Migdal-type renormalization-group calculation for the kinetic Ising model

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We make use of the calculation of Achiam and Kosterlitz and investigate the generalization of the Migdal-type renormalization-group calculation to critical dynamics, by looking at the one- and two-dimensional kinetic Ising model with no conserved magnetization. In the two-dimensional case the dynamical critical index Z_M (magnetic perturbation) $=2.064$ and Z_E (energylike perturbation) $=1.819$ for scale factor $\lambda=1$. Z_M involves the static β/ν exponent, while Z_E involves $1/\nu$, and therefore, it is not surprising that Z_M is closer to the high-temperature expansion results, since β/ν in the Migdal approximation for statics is much closer to the exact value than $1/\nu$. In one dimension we obtain $Z_M=Z_E=2$, which is the exact result of Glauber.

I. INTRODUCTION

The understanding of the dynamical critical properties have recently been enriched by the introduction of some of the renormalization-group techniques[1] that have shed so much light on the static critical properties. Following the extension of the $4-\epsilon$-type expansion to dynamics,[2] real-space renormalization-group calculations[3] are beginning to be applied to the dynamical critical phenomena.[4] In this paper we investigate the generalization of Migdal-type renormalization-group calculations[5] to critical dynamics by looking at the kinetic Ising model in one and two dimensions with no spin conservation.

It is well known that (at least in the static case) Migdal's method is accurate only near the critical dimensionality of the system,[5] which is one in the case of the Ising model. However, the results for the static critical properties of the two-dimensional Ising model are quite close to the exact results. If we assume that the results for the dynamics are as good as those of the statics, then in one dimension we should get back the exact result of Glauber for the dynamical critical exponent, $z=2$. In the two-dimensional case the error should be not higher than 25%. As we shall see in our method this is really the case, namely, in one dimension we get $z=2$, while in two dimensions we get z_M (for magnetic perturbation) $=2.064$ and z_E (for energylike perturbation) $=1.819$ if the scale factor is $\lambda \rightarrow 1$. It is easy to understand this difference between z_M and z_E. Our results in two dimensions can be written

$$z_M = 2 - \Delta + \pi_M \quad \left(\Delta = \frac{\beta}{\nu}\right) , \tag{1}$$

$$z_E = 2 - \Delta' + \pi_E \quad \left(\Delta' = 2 - \frac{1}{\nu}\right) , \tag{2}$$

where π_M and π_E are calculated in Eqs. (43) and (52). The static exponent Δ in Migdal's approximation is within 5% of the exact result, while Δ' differs from the exact result by 25%. Our results for z_M compare favorably with the available high-temperature series expansion results[6] ($z=2.13$) and Monte Carlo studies[7] ($z=2.3\pm0.3$). If one puts the exact values of $\Delta(=0.125)$ and $\Delta'(=1)$ in the above formulas, and uses the results of the Migdal-type calculation only in π_M and π_E, then $z_E - z_M = 0.007$.

The paper is organized as follows: In Sec. II Migdal's static calculation will be briefly reviewed. In Sec. III the dynamical model suitable for the Migdal-type calculation will be set up. In Secs. IV and V we derive and solve the recursion equations for the cases of magnetic and energylike perturbations, respectively. Finally in Sec. VI we discuss our results.

II. MIGDAL RECURSION EQUATION FOR THE STATIC PROPERTIES OF ISING-LIKE SYSTEMS

The renormalization-group transformation is defined by a recursion relation connecting domains of spins of different sizes. For definiteness we set up the recursion equation for an Ising spin system on a square lattice. We take a square domain in the lattice whose linear size is L. We define for this domain a normalized partition functional[5] W_L as

$$W_L(\sigma_\Gamma)=\frac{\sum\limits_{\{\sigma_{\text{int}}\}}\exp\left[-\frac{1}{kT}H\left(\sigma_\Gamma,\sigma_{\text{int}}\right)\right]}{\sum\limits_{\{\sigma_\Gamma\}}\sum\limits_{\{\sigma_{\text{int}}\}}\exp\left[-\frac{1}{kT}H\left(\sigma_\Gamma,\sigma_{\text{int}}\right)\right]}\ , \tag{3}$$

where σ_Γ are spins on the boundary of the domain (Γ denotes the boundary) and σ_{int} are spins inside the domain. Equation (3) implies that in the numerator we average the Boltzmann factor over all the internal spins. In this way $W_L[\sigma_\Gamma]$ is a functional of the spin distribution along the boundary. If we combine four domains with linear size L (Fig. 1) and take all the spins along the boundary lines with half weight, then upon averaging over spins along the common boundaries (σ_{CB}) we get an exact recursion equation,[5] namely,

$$W_{2L}(\sigma_\Gamma)=\frac{\sum\limits_{\{\sigma_{\text{CB}}\}}\prod\limits_{i=1}^{4}W_L(\sigma_{\gamma i})}{\left\langle\sum\limits_{\{\sigma_{\text{CB}}\}}\prod\limits_{i=1}^{4}W_L(\sigma_{\gamma i})\right\rangle}=R_2(W_L)\ . \tag{4}$$

In Eq. (4) $\langle\ \rangle$ means averaging over σ_Γ. The subscript 2 of R expresses that the scale factor in the transformation is 2. γ_i stands for the boundaries.

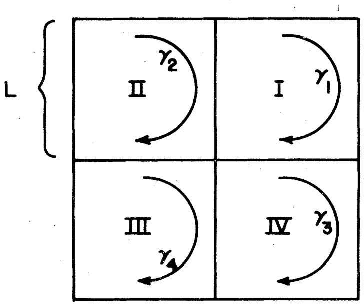

FIG. 1. Illustration of how four domains of spins are combined to form a "super-domain". Here γ_1 indicates the boundary of domain I, etc.

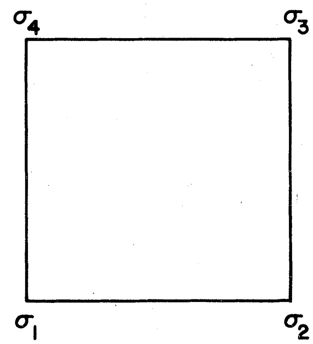

FIG. 2. Smallest domain, with four spins denoted by σ_1, σ_2, σ_3, and σ_4.

Equation (4) for the ferromagnetic Ising model has to be solved with the initial condition (see Fig. 2)

$$W_a=\frac{\exp\left(\frac{K}{2}\sum\limits_{i=1}^{4}\sigma_i\sigma_{i+1}+\frac{h}{2}\sum\limits_{i=1}^{4}\sigma_i\right)}{\sum\limits_{\{\sigma_i\}}\exp\left(\frac{K}{2}\sum\limits_{i=1}^{4}\sigma_i\sigma_{i+1}+\frac{h}{2}\sum\limits_{i=1}^{4}\sigma_i\right)}\ . \tag{5}$$

Here a is the lattice constant $K=J/kT$, where J is the exchange coupling and h is a dimensionless magnetic field.

Since Eq. (4) is strictly speaking a functional equation, there is little hope to solve it exactly. Migdal proposed the following approximations:

(a) Low-temperature approximation. This means that since we deal with ferromagnetic spin systems we may assume that at low temperatures the spins along a boundary line are almost all aligned. We replace them with one average spin (Fig. 3). With this the functional Eq. (4) is reduced to a simple equation for functions of four variables.

(b) Factorization hypothesis. This means that we use the following ansatz which is motivated partly by symmetry considerations,

$$W_L(\sigma_1\sigma_1'\sigma_2\sigma_2')=F_L(\sigma_1\sigma_1')F_L(\sigma_2\sigma_2')\ . \tag{6}$$

It is worthwhile to mention that even if one starts with the most symmetrical form for W_L, in the course of the renormalization one would always arrive at a system, where there is no coupling between the different directions.[8]

When substituting Eq. (6) into Eq. (4) we get for F_{2L}

$$F_{2L}(\sigma\sigma')=\frac{(F_L^2)^2_{\sigma\sigma'}}{\langle(F_L^2)^2_{\sigma\sigma'}\rangle}\ , \tag{7}$$

where

$$(F^2)_{\sigma\sigma'}=\sum_{\{\hat\sigma\}}F(\sigma\hat\sigma)F(\hat\sigma\sigma')\ . \tag{8}$$

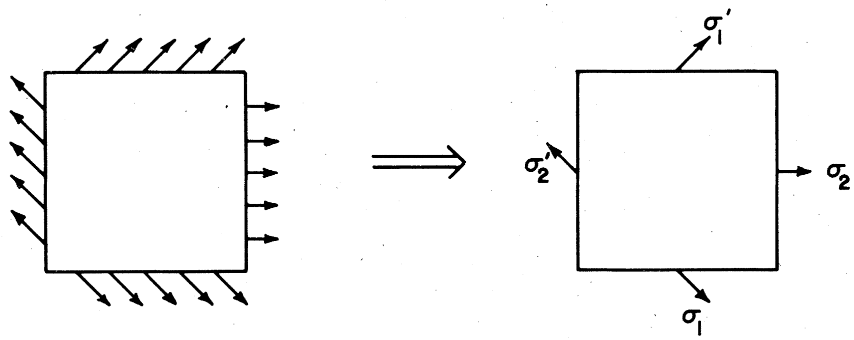

FIG. 3. Schematic representation of one of Migdal's approximations. The boundary spins on each side are all assumed to be parallel to each other and are represented by a single spin.

For a general scale factor λ and a general dimensionality d instead of Eq. (7), we have

$$F_{\lambda L}(\sigma\sigma') = \frac{(F_L^\lambda)_{\sigma\sigma'}^{\lambda^{d-1}}}{\langle (F_L^\lambda)_{\sigma\sigma'}^{\lambda^{d-1}} \rangle} . \quad (9)$$

In the case of the Ising model without a magnetic field the most general form for F_L can be written

$$F_L(\sigma\sigma') = \tfrac{1}{4}[1 + \sigma\sigma' \tanh(K_L)] = \frac{e^{K_L\sigma\sigma'}}{4\cosh(K_L)} . \quad (10)$$

(Remember that F_L has to be normalized to unity.) With this form the recursion Eq. (9) can be solved exactly and for $K_{\lambda L}$ we obtain

$$\tanh\left(\frac{K_{\lambda L}}{\lambda^{d-1}}\right) = \tanh^\lambda(K_L) , \quad K_a = K . \quad (11)$$

In the two-dimensional case, $\lambda \to 1$, the fixed point value K^* is

$$K^* = -\tfrac{1}{2}\ln[\tanh(K^*)] = -\tfrac{1}{2}\ln(2^{1/2}-1) , \quad (12)$$

which is just Onsager's result. In $d = 1 + \epsilon$, the first term in the ϵ expansion gives

$$K^* = \frac{1}{2\epsilon} . \quad (13)$$

Linearizing Eq. (11) around K^* one obtains[5]

$$\Delta' = d - \frac{1}{\nu} = \frac{1}{\ln\lambda} \ln \frac{\sinh(2K^*)}{\sinh\left(\frac{2K^*}{\lambda^{d-1}}\right)} , \quad (14)$$

where ν is the usual critical exponent. When $d = 2$ and $\lambda \to 1$,

$$\Delta' = 1.246 . \quad (15)$$

For $d = 1 + \epsilon$

$$\frac{1}{\nu} = \epsilon + 0(e^{-2/\epsilon}) . \quad (16)$$

In the case of a nonzero magnetic field, F_L is taken as

$$F_L(\sigma\sigma') = \frac{1}{4\cosh(K^*)} \exp[K^*\sigma\sigma' + h_L(\sigma+\sigma')] \quad (17)$$

and Eq. (9) is solved by linearizing in h_L. From scaling[9]

$$h_{\lambda L} = \lambda^{d-\beta/\nu} h_L . \quad (18a)$$

Equation (9) leads to[5]

$$\Delta = \frac{\beta}{\nu} = 1 + \frac{2K^*}{\ln\lambda}(\lambda^{1-d} - 1) . \quad (18b)$$

For $d = 2$ and $\lambda \to 1$

$$\Delta = 0.119 , \quad (19)$$

while in $d = 1 + \epsilon$

$$\Delta = O(e^{-2/\epsilon}) . \quad (20)$$

III. THE MODEL

The model we study is Glauber's kinetic Ising model.[10] The master equation for the probability distribution is

$$\tau_0 \frac{d}{dt} P(\{\sigma\},t) = -\sum_i w(\sigma_i) P(\{\sigma_i\},t) + \sum_i w(-\sigma_i) P(\{\sigma_i\}, -\sigma_i, t) . \quad (21)$$

The transition probabilities satisfy the condition of detailed balance and are chosen as

$$w(\sigma_j) = \frac{e^{-K\sigma_j \sum_\delta \sigma_{j+\delta}}}{e^{Kq}} . \quad (22)$$

Here q is the number of nearest neighbors, which is 2 of $d = 1$ and 4 for $d = 2$. The choice Eq. (22) ensures that $w(\sigma_j) \leq 1$. In the following we shall use the linear response theory proposed by Achiam and Kosterlitz. We assume that

$$P(\sigma,t) = \frac{\exp[\bar{H}(\sigma,t)]}{Z(t)} , \tag{23}$$

where

$$\bar{H}(\sigma,t) = +\frac{K}{2}\sum_{i,\delta}\sigma_i\sigma_{i+\delta} + h_0^M(t)\sum_i \sigma_i , \tag{24}$$

for magnetic perturbations and

$$\bar{H}(\sigma,t) = +\frac{K}{2}\sum_{i,\delta}\sigma_i\sigma_{i+\delta} + \frac{h_0^E(t)}{2}\sum_{i,\delta}\sigma_i\sigma_{i+\delta} \tag{25}$$

for energylike perturbations. Within linear response theory (in h), Eq. (21) can be written

$$\tau_0 \frac{d}{dt} P^M = -2h_0^M(t)\sum_i w(\sigma_i)\sigma_i \frac{\exp\left(+\frac{K}{2}\sum_{j,\delta}\sigma_j\sigma_{j+\delta}\right)}{Z} , \tag{26a}$$

$$\tau_0 \frac{d}{dt} P^E = -h_0^E(t)\sum_i w(\sigma_i)\sigma_i \sum_\delta \sigma_{i+\delta} \frac{\exp\left(+\frac{K}{2}\sum_{j,\delta}\sigma_j\sigma_{j+\delta}\right)}{Z} , \tag{26b}$$

respectively, for magneticlike and energylike perturbations. In Eqs. (26a) and (26b) w is given by Eq. (22). Note that since P is given by Eq. (23), where $\bar{H}$ is given by Eqs. (24) and (25), we can omit the h dependence of Z.

In order to arrive at such a form of the master equations (26a) and (26b) which is the most suitable for applying a Migdal-type method, we imagine Eqs. (26a) and (26b) to be valid for a domain with linear size L and as in Sec. I, we average both sides over the internal spins of that domain. According to Eq. (3), and because of the specific form of the $w(\sigma_i)$'s we obtain

$$\tau_0 \frac{d}{dt} W_L^M = -2h_0^M(t)\sum_{i\in B}\sigma_i \sum_{\{\sigma_{\rm int}\}}\left[w(\sigma_i)\frac{\exp\left(+\frac{K}{2}\sum_{j,\sigma}\sigma_j\sigma_{j+\delta}\right)}{Z_L}\right] , \tag{27a}$$

$$\tau_0 \frac{d}{dt} W_L^E = -h_0^E(t)\sum_{i\in B}\sigma_i \sum_{\{\sigma_{\rm int}\}}\left[w(\sigma_i)\sum_\delta \sigma_{i+\delta}\frac{\exp\left(+\frac{K}{2}\sum_{j,\delta}\sigma_j\sigma_{j+\delta}\right)}{Z_L}\right] . \tag{27b}$$

$\sum_{i\in B}$ means summation over the boundary spins. It is clear that all the boundary spins have less neighbors than the internal spins. This will have to be taken into account when writing the recursion equation for the right-hand sides of the above equations. According to approximation (a) in Sec. I, all the spins along the separate edges (in $d=2$) are assumed to be aligned. Therefore, taking a specific edge we have

$$\sum_{i\in B_S}\sigma_i \sum_{\{\sigma_{\rm int}\}} \to \sigma_S \sum_{i\in B_S}\sum_{\{\sigma_{\rm int}\}} .$$

Here B_S denotes a specific edge. In d dimensions the sum $\sum_{i\in B}$ can be divided into $2d$ separate sums, each one corresponding to one of the $2d$ average spins introduced in the Migdal approximation. Each separate sum contains L^{d-1} terms. If the expression $\sum_{\{\sigma_{\rm int}\}}[\ ]$ were the same for all the L^{d-1} terms, then we would have, instead of Eq. (27a) for example,

$$\tau_0 \frac{d}{dt} W_L^M = -2h_0^M(t) L^{d-1}\sum_{\eta=1}^{2d}\sigma_\eta Q_L(\sigma_\eta) , \tag{28}$$

where

$$Q_L(\sigma_\eta) = \sum_{\{\sigma_{\rm int}\}} w(\sigma_\eta)\frac{\exp\left(-\frac{K}{2}\sum_{j,\delta}\sigma_j\sigma_{j+\delta}\right)}{Z_L} . \tag{29}$$

By this we would obtain an extra L^{d-1} factor which changes the time scale of the system (without doing any renormalization-group transformation). Physically this can be easily understood. If all the boundary spins are the same, call it σ, clearly there will be L^{d-1} ways by which this configuration can be changed—namely, by the relaxation of any one of the L^{d-1} boundary spins. If any of those spins is relaxed, the block spin σ is flipped. Using approximation (b) of Sec. I we can write, for example, in the two-dimensional case (see Fig. 3)

$$\tau_0 \frac{d}{dt} W_L^M = \frac{-2h_L^M L \rho_L^M}{e^{K_L}(4\cosh K_L)} [(\sigma_1 + \sigma_1')F_L(\sigma_2\sigma_2') + (\sigma_2 + \sigma_2')F_L(\sigma_1\sigma_1')] \ , \tag{30a}$$

$$\tau_0 \frac{d}{dt} W_L^E = \frac{-h_L^E L \rho_L^E}{e^{K_L}(4\cosh K_L)} [\sigma_1\sigma_1' F_L(\sigma_2\sigma_2') + \sigma_2\sigma_2' F_L(\sigma_1\sigma_1')] \ , \tag{30b}$$

and we see that in Migdal's approximation Q_L is really the same along an edge. Obviously the right-hand sides of Eqs. (30a) and (30b) are the most general functional forms one can write in the Migdal approximation. Here F_L is given by Eq. (10). The denominator in the above formulas is the consequence of the specific choice of the bare transition probability, Eq. (22), and of the original normalization of P. Note that the spins along the edges have only one "nearest neighbor", that is why e^{K_L} appears instead of e^{Kq}.

We can now define

$$\hat{\tau}_L^M = \frac{\tau_0}{L\rho_L^M} \ , \tag{31}$$

$$\hat{\tau}_L^E = \frac{\tau_0}{L\rho_L^E} \ . \tag{32}$$

As we have emphasized the $\hat{\tau}$'s defined above are not the physically interesting quantities. In Migdal's approximation we deal with "clusters" of L aligned spins. $\hat{\tau}$ is the relaxation time of such a system of clusters. In order to obtain the physically interesting quantities we will have to calculate

$$\tau_L^M = L\hat{\tau}_L^M \quad (\tau_{\lambda L}^M = \lambda L \hat{\tau}_{\lambda L}^M) \ , \tag{33a}$$

$$\tau_L^E = L\hat{\tau}_L^E \quad (\tau_{\lambda L}^E = \lambda L \hat{\tau}_{\lambda L}^E) \ . \tag{33b}$$

IV. MAGNETIC PERTURBATION

In this case the master equation to which we apply the renormalization-group operator is

$$\hat{\tau}_L^M \frac{dW_L^M}{dt} = -\frac{2h_L^M}{e^{K_L}(4\cosh K_L)} \Pi_L^M \ . \tag{34}$$

Here Π_L^M is given by the square bracket in Eq. (30a). Below we shall work out the recursion equations in detail for $d=2$ and $\lambda=2$. The generalization to general d and λ is straightforward. Let us combine four blocks of linear size L and integrate over the spins on the common edges. Since (see Fig. 4)

$$W_{2L}^M = \frac{\sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4} W_L^M(I)\, W_L^M(II)\, W_L^M(III)\, W_L^M(IV)}{\langle \sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4} W_L(I)\, W_L(II)\, W_L(III)\, W_L(IV)\rangle} = \frac{R_2(W_L^M)}{\langle R_2(W_L)\rangle} \ , \tag{35}$$

on substituting this into an equation similar to Eq. (34) but corresponding to a cluster with linear size $2L$, and differentiating each factor in the numerator of Eq. (35) we obtain

$$\Pi_{2L}^M = C^M \sum_{s_1 s_2 s_3 s_4} \left[\Pi_L^M(I)\, W_L(II)\, W_L(III)\, W_L(IV) + \cdots + W_L(I)\, W_L(II)\, W_L(III)\, \Pi_L^M(IV) \right] . \tag{36}$$

In the above Eqs. (34)–(36) W_L^M contains h^M, but W_L is taken at $h^M=0$. This can be done because of the linear response theory we are using. C^M in Eq. (36) is given by

$$C^M = \frac{\hat{\tau}_{2L}^M h_L^M e^{K_{2L}} \cosh(K_{2L})}{\hat{\tau}_L^M h_{2L}^M e^{K_L} \cosh(K_L)} \frac{1}{\langle RW_L\rangle} \ . \tag{37}$$

It is easy to see however that Eq. (36) cannot be the correct recursion equation for Π. To clarify this point let us take the first term in Eq. (36).

From Fig. 4 one sees that

$$\Pi_L^M(I) = (s_3 + \sigma_2)F_L(\sigma_1' s_2) + (s_2 + \sigma_1')F_L(s_3\sigma_2) \ . \tag{38}$$

Because of the specific form of the transition probability, s_3 cannot be coupled to its nearest neighbors, which in our picture are σ_2 and σ_2', and so s_2 cannot be coupled to σ_1 and σ_1'. When averaging over s_3 and s_2 in the first term of Eq. (36) we should obtain

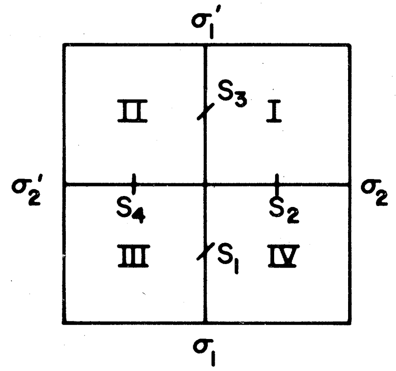

FIG. 4. Illustration of the construction of the recursion relation for Π_L in Eq. (36). Here s_3 is a spin on the common boundary of domain I and II, etc.

0. Therefore, in Eq. (36) instead of Eq. (38) we can use

$$\Pi_L^M(I) = \sigma_2 F_L(\sigma_1' s_2) + \sigma_1' F_L(s_3 \sigma_2) \ . \tag{39}$$

The recursion equation, when taking only the first term in Eq. (39) can be illustrated schematically as is shown in Fig. 5(a), where each dashed line corresponds to an F_L factor. Note that σ_2 is not coupled to its "nearest neighbors". Now s_1 is the "nearest neighbor" of $\hat{\sigma}_2$. If we put $\sigma_2 = \hat{\sigma}_2$ then s_1 becomes the "nearest neighbor" of σ_2 as well. Hence s_1 and $\sigma_2(=\hat{\sigma}_2)$ should not be coupled to each other and the recursion equation should take the form illustrated schematically in Fig. 5(b). Performing the actual calculation for general d and λ and requiring that Eq. (34) had the same form when written for a block of size λL we find

$$1 = \frac{\hat{\tau}_{\lambda L}^M h_L^M e^{K_{\lambda L}}}{\hat{\tau}_L^M h_{\lambda L}^M e^{K_L}} \lambda^{d-1} \left(\frac{1-\tanh^2 K_L}{1-\tanh^{22} K_L} \right)^{\lambda^{(d-1)/2}} \ . \tag{40}$$

Using Eqs. (11), (18a) and (33a) for very large L, according to dynamical scaling,

$$\frac{\tau_{\lambda L}^M}{\tau_L^M} = \lambda^{d-\Delta} \left[\frac{1-\tanh^2 K^*}{1-\tanh^2\left(\frac{K^*}{\lambda^{d-1}}\right)} \right]^{-\lambda^{(d-1)/2}} = \lambda^{z_M} \ ; \tag{41}$$

Eq. (41) is valid for $d > 1$.

For z_M we get

$$z_M = d - \Delta + \pi_M \ , \tag{42}$$

where

$$\pi_M = \frac{\lambda^{d-1}}{\ln\lambda} \ln \left[\frac{\cosh K^*}{\cosh\left(\frac{K^*}{\lambda^{d-1}}\right)} \right] \ . \tag{43}$$

For $d=2$ and $\lambda \to 1$, using Eq. (12) one gets

$$\pi_M = 0.183 \tag{44}$$

and finally using Eq. (19)

$$z_M = 2.064 \ . \tag{45a}$$

Formula (43) has a very interesting property. The value of π_M as $\lambda \to 1$ is independent of the factor λ^{d-1}. If we forget about this factor, we get the same duality property for π_M that is characteristic of the static formula of Migdal. Namely, Eq. (43) will be invariant under the transformation $\lambda' = (1/\lambda)$, $\beta' = \beta\lambda$.

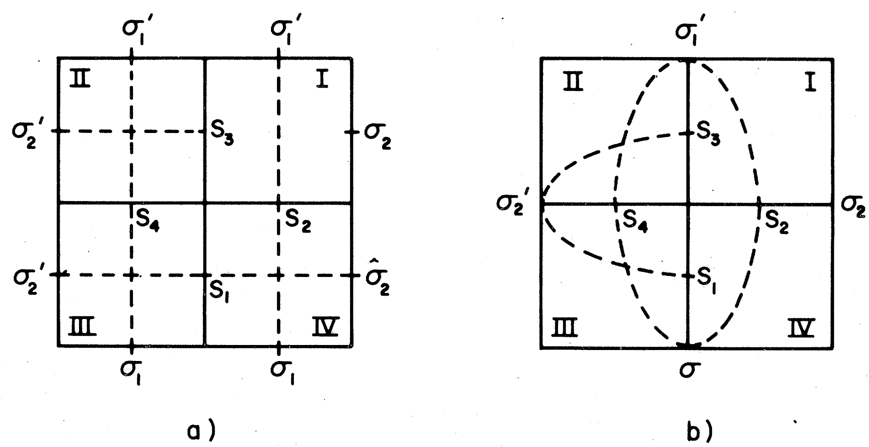

FIG. 5. (a) Schematic representation of the recursion equation for the first term in Π_L (I) before putting $\sigma_1 = \hat{\sigma}_1$. (b) Schematic representation of the recursion equation for the first term in Π_L (I) after taking $\sigma_1 = \hat{\sigma}_1$.

This was the reason why Migdal took the $\lambda \to 1$ limit. In Kadanoff's bond shifting formulation of the Migdal approximation, the $\lambda \to 1$ was necessary in order to restore the original isotropy of the system. Yet another argument can be given for taking the $\lambda \to 1$ limit. We would like our renormalization-group transformation to have the group (actually semigroup) properties, that is we would like the relationship

$$R_{\lambda_1}[R_{\lambda_2}(W_L)] = R_{\lambda_1\lambda_2}(W_L) \tag{45b}$$

to hold. It is easy to show [taking for example Eq. (11)] that the transformation has this property only if $\lambda_1 = 1+\epsilon_1$ and $\lambda_2 = 1+\epsilon_2$ with ϵ_1 and ϵ_2 very small.

We can now calculate $\tau_{\lambda L}^M$ for $d=1$. We will do this in the limit when $d = 1+\epsilon$. However in this case, using Eqs. (13) and (16) and scaling for the correlation length ξ_L

$$\xi_L(d=1) \sim \lim_{\epsilon\to 0}\left(1-\frac{K_L}{K^*}\right)^{-\nu} = \lim_{\epsilon\to 0}\left[1+\frac{2K_L}{\left(-\frac{1}{\epsilon}\right)}\right]^{-1/\epsilon} = e^{2K_L} . \tag{46}$$

Therefore, in Eq. (40) we have the factor

$$\frac{e^{K_{\lambda L}}}{e^{K_L}} = \left(\frac{\xi_{\lambda L}}{\xi_L}\right)^{1/2} = \lambda^{-1/2} .$$

Using this, and taking into account that $\Delta(d=1)=0$, we get

$$z_M = \tfrac{3}{2} + \lim_{\epsilon\to 0}\left[\frac{\lambda^{d-1}}{\ln\lambda}\ln\frac{\cosh K^*}{\cosh\left(\dfrac{K^*}{\lambda^{d-1}}\right)}\right] = 2 . \tag{47}$$

V. ENERGYLIKE PERTURBATION

In this case the master equation to which we apply the renormalization-group operator is

$$\hat{\tau}_L^E\frac{dW_L^E}{dt} = \frac{-h_L^E}{e^{K_L}(4\cosh K_L)}\Pi_L^E , \tag{48}$$

where Π_L^E is given by the square bracket in Eq. (30b). The recursion equation for Π_L^E is constructed analogously to that for Π_L^M. Now instead of Eq. (38) we have

$$\Pi_L^E(I) = s_3\sigma_2 F_L(\sigma_1' s_2) + s_2\sigma_1' F_L(s_3\sigma_2) . \tag{49}$$

Performing the calculation for general d and λ, and using the arguments of Sec. IV, we obtain

$$z_E = d - \Delta' + \pi_E . \tag{50}$$

When deriving Eq. (50) we have used $h_{\lambda L}^E = \lambda^{1/\nu} h_L^E$. Here π_E is given by

$$\pi_E = \pi_M + \frac{1}{\ln\lambda}\ln\frac{\tanh K^*}{\tanh\left(\dfrac{K^*}{\lambda^{d-1}}\right)} . \tag{51}$$

The value of π_E at $\lambda = 1$ is independent of the factor λ^{d-1} in π_M [i.e., Eq. (43)], and if we neglect it

$$\pi_E = \frac{1}{\ln\lambda}\ln\frac{\sinh K^*}{\sinh\left(\dfrac{K^*}{\lambda^{d-1}}\right)} . \tag{52}$$

For $d=2$, $\lambda \to 1$

$$\pi_E = 1.065 . \tag{53}$$

This together with Eq. (15) gives

$$z_E = 1.819 . \tag{54}$$

Obviously Eq. (52) also possesses the property of duality discussed in Sec. IV.

One expects $z_M = z_E$. That this is not so is very likely due to the large deviation of Δ' from the exact value. This conjecture is supported by the fact, that when we use the exact values for Δ and Δ', $z_E - z_M = 0.007$.

In $d=1$, analogously to the results of Sec. V we obtain $z_E = 2$, if we use the definition of the critical index ν given in Ref. 11, namely $\xi \sim (e^{2K})^\nu$ [when $\nu = 1$, i.e., Eq. (46)].

VI. DISCUSSION

Applying a Migdal-type recursion method to the two-dimensional kinetic Ising model we have obtained a result for the dynamical critical index which lies within the range of previous estimates obtained by high-temperature series expansion and Monte Carlo calculations. Our method is very similar to that Migdal used for the statics. However, the approximations introduced in the static case have more serious consequences in the dynamical case and care has to be exercised to apply them correctly. Because of the relative simplicity of this method, a whole vista of possibilities is opened up for applying the above method to other systems.

It is well known that the Migdal method gives very good results in the case of continuous spin systems, like the Heisenberg and xy models in $d = 2+\epsilon$ dimensions. Although the dynamics of the dual of the two-dimensional xy model has been looked at,[12] the effect of vortices on the dynamics of the two-dimensional xy model has not been clarified. It would be interesting to apply the above ideas to such systems. Another interesting possibility is the class of Blume-Emery-Griffith's models in both two and three dimensions. Migdal-type renormalization-group calculation seems to provide reasonable results when compared with experiments of submonolayer coverage of krypton on graphite.[13] Yet another area where the extension is quite straightforward is the study of the dynamics of spin glasses. The above are just a few of the potential applications of the direction which we are now taking.

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