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Quantum-statistical mechanics of extended objects. I. Kinks in the one-dimensional sine-Gordon system

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Making use of thermal-Green's-function technique, we study the quantum-statistical mechanics of a sine-Gordon system in $1+1$ dimensions. In the weak-coupling limit, the temperature dependences of the soliton energy, E_s, the soliton inertial mass, and the soliton density are determined. At high temperatures ($T > m$, where m is the mass of the fundamental field), E_s decreases monotonically as the temperature increases, and E_s jumps to zero around $T = T_{cr}$ ($\equiv e^{-1}E_s^0$), where E_s^0 is the soliton energy at $T=0$ K. The soliton density agrees with the classical statistical-mechanics results for $T_{cr} > T >> m$, if E_s in the classical theory is replaced by the temperature-dependent one of the present theory.

I. INTRODUCTION

Recently there has been great interest in kinks or extended objects in condensed-matter physics. Since a pioneering work by Krumhansl and Schrieffer[1] on kinks of the "ϕ^4" field theory, classical statistical mechanics has been applied to kinks of a variety of one-dimensional systems.[2] These analyses have firmly established kinks (i.e., localized nonlinear solutions) as a new class of elementary excitations in the one-dimensional system. Unfortunately, however, the above statistical-mechanics approaches are plagued by minor but persistent ambiguities. Furthermore in the case of the sine-Gordon system, inclusion of breathers as independent modes appears to completely destroy the excellent agreement between the exact transfer-matrix technique (TMT) result[3] and the heuristic ideal-gas model.[2] In parallel to the above development, there has been remarkable advance in the quantum field theory of nonlinear systems in $1+1$ dimensions.[4–8] In particular for the sine-Gordon system some exact results are known.[5,6]

The object of the present series of papers is to develop a quantum-statistical mechanics of the sine-Gordon system. We rely heavily on the earlier works on its quantum field theory. The finite-temperature effect is handled in terms of thermal Green's functions.[9] In the present paper we focus our attention on the kink (the soliton) related properties. In the second paper the breathers will be dealt with.

In Sec. II, we present the general formalism. We first divide the Hilbert space into sectors depending on the number of solitons N in the states. The partition function Z_0 of the $N=0$ sector and Z_1 of the $N=1$ sector are obtained within the weak-coupling limit. A moving soliton is also considered. From Z_0 and Z_1, we can derive m, E_s, E_l, and ζ_s, the physical mass of the fundamental boson (i.e., radiation), the soliton energy, the inertial mass of the soliton, and the fugacity of the soliton, which are discussed in Sec. III. We note that $E_s \neq E_l$ for $T \neq 0$ K, since the presence of thermal radiation breaks the Lorentz invariance of the system. In the higher-temperature region ($E_s > T >> m$), the above result is compared with the TMT results.[2,3]

The present result predicts a number of interesting corrections on the result obtained within the classical statistical mechanics. Of particular interest is the correction to the cos-cos correlation function obtained by Mikeska,[10] within the classical theory. Recent neutron scattering experimental data on the one-dimensional ferromagnet $CsNiF_3$ by Kjems and Steiner[11] are reanalyzed in the light of the present theory.

II. FORMULATION

In the field-theoretical study of the nonlinear system in $1+1$ dimensions, we generally encounter two principal problems. First, we have to renormalize a variety of divergences in a consistent way, in order to obtain finite sensible results, as the fluctuation of the field is usually divergent. Second, we have to deal with nonlinear modes (i.e., kinks) of the system, if any, which are not accessible by means of a simple perturbational analysis with respect to the small coupling constant.

Fortunately, in the case of the sine-Gordon system, these two problems are completely solved at zero temperature by Coleman[6] and by Dashen, Hasslacher, and Neveu (DHN).[5] In this section we shall capitalize on these results and build up the quantum-statistical mechanics of the sine-Gordon system.

The sine-Gordon system is described by the following Hamiltonian:

$$H=\frac{1}{2}\int dx\left[\pi^2(x)+\left(\frac{\partial\phi}{\partial x}\right)^2 -\frac{2(m^*)^2}{g^2}\cos g\phi+\frac{2m_0^2}{g^2}\right] , \qquad (1)$$

where $\pi(x)=\partial\phi/\partial t$, and m^* and g being the bare mass of the Bose field $\phi(x)$ and the coupling constant, respectively. We took here c_0 the limiting boson velocity as unity for simplicity. Also for later convenience we have added a constant term $2m_0^2g^{-2}$, with m_0 the physical mass of the ϕ field at the zero temperature.

First of all we shall note that the sine-Gordon system has infinitely degenerate vacua given by $\phi=0, \pm 2\pi g^{-1}, \pm 4\pi g^{-1},\ldots$. Furthermore, the topological conservation[7] of the system demands that the difference $\phi(x=+\infty)-\phi(x=-\infty)$ is independent of time and given by $2\pi N g^{-1}$ with N an integer. This follows simply from the fact that a local perturbation cannot change $\phi(x=+\infty)$ or $\phi(x=-\infty)$. Therefore, in order to investigate physical properties of the system, we can classify all possible states in terms of this integer N. The interger N corresponds to the number of the solitons N_s (or more precisely, $N_s-N_{\bar{s}}$, where $N_{\bar{s}}$ is the number of the antisolitons) in the state. We call the part of Hilbert space specified by N as sectors. We shall now examine the $N=0$ sector and $N=1$ sector separately. Note that intrinsic properties of a single soliton are extracted from the ratio Z_1/Z_0 where Z_1 and Z_0 are the partition functions of the sector $N=1$ and the sector $N=0$, respectively.[5]

A. Soliton-free state ($N=0$)

In this case, following Coleman,[6] we can completely eliminate the divergences from the simple boson loops even at finite temperatures. For this purpose it is convenient to introduce the "normal product" at finite temperatures as at $T=0$ K. This is defined as

$$N(A)=A-A_{\text{pairing}} , \qquad (2)$$

where all possible pairings[9] in the operators ϕ in A are subtracted. In particular $\langle N(A)\rangle=0$, if A does not contain a c-number term. Furthermore, we have

$$N([\phi(x)]^2)=[\phi(x)]^2-D , \qquad (3)$$

where

$$D=\langle[\phi(x)]^2\rangle \qquad (4)$$

the thermal Green's function of ϕ at the equal space time. At $T=0$ K, Eq. (2) reduces to the standard definition in the quantum field theory.[6] Then Eq. (1) is transformed into

$$H=\frac{1}{2}\int dx\left[\pi^2(x)+\left(\frac{\partial\phi}{\partial x}\right)^2 -\frac{2m^2}{g^2}N(\cos g\phi)+\frac{2m_0^2}{g^2}\right] , \qquad (5)$$

where

$$m^2=(m^*)^2\exp[-(\tfrac{1}{2}g^2)D] . \qquad (6)$$

Assuming that $\phi(x)$ is a free boson with mass m, D is evaluated as

$$D=T\sum_\nu\int\frac{dk}{2\pi}D^{(0)}(k,\omega_\nu) , \qquad (7)$$

where $D^{(0)}(k,\omega_\nu)=(\omega_\nu^2+k^2+m^2)^{-1}$ is the thermal Green's function of the free boson and $\omega_\nu=2\pi T\nu$ the Matsubara frequency with integer ν. In the diagrammatic language, we have renormalized all the interaction vertices in Eq. (1) by closing two dangling lines into a loop in all possible ways. For example the renormalized mass vertex in Eq. (5) is shown in Fig. 1(a). If we assume that the mass m in Green's function is the same as that in the left-hand side of Eq. (6), Eq. (6) gives a self-consistent equation for m.

Introducing a cutoff momentum Λ, Eq. (6) is evaluated as

$$D=\frac{1}{4\pi}\int_{-\Lambda}^{\Lambda}dk\,\frac{1}{\omega_k}\coth(\tfrac{1}{2}\beta\omega_k)$$
$$=\frac{1}{2\pi}\ln\left(\frac{2\Lambda}{m}\right)+f_0(\beta m) \qquad (8)$$

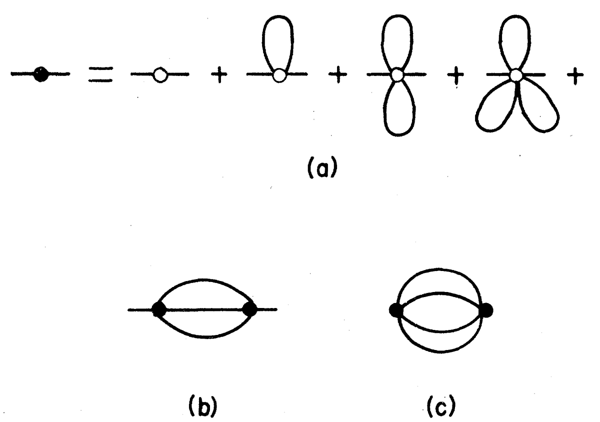

FIG. 1. Diagrammatic representations of the present analysis; the renormalized mass vertex m (a), and the higher-order corrections to m (b) and Ω_0 (c). The white and black circles represent the bare and renormalized interaction vertices, respectively.

and

$$f_0(\beta m) = \frac{1}{\pi}\int_0^\infty dk \frac{1}{\omega_k} N(\omega_k) = \frac{1}{\pi}\sum_{n=1}^{\infty} K_0(n\beta m) \quad , \tag{9}$$

where $\omega_k = (k^2+m^2)^{1/2}$, $\beta = (k_B T)^{-1}$, $N(E) = (e^{\beta E}-1)^{-1}$ the Bose distribution function, and $K_0(z)$ is the modified Bessel function. At $T=0$ K, Eq. (8) reduces to

$$D_0 = \frac{1}{2\pi}\ln\left[\frac{2\Lambda}{m_0}\right] \quad , \tag{10}$$

where m_0 is the mass of the boson at $T=0$ K.

Substituting Eq. (10) into Eq. (6), we have at $T=0$ K

$$m_0^2 = (m^*)^2 \exp\left[-\frac{g^2}{4\pi}\ln\left[\frac{2\Lambda}{m_0}\right]\right] \quad . \tag{11}$$

This relation obtained by Coleman[6] relates the physical mass at $T=0$ K to the bare mass m^*. Now making use of Eq. (11), we can completely eliminate the divergence from Eq. (6). Expressing m^* in terms of m_0, we have

$$m^2 = m_0^2 \exp[-(\tfrac{1}{2}g^2)D'] \quad , \tag{12}$$

where $D' = D - D_0$, which is free of the divergence. Finally making use of Eq. (8) the mass m at finite temperatures is given by

$$m = m_0 \exp[-(\tfrac{1}{4}g'^2) f_0(\beta m)] \quad , \tag{13}$$

with

$$g'^2 = \frac{g^2}{1-g^2/8\pi} \quad . \tag{14}$$

Equation (13) shows that at finite temperatures the physical mass depends on temperature and decreases as the temperature is increased. This is one of our principle results in the present analysis.

Making use of Eq. (5), the thermodynamic potential of the $N=0$ sector is given by

$$\Omega_0 (\equiv -\beta^{-1}\ln Z_0) = \beta^{-1}\sum_k \ln(2\sinh\tfrac{1}{2}\beta\omega_k) - \left[(m^2-m_0^2)g^{-2} + \frac{m^2}{2}D\right]L + O(g^4) \quad , \tag{15}$$

where Z_0 is the partition function of the $N=0$ sector and L is the length of the system. Substracting from Eq. (15) the ground-state energy E_0 at $T=0$ K,

$$E_0 = \frac{1}{2}\sum_k \omega_k^0 - \frac{1}{2}m_0^2 D_0 L \quad , \tag{16}$$

where $\omega_k^0 = (k^2+m_0^2)^{1/2}$, we obtain

$$\frac{\Omega_0 - E_0}{L} = \beta^{-1}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\ln(1-e^{-\beta\omega_k}) + \frac{g'^2}{8}m^2 f_0^2 + O(g'^4) \quad , \tag{17}$$

where the first term is an ordinary thermodynamic potential for a free boson system and the second term gives a finite correction due to interaction for small g'^2. Therefore the mass renormalization completely eliminates the divergence from the theory at least for the $N=0$ sector.

The above results for m and Ω_0 are in fact the lowest-order terms in perturbation expansion in powers of g^2. The next-order corrections of the order of g^4 to m and Ω_0 are shown in Fig. 1(b) and Fig. 1(c), respectively. All these contributions are convergent and therefore neglected in the present work. Therefore our results are exact in the weak-coupling limit.

B. One-soliton state ($N=1$)

As is well known the sine-Gordon equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{\hat{m}^2}{g}\sin g\phi = 0 \quad , \tag{18}$$

which can be derived from Eq. (1) allows a class of classical solutions of the form

$$\phi_s = \frac{4}{g}\tan^{-1}\left[\exp\frac{\hat{m}(x-vt)}{(1-v^2)^{1/2}}\right] \quad , \tag{19}$$

which describes a moving soliton with velocity v. Here we have replaced m^* in Eq. (1) by $\hat{m}$, which will be determined self-consistently (see Sec. II A).

According to DHN,[5] the $N=1$ sector is treated by substituting for ϕ in Eq. (1) by

$$\phi = \phi_s + \hat{\phi} \quad , \tag{20}$$

where ϕ_s is the classical solution for a single soliton. In this subsection let us consider a static soliton; the soliton with $v=0$.

Substituting Eq. (20) into Eq. (1), we obtain

$$H = E_s^{\rm Cl} + \frac{1}{2}\int dx\left[\hat{\pi}^2(x) + \left[\frac{\partial\hat{\phi}}{\partial x}\right]^2 + \hat{m}^2(\cos g\phi_s)\hat{\phi}^2\right] + \int dx\left[\frac{m_0^2-\hat{m}^2}{g^2} - \frac{(m^*)^2}{g^2}\cos[g(\phi_s+\hat{\phi})] + \frac{\hat{m}^2}{g^2}\cos(g\phi_s) - \frac{\hat{m}^2}{g}\sin(g\phi_s)\hat{\phi} - \frac{\hat{m}^2}{2}\cos(g\phi_s)\hat{\phi}^2\right] \quad , \tag{21}$$

where $\hat{\pi}(x) = \partial\hat{\phi}/\partial t$ and E_s^{Cl} is the classical soliton energy given by

$$E_s^{\text{Cl}} \equiv \frac{1}{2}\int dx\left[\left(\frac{\partial\phi_s}{\partial x}\right)^2 + \frac{2\hat{m}^2}{g^2}(1-\cos g\phi_s)\right] = \frac{8\hat{m}}{g^2}\ , \tag{22}$$

The field $\hat{\phi}$ is now expanded in terms of the normal modes of the eigen equation

$$\omega_n^2 u_n(x) = -\frac{\partial^2}{\partial x^2}u_n(x) + \hat{m}^2\cos[g\phi_s(x)]u_n(x) \tag{23}$$

of which normal modes are well known.[12] In particular, there is one bound state with $\omega_b = 0$ and scattering states with $\omega_k = (k^2+\hat{m}^2)^{1/2}$ where k is the wave number of the mode.

Making use of the normal mode, we can carry out the mass renormalization as in Sec. II A. For $\hat{m}$, we have an equation identical to Eq. (6), except that D in Eq. (6) is replaced by $D_s(x)$ the equal space-time propagator for the $\hat{\phi}$ field. $D_s(x)$ is calculated in Appendix A as

$$D_s(x) = D - \left(\frac{1}{2\pi} + f_1(\beta m)\right)\text{sech}^2(\hat{m}x) \tag{24}$$

and

$$f_1 = \frac{m^2}{\pi}\int_0^\infty dk\ \omega_k^{-3}N(\omega_k)\ . \tag{25}$$

The first term in Eq. (24) is the propagator for the free boson with mass $\hat{m}$. Therefore $\hat{m}$ has to be identical to m, except in a narrow region around the static soliton (i.e., $x=0$). Since $\hat{m}$ deviates from m in the narrow region, we can put $\hat{m}=m$, as long as we neglect terms of the order of L^{-1}. Although this argument is sufficient for $\hat{m}$, a more careful analysis is required to calculate the soliton energy, as the soliton energy is expressed as a difference of two quantities of the order of L. The details of this analysis are again given in Appendix A. Finally the thermodynamic potential of the $N=1$ sector is given by

$$\Omega_s \equiv \Omega_1 - \Omega_0 = \frac{8m}{g^2} + \beta^{-1}\left[\sum_n \ln\left(2\sinh\frac{\beta}{2}\omega_n\right) - \sum_k \ln\left(2\sinh\frac{\beta}{2}\omega_k\right)\right] + \frac{m^2}{2}\int dx\,(1-\cos g\phi_s)D\ , \tag{26}$$

where $\Omega_1 \equiv -\beta^{-1}\ln(Z_1|_{v=0})$, and the sum over n has to be carried out over all the eigenmodes with eigenfrequency ω_n of Eq. (23). Here we have subtracted Ω_1 by Ω_0 the thermodynamic potential of the $N=0$ sector.

In the dilute limit the probability of one soliton state in the grand canonical ensemble is given by

$$n_s = \frac{Z_1}{Z_0+Z_1+\ldots} \simeq \frac{Z_1}{Z_0}\ . \tag{27}$$

In particular, the probability of one soliton state with $v=0$ (or $p=0$ where p is the momentum) is given by

$$n_s(0) = \exp(-\beta\Omega_s)\ . \tag{28}$$

The second term in Eq. (26) is transformed following DHN (see also Appendix A) as

$$\sum_n \ln(2\sinh\tfrac{1}{2}\beta\omega_n) - \sum_k \ln(2\sinh\tfrac{1}{2}\beta\omega_k) = -\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} dk\,\Delta(k)\frac{\partial}{\partial k}\ln(2\sinh\tfrac{1}{2}\beta\omega_k)\ , \tag{29}$$

where

$$\Delta(k) = 2\tan^{-1}(m/k) \tag{30}$$

is the phase shift of the $\hat{\phi}$ field with wave number k scattered by the static soliton. The integral in Eq. (29) is logarithmically divergent, which we have cut off at $|k| = \Lambda$, consistently with Eq. (8). Then as in the case of zero temperature[5] the logarithmic divergence of Eq. (29) is exactly cancelled with the last term in Eq. (26). Substituting Eqs. (29) and (30) into Eq. (26), we can simplify Eq. (26) as

$$\Omega_s = \frac{8m}{g'^2} + 2mf_0 + 2Tf_2\ , \tag{31}$$

where g'^2 and f_0 have been already defined in Eqs. (14) and (9), respectively, and

$$f_2(\beta m) = -\frac{m}{\pi}\int_0^\infty dk\ \omega_k^{-2}\ln(1-e^{-\beta\omega_k})\ , \tag{32}$$

which implies

$$-\frac{\partial f_2(z)}{\partial z} = f_0(z)\ . \tag{33}$$

At $T=0$ K, Ω_s is nothing but the soliton energy

$$E_s^0 = \frac{8m_0}{g'^2}\ , \tag{34}$$

which is one of the DHN results.

At finite temperatures, we can extract the soliton energy from Ω_s by

$$E_s = \Omega_s - T\left(\frac{d\Omega_s}{dT}\right) = \frac{8m}{g'^2}\ . \tag{35}$$

In arriving at the last expression, we have made use of Eq. (13) as well as Eq. (33).

Equation (35) is surprisingly simple; the soliton energy is given by the same expression as that at $T=0$ K, except that the boson mass m_0 is replaced by the temperature-dependent (self-consistent) mass.

C. Moving-soltion state ($N=1$)

At the absolute zero of temperature, the energy of the moving soliton is readily obtained by the Lorentz transformation

$$E_s^0(v)=\frac{E_s^0}{(1-v^2)^{1/2}}\simeq E_s^0(1+\tfrac{1}{2}v^2) \ . \tag{36}$$

However, at finite temperatures, this is not necessarily true, as the thermal bosons establish a preferred frame (i.e., $v=0$). Therefore it is worthwhile to check the effect of this background fluctuation on a moving soliton. For this purpose as a classical solution, we take a time-dependent solution given in Eq. (19), then as in Sec. II B, we work out the fluctuation corrections to the classical solution. In our calculation it is very convenient to express everything in the rest frame of the soliton. Denoting the operators in the rest frame of the soliton by primes, we have, for example

$$H=\frac{H'+vP'}{(1-v^2)^{1/2}} \ , \tag{37}$$

where H is the Hamiltonian and P is the momentum operator

$$P=\int dx \, \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} \ . \tag{38}$$

The thermodynamic potential for a moving soliton with velocity v is given by

$$\Omega_s(v)\equiv\Omega_1(v)-\Omega_0$$
$$=\frac{8m}{g^2}(1-v^2)^{-1/2}+\frac{L}{2\pi}\int_{\Lambda}^{\Lambda}dk'\,\Delta(k')\frac{\partial}{\partial k'}\left[\ln\left(2\sinh\frac{\beta}{2}\omega_k\right)\right]+2m(1-v^2)^{-1/2}D \ , \tag{39}$$
$$=\frac{8m}{g'^2}(1-v^2)^{-1/2}+2m(1-v^2)^{-1/2}f_0+2Tf_2(v) \ , \tag{40}$$

where

$$\omega_k=(\omega_k'+vk')(1-v^2)^{-1/2} \ ,$$
$$f_2(v)=-\frac{m}{2\pi}(1-v^2)^{1/2}\int_{-\infty}^{\infty}dk\,[\omega_k(\omega_k+vk)]^{-1}\ln(1-e^{-\beta\omega_k}) \ . \tag{41}$$

In deriving the above expression, we have taken into account the fact that in the moving frame the length of the system is contracted by a factor $(1-v^2)^{1/2}$ [i.e., $L'=(1-v^2)^{1/2}L$] and that the mass of the ϕ field is the same as that in the rest frame.

The energy of the moving soliton with velocity v is then given as before

$$E_s(v)\equiv\Omega_s(v)-T\left(\frac{d\,\Omega_s(v)}{dT}\right)$$
$$=E_s(1-v^2)^{-1/2}+\delta E_s(v) \ , \tag{42}$$

where

$$\delta E_s(v)=2m(1-v^2)^{-1/2}f_0-2T^2\frac{\partial f_2(v)}{\partial T} \ . \tag{43}$$

For small v ($<<1$), Eq. (40) can be rewritten

$$E_s(v)=E_s+\tfrac{1}{2}E_I v^2 \ , \tag{44}$$

where

$$E_I=E_s+4mf_1 \tag{45}$$

and f_1 has been defined in Eq. (25). Since $f_1>0$, Eq. (45) implies $E_I>E_s$. However, it should be remembered that Eq. (45) does not mean that there are two distinct masses involved. Rather Eq. (45) implies that at finite temperatures the limiting velocity of the soliton is reduced by a factor $(E_s/E_I)^{1/2}$, although we shall refer to E_I as the inertial mass of the soliton.

Sometimes it is more convenient to write the soliton energy as a function of momentum p. In this case Eq. (44) can be written $E_s(p)=E_s+\frac{1}{2}E_I^{-1}p^2$ and

$$p=E_I v \ . \tag{46}$$

A final remark is in order; as one can see from Eqs. (13), (17), (31), and (35) in the weak-coupling limit, the proper expansion parameter appears to be g'^2 rather than g^2. This is again in quite harmony with an observation of DHN at $T=0$ K.

III. SOLITONS AT FINITE TEMPERATURES

A. Energy, inertial mass, and fugacity

The soliton energy at finite temperatures is given by Eq. (35); the temperature dependence is exactly given by that of the ϕ field mass m. Within the weak-coupling limit m is given self-consistently by Eq. (13). In this limit m deviates significantly from the zero-temperature value m_0 only when $T >> m$. In this limit f_0 in Eq. (9) is expanded (see Appendix B) as

$$f_0 = \tfrac{1}{2}(\beta m)^{-1} + \frac{1}{2\pi}\ln\left(\frac{\gamma\beta m}{4\pi}\right) - \frac{\zeta(3)}{2(2\pi)^3}(\beta m)^2 \ , \tag{47}$$

where $\gamma = 1.76...$ the Euler constant and $\zeta(3)$ is Riemann's zeta function.

The self-consistent equation is then approximately given by

$$R = \exp[-(T/E_s^0)R^{-1}] \ , \tag{48}$$

where $R \equiv m/m_0$ and E_s^0 is the soliton energy at the absolute zero temperature. The mass ratio R decreases monotonically with increasing temperature. But around $T = T_{cr}$ $(\simeq e^{-1}E_s^0)$, R drops discontinuously to zero, where E_s^0 is the soliton energy at $T = 0$ K.[13] This feature is clearly seen in Fig. 2, where R obtained numerically starting from Eq. (13) for some values of g^2 are shown. The critical temperature, at which the discontinuous drop occurs, appears to increase slightly as g^2 is increased. Of course, the details of temperature dependence of R near $T = T_{cr}$ can be quite different, as our weak-coupling approximation starts to break down around this temperature region [the expansion parameter is no longer g^2 itself but rather $g^2(\beta m)^{-1}$ in this temperature region]. However, the above behavior strongly suggests that the cosine potential of the ϕ field is completely washed out in this temperature region ($T \simeq T_{cr}$), due to the fluctuation of the ϕ field itself; the potential necessary for the soliton and the mass of the ϕ field may disappear around $T \simeq T_{cr}$. In order to characterize completely the temperature dependence of the mass ratio R near $T = T_{cr}$, further work is certainly required.

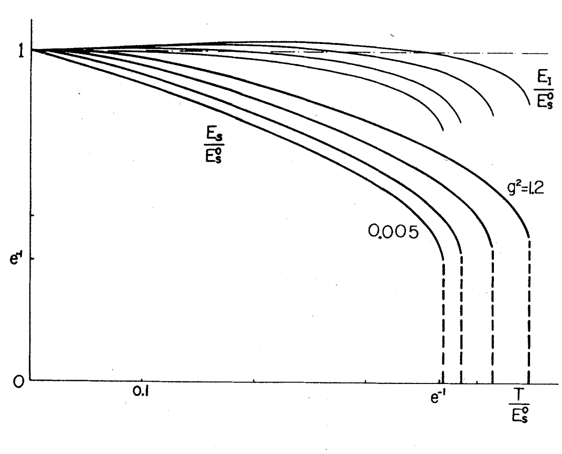

FIG. 2. Temperature dependences of E_s (thick line) and E_I (thin line). Both lines are from the top for $g^2 = 1.2$, 0.62, 0.2, and 0.005.

The inertial mass of the soliton is given, on the other hand, by Eq. (45). Making use of the high-temperature expansion of f_1,

$$f_1 = \tfrac{1}{4}[(\beta m)^{-1} - 2/\pi + \tfrac{1}{3}(\beta m)^2 + ...] \ , \tag{49}$$

we obtain for $\beta m << 1$,

$$E_I = E_s^0\left(\frac{2\pi}{\gamma\beta m}\right)^{g'^2/8\pi} - \frac{2}{\pi}m - \tfrac{1}{2}(RE_s^0)^{-1}T^2 \simeq E_s^0 \ . \tag{50}$$

Therefore unlike the soliton energy E_s, the inertial mass is only weakly temperature dependent except for near T_{cr}. E_I/E_s^0 are also shown for some values of g^2 as functions of T/E_s^0 in Fig. 2.

From $\Omega_s^{(v)}$ obtained in Eq. (40), we can calculate the chemical potential of the soliton

$$\mu_s^{(v)} \simeq \mu_s + v^2 m f_1 = -T\left(\frac{d\,\Omega_s(v)}{dT}\right) = TS_s$$
$$= -2T\left(f_2(v) + T\frac{df_2(v)}{dT}\right) \ , \tag{51}$$

or fugacity

$$\zeta_s(v) = e^{\beta\mu_s} = \zeta_s e^{\beta m v^2 f_1} \ . \tag{52}$$

Here μ_s and ζ_s are those for $v = 0$, which are exponentially small at low temperatures ($T << m$). At high temperatures ($T >> m$), on the other hand, we have

$$\mu_s = T\left(\ln(2\beta m) - 1 + \frac{2}{3}\frac{\zeta(3)}{(2\pi)^3}(\beta m)^3\right) \tag{53}$$

and

$$\zeta_s = \left(\frac{2\beta m}{e}\right)[1 + O((\beta m)^3)] \ , \tag{54}$$

where we have made use of the asymptotic expansion of f_2 for $\beta m << 1$ (see Appendix B),

$$f_2 = -\tfrac{1}{2}\ln(2\beta m) - \frac{1}{2\pi}(\beta m)\ln\left(\frac{\gamma\beta m}{4\pi}\right) + \frac{1}{6}\frac{\zeta(3)}{(2\pi)^3}(\beta m)^3 \ . \tag{55}$$

B. Soliton density

We have already seen that the probability of finding one soliton with zero velocity is given by $n_s(0)=e^{-\beta\Omega_s}$ in the dilute soliton limit. Similarly the probability of finding one soliton with velocity v is given as

$$n_s(v)=e^{-\beta\Omega_s(v)} , \qquad (56)$$

where $\Omega_s(v)$ is given in Eq. (40).

Then the total probability of finding one soliton (i.e., the soliton density) is given by

$$\bar{n}_s=\frac{1}{2\pi}\int dp\; n_s(v)=\frac{\zeta_s}{(2\pi)^{1/2}}\left(\frac{E_l}{\beta}\right)^{1/2}\left(\frac{E_l}{E_l{}^*}\right)^{1/2}e^{-\beta E_s} , \qquad (57)$$

where ζ_s, E_l, and E_s have been already given. The new parameter $E_l{}^*$ arises from the v dependence of the fugacity and is given by

$$E_l{}^*=E_l-2mf_1\simeq E_s^0-\tfrac{1}{2}T , \qquad (58)$$

where the last expression is for $\beta m \ll 1$.

First, at low temperatures ($T \ll m$) Eq. (57) reduces to

$$\bar{n}_s=\left(\frac{E_s^0}{2\pi\beta}\right)^{1/2}e^{-\beta E_s^0} , \qquad (59)$$

which agrees with the one obtained by Trullinger.[14] He has made use of a heuristic ideal gas model. At high temperatures ($T \gg m$) on the other hand, we have

$$\bar{n}_s=\frac{2m}{e}\left(\frac{\beta E_l}{2\pi}\right)^{1/2}\left(\frac{E_l}{E_l{}^*}\right)^{1/2}e^{-\beta E_s} . \qquad (60)$$

This agrees within a numerical factor, with the result of the classical statistical mechanics,[2] if we replace E_s, $E_l{}^*$, and E_l by the temperature-independent soliton energy E_s^0. The important difference between the classical result and the present result is the fact that E_s in the exponent is the temperature-dependent soliton energy. In the classical statistical mechanics, the fluctuations of the ϕ field, which reduces E_s at high temperatures, are completely neglected. Another minor difference is an overall numerical factor of $1/e$, of which the origin is not clear at the moment.

C. Correlation function

The present technique is easily applied to calculate dynamical correlation functions of the ϕ field. Of particular interest is the cos-cos correlation function defined by

$$S(q,\omega)=\frac{1}{(2\pi)^2}\int\int dt\,dx\,\exp[i(qx-\omega t)] \times\langle\cos[g\phi(x,t)] \times\cos[g\phi(0,0)]\rangle . \qquad (61)$$

Here we focus our attention to the contribution to $S(q,\omega)$ arising from the scattering of the soliton $S_s(q,\omega)$. Making use of the soliton density given in Eq. (56), we obtain

$$S_s(q,\omega)\propto\frac{\pi E_l}{32m^2q}\left(\frac{2m}{eT}\right)e^{-\beta E_s} \times\left[\frac{16}{\pi}\exp\left(-\frac{E_l{}^*\omega^2}{4Tq^2}\right)\frac{\pi q/2m}{\sinh(\pi q/2m)}\right]^2 , \qquad (62)$$

which reduces to Mikeska's classic result,[9] if we put $E_s=E_l=E_s^0$. Equation (62) gives the half-width for ω at half maximum (HWHM) ω_H and the integrated intensity $S_s(q)$ of the structure factor $S_s(q,\omega)$ of Eq. (62),

$$\omega_H=(2\ln 2)^{1/2}\left(\frac{T}{E_l{}^*}\right)^{1/2}q \qquad (63)$$

and

$$S_s(q)\propto 8\left(\frac{2}{\pi}\right)^{1/2}\frac{\zeta_s}{m^2}(TE_l)^{1/2}\left(\frac{E_l}{E_l{}^*}\right)^{1/2} \times\left(\frac{\pi q/2m}{\sinh(\pi q/2m)}\right)^2 e^{-\beta E_s} . \qquad (64)$$

The HWHM is determined by $E_l{}^*$, while the exponential factor of $S_s(q)$ is controlled by E_s. The present theory predicts that at high temperatures ω_H deviates slightly from the classical result[10] (note $E_l\simeq E_s^0$), while the exponential term in $S_s(q)$ can be much larger than the classical result in particular at the high-temperature region ($T \gg m$).

IV. CONCLUDING REMARKS

Extending the field theoretic approach by Coleman and by DHN to finite temperatures, we have studied the quantum-statistical mechanics of the sine-Gordon system in 1+1 dimensions. We find that with the single mass renormalization, the divergences of the system are completely eliminated. This results in a self-consistent equation for m the mass of the ϕ field. The quantum correction to the soliton energy is obtained. We find at all temperatures $E_s=8m/g'^2$ with

$$g'^2=g^2\Big/\left(1-\frac{1}{8\pi}g^2\right) .$$

This is a simple extension of the DHN result. We have also calculated the soliton density, which agrees at high temperatures with the result of the classical statistical mechanics, if the soliton masses in the classical theory are reinterpreted as E_s, E_l, and E_l^*.

Recently very interesting neutron scattering data for quasi-one-dimensional ferromagnet $CsNiF_3$ were reported by Kjems and Steiner.[11] The observed central peaks in the presence of high magnetic fields were analyzed in terms of the classical theory of Mikeska.[10] They obtained a rather satisfactory agreement for the HWHM, by making use of the value $E_s^0 = 34$ K and $g^2 = 0.62$. However, they found that the integrated intensity was better described with $E_s = 27$ K rather than $E_s^0 = 34$ K. In fact the present theory predicts $E_s \simeq 25$ K at $T = 10$ K, if we choose $E_s^0 = 34$ K and $g^2 = 0.62$. Therefore the above discrepancy may be accounted for by the quantum corrections. In any case further experiments are certainly desirable.

If we limit ourselves to $CsNiF_3$, a more direct comparison with the present theory can be done either by measuring the spin-wave energy gap m as a function of magnetic fields or temperatures. Alternatively the magnetization in $CsNiF_3$ in high magnetic fields is also proportional to m. Therefore the predicted temperature dependence of m should be directly accessible to experiments.

Note added in proof. The discontinuous change in the soliton energy found in this work is very likely the artifact of the single-loop (or the self-consistent harmonic phonon) approximation used here. What actually takes place around T_{cr} in this one-dimensional system appears to be a crossover (continuous change) from a symmetry-broken state at lower temperatures to a symmetric state at higher temperatures. In a separate publication we will reexamine the role of solitons in such crossover behavior in the one-dimensional system.

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APPENDIX A: EQUAL-SPACE-TIME GREEN'S FUNCTION IN THE PRESENCE OF A SOLITON

We shall derive here Eq. (24) and Eq. (26) for the $N=1$ sector. First let us note that the normal modes of Eq. (23) consist of one bound state $u_b \propto \operatorname{sech}\hat{m}x$, with $\omega_b = 0$ and a set of scattering states, which is given by[12]

$$u_k = \frac{1}{\sqrt{L}} C_k e^{ikx}(k + i\hat{m}\tanh\hat{m}x) \quad , \tag{A1}$$

with $\omega_k = (k^2 + \hat{m}^2)^{1/2}$, and the normalization constant C_k up to $O(L^{-1})$ is

$$C_k = \left[\omega_k^2 - \frac{2\hat{m}}{L}\right]^{-1/2} \quad . \tag{A2}$$

The allowed values of k are defined by the Born–von Karman boundary condition, which for the scattering states, is given by[5]

$$Lk_n + \Delta(k_n) = 2\pi n \quad , \tag{A3}$$

n being an integer, and the phase shift $\Delta(k)$ is given by Eq. (30). The field operator $\hat{\phi}$ is then expanded as

$$\hat{\phi}(x) = \sum_n \left[\frac{1}{2\omega_n}\right]^{1/2} [a_n u_n{}^*(x) + a_n^\dagger u_n(x)] \quad ,$$

where $\omega_n = \omega_{k_n}$. Here we have discarded the bound-state mode $u_b(x)$, since this mode represents the translational degree of freedom of the soliton. The thermal average of $[\hat{\phi}(x)]^2$ is then given by

$$\langle[\hat{\phi}(x)]^2\rangle \equiv D_s(x) = \frac{1}{L} T \sum_\nu {\sum_n}' \frac{C_n^2}{\omega_\nu^2 + \omega_n^2} \times (k_n^2 + \hat{m}^2 \tanh^2 \hat{m}x) \quad . \tag{A5}$$

Taking into account Eqs. (A2) and (A3) as well as the fact that $n=0$ term is missing in the sum $\sum_n'$, we obtain

$$D_s(x) = D - [1/2\pi + f_1(\beta\hat{m})] \times \operatorname{sech}^2(\hat{m}x) + O(L^{-2}) \quad , \tag{A6}$$

which is Eq. (24). Note that terms of the order of L^{-1} cancel exactly with each other.

It is then natural to neglect the inhomogeneous term (i.e., the second term) in Eq. (A6) in the self-consistent equation for $\hat{m}$, but to include it in defining the normalized potential;

$$\hat{m} = m + O(L^{-2}) \tag{A7}$$

but

$$(m^*)^2 \cos[g(\phi_s + \hat{\phi})] = m^2 \exp\left[\frac{g^2}{4\pi} F(x)\right] \times N\{\cos[g(\phi_s + \hat{\phi})]\} \quad , \tag{A8}$$

where

$$F(x) = [1 + 2\pi f_1(\beta m)] \operatorname{sech}^2 mx \quad . \tag{A9}$$

Since this inhomogeneous correction is of the order of g^2, its effect can be treated perturbationally. In fact, expanding the thermodynamic potential thus obtained in powers of g, we can show that the leading terms in $(g^2/4\pi)F(x)$ are exactly cancelled out and we obtain Eq. (26).

APPENDIX B: HIGH-TEMPERATURE BEHAVIORS OF f_0, f_1, AND f_2

First let us consider f_0 defined in Eq. (9)

$$f_0(z)=\frac{1}{\pi}\int_0^\infty d\theta(e^{z\cosh\theta}-1)^{-1} \ , \tag{B1}$$

where $z=\beta m$. Here we have changed the integral variable to θ by $k=m\sinh\theta$. Equation (B1) is transformed as

$$f_0(z)=\frac{1}{2\pi}\int_0^\infty d\theta\left[\coth\left(\frac{z}{2}\cosh\theta\right)-1\right]=\frac{1}{2\pi}\int_0^\infty d\theta\left(\sum_{\nu=-\infty}^{\infty}\frac{(z/2)\cosh\theta}{(\pi\nu)^2+[(z/2)\cosh\theta]^2}-1\right) , \tag{B2}$$

where the summation is over integer ν. In order to evaluate two terms separately, we have to introduce a cutoff in θ, although there should be no cutoff dependence in the final expression. Then Eq. (B2) is transformed as

$$f_0(z)=\frac{1}{2\pi}\left(\int_0^{\theta_0} d\theta\frac{2}{z\cosh\theta}+4\sum_{\nu=1}^{\infty}\int_0^{\theta_0}d\theta\frac{z\cosh\theta}{(2\pi\nu)^2+(z\cosh\theta)^2}-\theta_0\right)=\frac{1}{2\pi}\left(\frac{\pi}{z}+2\pi\sum_{\nu=1}^{\nu_0}[(2\pi\nu)^2+z^2]^{-1/2}-\theta_0\right) , \tag{B3}$$

with $\nu_0=(z/2\pi)\cosh\theta_0$.

In the second term of Eq. (B3), we have transferred the cutoff in θ_0 to the cutoff in ν, as in commonly done in the theory of superconductivity.[9] In general, where the divergence is logarithmic this procedure is justified. Finally expanding the second term in powers of z we have

$$\begin{aligned}f_0(z)&=\frac{1}{2z}+\sum_{\nu=1}^{\nu_0}\left(\frac{1}{(2\pi\nu)}-\frac{1}{2}\frac{z^2}{(2\pi\nu)^3}+O(z^4)\right)-\frac{1}{2\pi}\theta_0=\frac{1}{2z}+\frac{1}{2\pi}\ln(\gamma\nu_0)-\frac{\zeta(3)}{2(2\pi)^3}z^2-\frac{1}{2\pi}\theta_0\\&=\frac{1}{2z}+\frac{1}{2\pi}\ln\left(\frac{\gamma z}{4\pi}\right)-\frac{\zeta(3)}{2(2\pi)^3}z^2 \ .\end{aligned} \tag{B4}$$

This is exactly Eq. (47) in the text. As expected the logarithmic divergence cancels out exactly.

The function $f_1(z)$ defined in Eq. (25) can be analyzed similarly. First we write

$$\begin{aligned}f_1(z)&=\frac{1}{2\pi}\int_0^\infty d\theta\,\text{sech}^2\theta\left(\frac{2}{z\cosh\theta}+4\sum_{\nu=1}^{\infty}\frac{z\cosh\theta}{(2\pi\nu)^2+(z\cosh\theta)^2}-1\right)\\&=\frac{1}{2\pi}\left[\frac{\pi}{2z}+2\pi z\sum_{\nu=1}^{\infty}\frac{1}{(2\pi\nu)^2}\left(1-\frac{z}{[(2\pi\nu)^2+z^2]^{1/2}}\right)-1\right]\\&=\frac{1}{4z}-\frac{1}{2\pi}+\frac{\pi}{12}z-\frac{\zeta(3)}{(2\pi)^2}z^2+O(z^3) \ .\end{aligned} \tag{B5}$$

Finally $f_2(z)$ defined in Eq. (32) is evaluated as follows:

$$\begin{aligned}f_2(z)&=-\frac{1}{\pi}\int_0^\infty d\theta\,\text{sech}\theta\ln\left[2e^{-(z/2)\cosh\theta}\sinh\left(\frac{z}{2}\cosh\theta\right)\right]\\&=-\frac{1}{\pi}\int_0^\infty d\theta\,\text{sech}\theta\left[\ln(z\cosh\theta)-\frac{z}{2}\cosh\theta+\sum_{\nu=1}^{\infty}\ln\left(1+\frac{z^2\cosh^2\theta}{(2\pi\nu)^2}\right)\right]\\&=-\frac{1}{\pi}\left[\frac{\pi}{2}\ln(2z)-\frac{z}{2}\theta_0+\pi\sum_{\nu=1}^{\nu_0}\ln\left(\frac{[(2\pi\nu)^2+z^2]^{1/2}+z}{2\pi\nu}\right)\right] ,\end{aligned} \tag{B6}$$

where we have introduced a cutoff θ_0 as before and in the last term the cutoff is transferred to that on ν with

$$\nu_0=\frac{z}{2\pi}\cosh\theta_0 \ .$$

Then expanding the last term in powers of z, we obtain

$$f_2(z)=-\tfrac{1}{2}\ln(2z)-\frac{z}{2\pi}\ln\left(\frac{\gamma z}{4\pi}\right)+\frac{\zeta(3)}{6(2\pi)^3}z^3 \ . \tag{B7}$$

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