equiv e^{Y(k)}\,,\quad \mathrm{Im}k>0\,. \tag{B1}$$

The quantity in the exponent is evaluated by contour integration. Note that for $|k'|\to\infty$, $\psi(k')$ increases as k'^2 so that the integrand behaves like $1/k'$. In closing the contour over the real axis with a semicircle on the upper-half plane the latter yields a negligible contribution. However, there exists a branch cut originating at $k=b$ and extending to $b\infty$ due to the presence of the logarithmic term in $\psi(k')$ [Eqs. (2.11) and (2.13)], and two additional branch points associated with the zeros of $\psi(k')$. Let the latter be at k_1 and k_2. We have already seen in Appendix A that for low frequencies there are only two zeros of $\psi(k)$ in the upper-half complex k plane.

We proceed by explicitly removing the two zeros of

$\psi(k')$ by writing[34,38]

$$Y(k)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-k}\left[\ln\left(\frac{\psi(k')k'^2}{(k'^2-k_1^2)(k'^2-k_2^2)}\right)+\ln\left(\frac{(k'^2-k_1^2)(k'^2-k_2^2)}{k'^4}\right)\right],$$

$$\equiv Y_1(k)+Y_2(k)\ . \qquad \text{(B2)}$$

Using

$$\ln\left(\frac{k'^2-k_1^2}{k'^2}\right)=-\int_0^{k_1}\frac{2k_1dk_1}{k'^2-k_1^2}\ ,$$

we have for $Y_2(k)$,

$$Y_2(k)=\ln\left(\frac{(k+k_1)(k+k_2)}{k^2}\right). \qquad \text{(B3)}$$

Now we evaluate $Y_1(k)$ by closing the contour of integration by a semicircle in the upper-half plane with an indentation for going around the branch cut emanating from $k'=b$ (see Fig. 3).

With

$$\tilde{\psi}(k)=\frac{\psi(k)k^2}{(k^2-k_1^2)(k^2-k_2^2)}\ , \qquad \text{(B4)}$$

we have

$$Y_1(k)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dk'}{k'-k}\ln\tilde{\psi}(k')$$
$$=\ln\tilde{\psi}(k)+\int_{C_1+C_2}\frac{dk'}{k'-k}\ln\tilde{\psi}(k')\ , \qquad \text{(B5)}$$

where $\ln\tilde{\psi}(k)$ is the residue at $k'=k$. The contour C_1 refers to the semicircular contour at infinity on which the integrand tends to zero and the contour C_2 refers to the path around the branch cut from b to $b\infty$. The parts of C_2 above and below the branch cut can be combined into a single expression. The remaining small circular contour around the branch point can be shown to give a zero contribution by noting that the phases of $\tilde{\psi}^{(+)}(b)$ just above the branch cut and $\tilde{\psi}^{(-)}(b)$ just below the branch cut are the same.[37] This can be proved using Rouche's theorem.[39] Thus

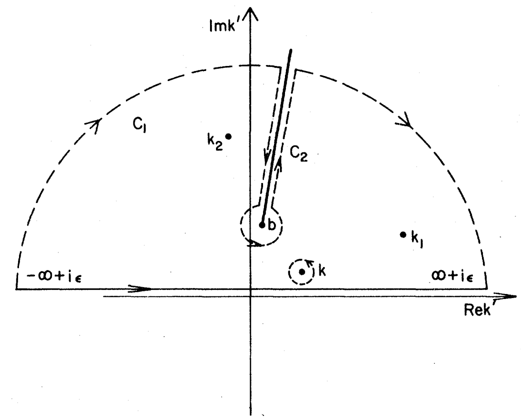

FIG. 3. Contour of integration for $Y(k)$ defined in Eq. (B1). The original contour from $-\infty+i\epsilon$ to $\infty+i\epsilon$ is distorted to one around the branch cut together with the circular contour around k. The positions of the zeros of $\psi(k)$ are indicated.

$$Y_1(k)=\ln\tilde{\psi}(k)+\int_b^{b\infty}\frac{dk'}{2\pi i(k'-k)}\ln\left(\frac{\tilde{\psi}^{(-)}(k')}{\tilde{\psi}^{(+)}(k')}\right)$$
$$\equiv\ln\tilde{\psi}(k)+I(k)\ . \qquad \text{(B6)}$$

Writing

$$Y(k)=\ln\left(\frac{(k+k_1)(k+k_2)}{k^2}\right)+\ln[\tilde{\psi}(k)]+I(k)\ , \qquad \text{(B7)}$$

we have from Eqs. (B4) and (B7)

$$X^{(+)}(k)=\frac{\psi(k)}{(k-k_1)(k-k_2)}e^{I(k)}\ . \qquad \text{(B8)}$$

The integral $I(k)$ is evaluated numerically for any value of k. We shall need $X^{(+)}$ for real k only.

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