Logarithmic temperature contributions to long-wavelength spin susceptibility from particle-particle scattering

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Based on a second-order perturbation calculation with a δ-function interaction, Geldart and Rasolt have recently shown that the long-wavelength static spin susceptibility is nonanalytic in T at low temperatures. We find the nonanalyticity is suppressed by the summation of particle-particle scattering terms to all orders. Commonly held analyticity assumptions regarding the long-wavelength and low-temperature behavior of the susceptibility are thus not invalidated. We suggest that for the nonuniform susceptibility, a Kohn-Luttinger-Ward-like equivalence between the zero- and low-temperature perturbation series exists between classes of terms, rather than order by order.

I. INTRODUCTION

The question of the existence of $T^n \ln T$ contributions to the spin susceptibility χ of a many-fermion system has aroused considerable interest in recent years. For the uniform case several authors[1–3] have conjectured a $T^2 \ln T$ term. It has now been shown[4,5] that $T^2 \ln T$ terms cannot arise from certain processes (long-wavelength particle-hole scattering).

For the nonuniform case, in the static long-wavelength limit we may expand χ as

$$\chi(k) = \chi(0) - \alpha k^2 + O(k^4) . \quad (1)$$

Geldart and Rasolt (GR) have recently investigated[6] the coefficient α in second-order perturbation theory for the case of a δ-function two-body interaction with strength I, and find

$$\alpha = \alpha' + \left(\frac{I^2 \mu_B^2 m^3 k_F}{288 \pi^6 \hbar^6}\right) \ln \left(\frac{k_B T}{\epsilon_F}\right) . \quad (2a)$$

Their calculation involves the evaluation of multiple integrals from which the subtle logarithmic singularity is extracted after much effort. GR have argued that the logarithmic divergence persists to all orders in perturbation theory and for finite-range interactions, and is not a result confined to a second-order calculation. The logarithmic dependence on T has the following remarkable implications, as they point out: (i) As $T \to 0$, $\chi(k)$ cannot be expressed as a regular series in powers of k^2. (ii) $\chi(k)$ may attain a maximum at a nonzero k. (iii) Corrections are not small [i.e., $O((T/T_F)^2)$] at low temperatures. These conclusions contradict widely held assumptions regarding the low-temperature behavior of $\chi(k)$.

The implications of GR's claims put the nonuniform susceptibility on a different footing from the uniform one. In the uniform case one has a well-known theorem due to Kohn, Luttinger, and Ward.[7] This theorem implies that the inverse of the susceptibility (related to derivatives of the free energy) calculated (a) as the limit of the finite-temperature theory, and (b) within the zero-temperature theory, agree *order by order*. The theorem applies to spherically symmetric interactions, and no initial distortion of the Fermi surface, with the thermodynamic limit being taken before $T \to 0$. Thus

$$\lim_{T \to 0} \chi_n^{-1}(0,T) = \chi_n^{-1}(0,0) ,$$

where the subscript n refers to the order-of-perturbation theory and we explicitly display the momentum and temperature dependences. For a normal system, this equation is valid for χ_n itself since one can invert without encountering singularities. Ideally one would like to have the generalization to nonzero wave vectors as

$$\lim_{T \to 0} \lim_{k \to 0} \chi_n(k,T) = \lim_{k \to 0} \lim_{T \to 0} \chi_n(k,T) = \chi_n(0,0) . \quad (2b)$$

In view of the behavior of α given in Eq. (1), the above relation is meaningless since the limits $T \to 0$ and $k \to 0$ cannot be interchanged. The singular dependence of $\chi(k,T)$ on k and T near $k=0, T=0$ is quite startling since there is no elementary reason to think of k and T as coupled variables.

In this work we derive the second-order perturbation result of GR from a new point of view. The magnetic susceptibility is basically a (triplet) particle-hole (p-h) propagator. However, in perturbation theory one comes across intermediate states which involve particle-particle (p-p) scattering. These make their *first appearance* in a second-order perturbation calculation. It is well known in the theory of superconductivity that the p-p propagator has a logarithmic dependence on T for small total momentum Q of the colliding particles.[8] Moreover, in the $T=0$ series also, the p-p propagator gives rise to a $\ln Q$ divergence.[9–11] Therefore we examine the possibility that the $\ln T$ found by GR is a consequence of the singularities in the p-p propagator.

The plan of the paper is as follows. In Sec. II the nonuniform spin susceptibility $\chi(k)$ is expressed in terms of the particle-particle propagator $R(Q)$. The p-p propagator is then explicitly evaluated in Sec. III for small Q and T and the logarithmic singularities of R in Q, T are displayed. In Sec. IV we extract the leading logarithmically singular contributions to $\chi(0)$ as $T\to 0$. Useful properties of the polygamma functions that arise in this calculation are given in the Appendix. Whereas (due to cancellation) there is no net $\ln T$ in $\chi(0)$, a $\ln T$ survives in the result for α. In Sec. V, we discuss the implications of the $\ln T$ singularity, and techniques for suppressing it. This suggests the appropriate generalization to the nonuniform case of the theorem connecting zero-temperature and low-temperature perturbation terms for $\chi(k)$.

II. χ IN TERMS OF THE p-p PROPAGATOR

The wave-vector-dependent static spin susceptibility $\chi(k)$ can be expressed in terms of the triplet vertex part $\Lambda(p,k)$ and the single-particle temperature Green's function $G(K)$ as

$$\chi(k)=2\mu_B^2(-T)\sum_p G(p+\tfrac{1}{2}k)\Lambda(p,k)G(p-\tfrac{1}{2}k)\ . \tag{3}$$

We use the usual four-vector notation $p\equiv(\vec{p},i\omega_n)$ where ω_n is the discrete frequency $(2n+1)\pi T$. We use units $\hbar=k_B=1$ throughout. Here $\sum$ stands for summation over momentum and the discrete frequency. Thus a perturbation expression for G and Λ generates one for χ. In Fig. 1, we display all the Feynman diagrams for G and Λ up to the second order in I. In Fig. 2, we exhibit the labelled diagrams for G and Λ from which the singular part of the susceptibility arises. The relevant part of the susceptibility χ which contains the singularity can be explicitly written

$$\chi(k)=2\mu_B^2T^2I^2\sum_{p_1,p_2}G_0(p_1+\tfrac{1}{2}k)G_0(p_1-\tfrac{1}{2}k)R(p_1+p_2)\times[G_0(p_2+\tfrac{1}{2}k)G_0(p_2-\tfrac{1}{2}k)-G_0(p_1+\tfrac{1}{2}k)G_0(p_2-\tfrac{1}{2}k)-G_0(p_1-\tfrac{1}{2}k)G_0(p_2+\tfrac{1}{2}k)]\ , \tag{4}$$

where

$$G_0(p)=(i\omega_n+\mu_0-p^2/2m)^{-1}$$

and

$$R(Q)=(-T)\sum_{p_1}G_0(p_1)G_0(Q-p_1)\ ;$$

$$Q\equiv(\vec{Q},i\Omega)\ . \tag{5}$$

R is clearly the particle-particle propagator. Here $\Omega=2\pi\nu T$, ν being an integer.

It is convenient to relabel the variables in Eq. (4) and write it

$$\chi(k)=2\mu_B^2T^2I^2\sum_{Q,p}R(Q)G(p)G(Q-p)\{G(p-k)G(k+Q-p)-G(p)[G(p+k)+G(p-k)]\}\ . \tag{6}$$

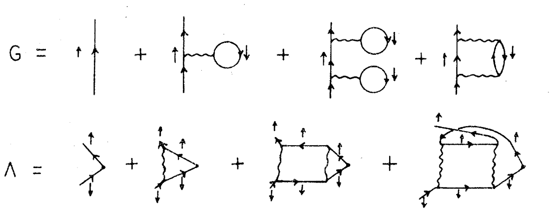

FIG. 1. All diagrams for G and Λ up to second order. The interaction is only between electrons of opposite spin.

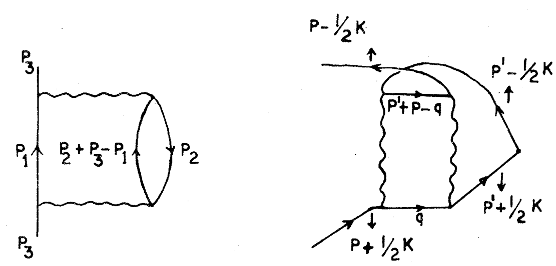

FIG. 2. Dangerous diagrams.

In Eq. (6) we have omitted the subscript on G, the free-particle propagator. Expanding Eq. (6) in powers of k, we find the following expressions for $\chi(0)$ and α:

$$\chi(0)=2\mu_B^2T^2I^2\sum_{Qp}R(Q)[G^2(p)G^2(Q-p)-2G^3(p)G(Q-p)] \ , \tag{7}$$

$$\alpha=-\left[\frac{\mu_B^2T^2I^2}{m}\right]\sum_{Qp}R(Q)G^2(p)G(Q-p)\left[G(Q-p)G(p)+G^2(Q-p)-2G^2(p)+\tfrac{4}{3}\epsilon_pG^2(p)G(Q-p)\right.$$
$$\left.+\tfrac{4}{3}\epsilon_{Q-p}G^3(Q-p)-\frac{4}{3}\frac{\vec{p}\cdot(\vec{Q}-\vec{p})}{2m}G(p)G^2(Q-p)-\tfrac{8}{3}\epsilon_pG^3(p)\right] . \tag{8}$$

In obtaining Eq. (8), we have used the rotational invariance of χ and averaged over the direction of $\vec{k}$.

GR state their result for χ [in Eqs. (11) and (12) of Ref. 6] in terms of the p-h propagator P_0 defined as

$$P_0(q)=(-T)\sum_p G(p+q)G(p) \ . \tag{9}$$

We note that if their nonsingular terms (P_0^2, P_0^3 and $\delta\mu$) in their Eqs. (11) and (12) are dropped, then their result yields precisely our Eq. (4) after a suitable change of variables. Thus the same second-order expression can be viewed in either the p-p or p-h "view points". As another example consider the self-energy contribution from the diagram of Fig. 2. This can be written

$$\Sigma(p_3)=-T^2I^2\sum_{p_1,p_2}G(p_1)G(p_2)G(p_2+p_3-p_1) \ . \tag{10}$$

This can be viewed either as a p-p term with total momentum $Q=p_2+p_3$, or as a p-h term with momentum transfer $q=p_3-p_1$. Thus

$$\Sigma(p_3)=TI^2\sum_{p_2}R(p_2+p_3)G(p_2)$$
$$=TI^2\sum_{p_1}P_0(p_3-p_1)G(p_1) \ . \tag{11}$$

It is noteworthy that GR attribute their singularity in χ to the behavior of P_0 in the large momentum transfer (i.e., $q\simeq 2k_F$) region of phase space. Since at low T $G(p)$ is largest for $|\vec{p}|=k_F$, it follows that in Eq. (10) $\vec{p}_1,\vec{p}_2$ and $\vec{p}_2+\vec{p}_3-\vec{p}_1$ should all be close to the Fermi momentum. Setting

$$|\vec{p}_1|=|\vec{p}_2|=|\vec{p}_2+\vec{p}_3-\vec{p}_1|=k_F$$

we can readily see that $|\vec{p}_1-\vec{p}_3|=0$ implies $|\vec{p}_2+\vec{p}_3|=0$. Hence the $|\vec{q}|=2k_F$ region of phase space in the p-h view point is contained within the $|\vec{Q}|=0$ region in the p-p view point. Hence we may expect the GR singularity to arise from the $Q=0$ behavior of $R(Q)$.

From Eq. (4) the p-p propagator can be written

$$R(Q)=(-T)\sum_p G(p)[G(Q-p)-G(-p)]+R(0). \tag{12}$$

Here $R(0)$ (after doing the frequency summation) is given by[8]

$$R(0)=-N(0)\int_0^{\omega_L}\tanh\left[\frac{\xi}{2T}\right]\frac{d\xi}{\xi} \ . \tag{13}$$

Here $N(0)$ is the density of states $mk_F/2\pi^2$ and ξ is the energy measured with respect to the Fermi energy ϵ_F. The high-energy cutoff ω_L is implicit in the use of δ function as discussed in Sec. V. It is clear that

$$R(0)=N(0)\ln(T/T_F)+\text{constant} \ . \tag{14}$$

For $T \ll T_F$, the G's are sharply peaked at the Fermi surface and we can use the approximation

$$\sum_p \to N(0)\int_{-\infty}^{+\infty}d\xi_p\int\frac{d\Omega_p}{4\pi} \ , \tag{15}$$

where Ω_p is the solid angle. In using Eq. (15) any slowly varying p dependence of the integrand should be approximated by

$$p=(2m\epsilon_F)^{1/2}(1+\xi_p/\epsilon_F)^{1/2}$$
$$\simeq k_F(1+\tfrac{1}{2}\xi_p/\epsilon_F)$$

ensuring the function has the correct value and derivative at the Fermi surface. However, for extracting leading temperature singularities in the small Q region it is sufficient to put $p\simeq k_F$ in the integrand, since corrections can be shown to be higher order in T.

In Eq. (12) we perform the ξ integral first and find

$$R(Q)=R(0)-\tfrac{1}{2}N(0)\int_{-1}^{1}\tfrac{1}{2}dx\sum_n\left[\frac{\Theta(\omega_n)-\Theta(\Omega-\omega_n)}{n+\frac{1}{2}-\nu+i\dfrac{Qx}{2\pi T}}-\frac{\Theta(\omega_n)-\Theta(\Omega-\omega_n)}{n+\frac{1}{2}}\right] . \tag{16}$$

Using the standard properties of the diagamma functions (see Appendix) we find[12]

$$R(Q) = N(0)[R_1 + R_2(Q)] \ , \tag{17}$$

where

$$R_1 = \ln T - \psi(\tfrac{1}{2}) + \text{constant} \ , \tag{18}$$

$$R_2(Q) = \frac{1}{y} \operatorname{Im} \ln \Gamma(\tfrac{1}{2} - \tfrac{1}{2}\nu + iy) \ , \tag{19}$$

and

$$y = Q/2\pi T \ . \tag{20}$$

Equations (17)–(20) contain the structure of R in a form particularly useful for manipulations. However the asymptotic singularity structure is more transparent in the approximate form

$$R_2(Q) = \tfrac{1}{2}\left\{ \ln(Q^2 + \tfrac{1}{4}\Omega^2) - \frac{\ln[(Q/2T)^2 + (\Omega/4T)^2]}{1 + (Q/2T)^2 + (\Omega/4T)^2} \right\} . \tag{21}$$

This can be obtained from Eq. (4) by first carrying out the frequency sum, then the angular average, and finally the momentum sum, retaining only the terms which are singular as any one of Q, Ω, or T goes to zero. The asymptotic behavior of Eqs. (17) or (21) is as follows:

$$\lim_{\Omega \to 0;\, T=0,\, Q=0} R(Q) = N(0) \ln(\tfrac{1}{2}\Omega) \ ,$$

$$\lim_{Q \to 0;\, T=0,\, \Omega=0} R(Q) = N(0) \ln Q \ , \tag{22}$$

$$\lim_{T \to 0;\, Q=0,\, \Omega=0} R(Q) = N(0) \ln T \ .$$

IV. SINGULAR PART OF χ

A. General procedure

We have seen in Sec. III that $R(Q)$ is singular for small (Q, Ω, T) and would expect this to be reflected in χ, which from Eqs. (7) and (8) is expressed as a convolution of R with some function of Q, Ω, and T. For orientation, let us consider a model problem which involves Q and T only and simulates the logarithmic singularity in Eq. (22). Let R_m be singular for small Q,T as follows:

$$R_m(Q,T) = \ln(Q+T) \ . \tag{23}$$

Typically we are interested in integrals of the type

$$J = \int_0^{Q_c} R_m(Q,T) F\left(Q, \frac{Q}{T}\right) dQ \ . \tag{24}$$

Here Q_c is some arbitrary cutoff and we are interested in the leading temperature correction to J. For example consider $F=1$ in which case we get $-T \ln T$ from the lower limit. (This is independent of the cutoff Q_c as expected.) Setting $F = Q^m$, we see that the leading correction goes as $T^{m+1} \ln T$. Thus the contribution comes from Q of order T. If $F = F(Q, Q/T + Q)$ then the leading dependence is given by $F(Q, Q/T)$. Hence Q may be neglected in comparison with Q/T in a combination $Q + Q/T$. Note also that if F can be expanded in a Taylor series in the unscaled variables Q, then the leading contribution comes from the smallest power of Q. The general method followed here is to substitute $Q = Ty$ which gives an integral going from 0 to Q_c/T. The singularities arising from the low-temperature behavior are then obtained from the behavior of the integral from the upper limit (thus requiring the asymptotic, i.e., $y \to \infty$ behavior of the integrand).

B. Uniform susceptibility

Scaling momenta by k_F, temperatures by T_F, and energies by ϵ_F, we can write $\chi(0)$

$$\chi(0) = \Delta_1 T^2 \sum_{Q,p} [R_1 + R_2(Q)] \times [G^2(Q-p) G(p) - 2G^3(p) G(Q-p)] \ , \tag{25}$$

where

$$\Delta_1 = 4\mu_B^2 I^2 m^3 k_F^3 / \pi^2 \ . \tag{26}$$

Using Eq. (15) and partially integrating with respect to ξ, the integrand reduces to a single term $\sim R G(p) G^3(Q-p)$. Then, using identity of Eq. (A11) of the Appendix, dropping Q relative to Q/T (as mentioned above), and also keeping only the lowest-order terms in Q, we find

$$\chi(0) = \frac{\Delta_1 T^2}{\pi^2} \sum_\nu \int_0^{y_c} y\, dy [R_1 + R_2(y,\nu)] \operatorname{Im} \psi^{(1)}(\tfrac{1}{2} + \tfrac{1}{2}\nu + iy) \ . \tag{27}$$

Here $y_c = Q_c/2\pi T$; Q_c being some cutoff. By Eq. (A9), the polygamma function can be shown to be an even function of ν. We separate the $\nu = 0$ term and convert the summation to an integration by using the Euler-Maclaurin[13] formula. This gives

$$\chi(0) = \frac{4\Delta_1 T^2}{\pi^2} \int_0^\infty d\phi \int_0^{y_c} y\, dy [R_1 + R_2(y, 2\phi)] \operatorname{Im} \psi^{(1)}(\tfrac{1}{2} + \phi + iy) \ . \tag{28}$$

The correction terms are neglected as they are nonsingular. The integral is trivial since $\psi^{(1)}$ is a perfect differential. Only the boundary term from $\phi=0$ is nonzero. Using the asymptotic forms for the polygamma functions (see Appendix) in the large y limit, we find the logarithmic contributions from R_1 and R_2

$$\chi_1=(4\Delta_1 T^2/\pi^2)(\tfrac{1}{4}\pi^2 y_c^2 \ln T) \; ,$$

$$\chi_2=-\chi_1 \; . \tag{29}$$

Thus each of the two terms in Eq. (29) is potentially a $\ln T$ term but the sum of χ_1 and χ_2 is independent of $\ln T$. Note that this does not necessarily imply the absence of $T^2 \ln T$ contributions.[1] However, we shall not pursue this here for reasons discussed in Sec. V.

C. Coefficient of k^2

In Eq. (8) we make the following simplifications: (i) We neglect terms with five G's in comparison with those with six G's since each G contains a T dependence of $O(1/T)$ (This is similar to the arguments of GR and can be justified by a detailed examination). (ii) We set $\epsilon_{Q-p} \simeq \epsilon_F \simeq \epsilon_p$ since the G's are sharply peaked at the Fermi surface [see the discussion below Eq. (15)]. Converting to dimensionless variables we get

$$\alpha=-\Delta_2 T^2 \sum_{Q,p}[R_1+R_2(Q)][G^3(p)G^3(Q-p)+2G^2(p)G^4(Q-p)-2G^5(p)G(Q-p)] \; , \tag{30}$$

where

$$\Delta_2=\tfrac{32}{3}N(0)\mu_B^2 I^2 m^2 \; .$$

From the p sum we go over to ξ integration [see Eq. (15)] and on integrating by parts, we find that Eq. (30) is equivalent to

$$\alpha=4\Delta_2 T^2 \sum_{Q,p}[R_1+R_2(Q)]G(p)G^5(Q-p) \; . \tag{31}$$

The ξ integration and the ω_n summation is carried out using Eq. (A11). The remaining angular integral is trivial since the polygamma functions are perfect differentials and we find

$$\alpha=-\frac{\Delta_3}{\pi}\sum_{\nu}\int_0^{y_c} y\,dy[R_1+R_2(y,\nu)] \times \mathrm{Im}\psi^{(3)}(\tfrac{1}{2}+\tfrac{1}{2}\nu+iy) \; , \tag{32}$$

$$\Delta_3=\mu_B^2 I^2 k_F m^3/288\pi^6 \; . \tag{33}$$

Since R_1 and R_2 contains logarithmic terms, it is plausible that α will contain $\ln T$ also, barring cancellations between R_1 and R_2 such as those which occurred in $\chi(0)$. As before, we can convert the ν summation to an integration and split up the contribution as follows:

$$\alpha=\alpha_1+\alpha_2 \; , \tag{34}$$

$$\alpha_1=-\frac{4\Delta_3}{\pi}\ln T\int_0^{\infty}d\phi\int_0^{y_c}y\,dy \times \mathrm{Im}\psi^{(3)}(\tfrac{1}{2}+\phi+iy) \; , \tag{35}$$

$$\alpha_2=-\frac{4\Delta_3}{\pi}\int_0^{\infty}d\phi\int_0^{y_c}dy\,\mathrm{Im}\ln\Gamma(\tfrac{1}{2}+\phi+iy) \times \mathrm{Im}\psi^{(3)}(\tfrac{1}{2}+\phi+iy) \; . \tag{36}$$

Note that

$$\mathrm{Im}\psi^{(3)}(\tfrac{1}{2}+\phi+iy)=-\frac{\partial^3}{\partial\phi\partial y^2}\mathrm{Im}\psi(\tfrac{1}{2}+\phi+iy) =-\frac{\partial^2}{\partial\phi\partial y}\mathrm{Re}\psi^{(1)}(\tfrac{1}{2}+\phi+iy) \; . \tag{37}$$

The α_1 term has an explicit $\ln T$ dependence. Using Eq. (37) the ϕ integral in α_1 can be performed, followed by a partial integration over y. (The boundary term in the y integration vanishes.) Using Eq. (A7) we find

$$\alpha_1=2\Delta_3 \ln T \; . \tag{38}$$

The singularity in α_2 is more subtle, and can be uncovered only after doing the multiple integrals. We use Eq. (37) to integrate with respect to ϕ and y and obtain

$$\alpha_2=\frac{4\Delta_3}{\pi}\int_0^{\infty}d\phi\int_0^{y_c}dy[\mathrm{Re}\psi^{(1)}(\tfrac{1}{2}+\phi+iy)]^2 \; . \tag{39}$$

In order to evaluate the singular part of Eq. (39) it is sufficient to use the asymptotic form of Eq. (A4) which gives

$$\alpha_2=\frac{4\Delta_3}{\pi}\int_0^{\infty}d\phi\int_0^{y_c}dy\left[\frac{\phi}{\phi+y}\right]^2 \; . \tag{40}$$

We can separate the ϕ and y integrals by using the substitution $\phi=y\tan\theta$ and find

$$\alpha_2=-\Delta_3 \ln T \; . \tag{41}$$

Adding Eqs. (38) and (41) we get the final expression for the singular part of α

$$\alpha=(I^2\mu_B^2 m^3 k_F/288\pi^6)\ln T \; . \tag{42}$$

Clearly Eqs. (42) and (2) are identical (with $\hbar=1$ and neglecting constant additive terms). In fact our Eq. (39) is (apart from factors) the same as Eq. (A20) of GR. The $\ln T$ dependence of α can also be extracted by directly using the asymptotic forms for R, Eq. (21), but the full form is required to get the right coefficient.

V. DISCUSSION AND CONCLUSIONS

In summary we find that the logarithmic singularities of R lead to (a) no $\ln T$ term in the uniform susceptibility due to cancellations between two terms in R; (b) a $\ln T$ term in α with precisely the same coefficient as in the paper of GR. Note that the singularity requires a frequency summation in this case, in contrast to the case of superconductivity (where $\nu=0$ is the most important term). The reason is that R is singular for $\Omega \simeq 0$ which implies that all ν up to $O(1/T)$ contribute to the singularity in χ.

Thus the $\ln T$ term obtained by GR can be viewed as originating from p-p scattering. The considerable reduction in effort in obtaining α indicates that this is indeed the *natural* view point to understand this singularity. This immediately suggests the well-known cure for this singularity disease. One should not stop at second-order perturbation theory since for low enough temperatures, the effective coupling ($\simeq -I^2 \ln T$) becomes very large. Instead, the p-p t matrix should be used in place of R. This, in the ladder approximation, is given by

$$t(Q)=I/[1-IR(Q)] \ . \tag{43}$$

Thus the perturbation series in I has to be reshuffled to give one in $t(Q)$, a possibility which is well recognized in the theory of nuclear matter.[14] The lowest-order self-energy[15] and vertex part[16] in this approach are well known in the literature,

$$\Sigma(p_1)=T\sum_{p_2} t(p_1+p_2)G_0(p_2) \ ,$$

$$\Lambda(p_1,k)=1+(-T)\sum_{p_2} t(p_1+p_2)G_0(p_2+\tfrac{1}{2}k) \times G_0(p_2-\tfrac{1}{2}k) \ . \tag{44}$$

Thus α expressed in terms of $t(Q)$ instead of $R(Q)$ will not show any logarithmic singularity since t is small in the region where R is large. In fact the t matrix in this approximation has a zero as $T\to 0$. (For an attractive interaction, it would have a pole at some T_c). GR view the second-order expression as an element in an infinite p-h series, and find that the singularity persists on including higher-order paramagnon terms in this series. However, at low temperature it is precisely the p-p scattering for which perturbation theory fails. Hence we should view the second-order term as an element in the infinite p-p series as stated above. The replacement of R by a t matrix, which suppresses the second-order $\ln T$ contribution to α, will also affect any second-order $\ln T$ contributions to $\chi(0)$ that comes from R. It is clear that any possible such $T^2 \ln T$ contribution to $\chi(0)$ would be suppressed in a similar manner. This complements the findings of Refs. 4 and 5 where it is shown that long-wavelength p-h scattering does not give a $T^2 \ln T$ contribution. This does not of course rule out the possiblity of $T^2 \ln T$ terms coming from other regions of phase space not considered by us.

We should mention that within the $T=0$ formalism it is well known that p-p scattering leads to $\ln Q$ type terms (for example in the Landau interaction function[9,10]). The cure in this case is again well known—a p-p ladder summation should be invoked.[9–11]

It is interesting to note that if the large energy cutoff ω_L implicit in the use of δ function interactions, is allowed to go to infinity, the function R goes to infinity as $\sim \omega_L^{1/2}$ in three dimensions. This implies that perturbation theory breaks down. The cure in this case is once again the use of a t matrix which vanishes in this limit. Hence the self-energy and vertex corrections vanish, yielding for pure δ function interactions a susceptibility identical with the free value. This is an extension of Herring's comments[17] regarding the absence of an energy shift in three-dimensions for a pure δ function interaction.

In conclusion we stress that a Kohn-Luttinger-Ward-like theorem for the nonuniform susceptibility may not be valid order by order, as the singularity in α indicates. We suggest instead, that the $T=0$ and $T\to 0$ perturbation theories coincide *not between terms of a given order, but rather between (infinite) classes of perturbation terms.* Thus, denoting by χ_c the result of a class summation, we suggest that Eq. (2b) should be replaced by

$$\lim_{T\to 0}\lim_{k\to 0}\chi_c(k,T)=\lim_{k\to 0}\lim_{T\to 0}\chi_c(k,T) =\chi_c(0,0) \ .$$

APPENDIX: SOME USEFUL DEFINITIONS AND IDENTITIES

The expressions containing finite temperature Green's functions in the text can be usefully reduced by using the properties of polygamma functions[18] given below.

The pth-order polygamma function ($p=0,1,2$) is a logarithmic derivative of the Γ function

$$\psi^{(p)}(z)=\frac{d^{p+1}}{dz^{p+1}}\ln\Gamma(z) \ . \tag{A1}$$

For $p=1,2,3,\ldots$, it is related to a discrete sum ($z\neq 0,-1,-2,\ldots$)

$$\psi^{(p)}(z)=(-1)^{p+1}p!\sum_{n=0}^{\infty}\frac{1}{(n+z)^{p+1}}\ , \quad \text{(A2)}$$

with the digamma function $\psi^{(0)}\equiv\psi(z)$ being

$$\psi(z+1)=\psi(z)+\frac{1}{z}$$
$$=-\gamma+\sum_{n=0}^{\infty}\frac{z}{n(n+z)}\ , \quad \text{(A3)}$$

where $\gamma=0.5772$ is Euler's constant. The asymptotic forms for $z\to\infty$, $|\arg z|<\pi$ are

$$\psi^{(p)}(z)\to(-1)^{p-1}(p-1)!z^{-p}+O(z^{-p-1})\ , \quad \text{(A4)}$$

$$\psi(z)\to\ln z+O(1/z)\ , \quad \text{(A5)}$$

$$\ln\Gamma(z)\to(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln(2\pi)+O(z^{-1})\ . \quad \text{(A6)}$$

For certain values of the argument, the imaginary parts of the functions are simple,

$$\operatorname{Im}\psi(\tfrac{1}{2}+iy)=(\tfrac{1}{2}\pi)\tanh\pi y \quad \text{(A7)}$$

and for $y\to\infty$

$$\operatorname{Im}\ln\Gamma(\tfrac{1}{2}+iy)\to y\ln y\ . \quad \text{(A8)}$$

A useful symmetry relation for p odd and $z=\frac{1}{2}-\frac{1}{2}\nu+iy$ is

$$\operatorname{Im}\psi^{(p)}(1-z)+\operatorname{Im}\psi^{(p)}(z)=0=(-1)\,\pi\operatorname{Im}\frac{d^p}{dz^p}\cot\pi z\ . \quad \text{(A9)}$$

Finally, we prove a useful identity. Using contour integration we have

$$\sum_n\int_{-\infty}^{+\infty}\frac{d\xi}{(2\pi i)}\,\frac{1}{(\xi-i\omega_n)[\xi-i(\Omega-\omega_n)-x]^{p+1}}=\sum_n\frac{\theta(\omega_n)-\theta(\Omega-\omega_n)}{(2i\omega_n-i\Omega-x)^{p+1}}\ . \quad \text{(A10)}$$

With a change of variable $n\to-n-1-\nu$ for the second term and using Eq. (A2), we obtain the identity

$$\sum_n\int_{-\infty}^{+\infty}\frac{d\xi}{(2\pi i)}\,\frac{1}{(\xi-i\omega_n)[\xi-i(\Omega-\omega_n)-x]^{p+1}}=\frac{(-1)^{p+1}}{p!(4\pi iT)^p}\left[\psi^{(p)}\left(\frac{1}{2}-\frac{1}{2}\nu+\frac{ix}{4\pi T}\right)+(-1)^p\psi^{(p)}\left(\frac{1}{2}-\frac{1}{2}\nu-\frac{ix}{4\pi T}\right)\right]\ . \quad \text{(A11)}$$

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[13]See for example A. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), p. 16.
[14]Formal device used here is to invert Eq. (43) and write $I=t/1+tR$. Then expand I in powers of t, taking care not to double count. (We thank Professor C. S. Warke for this comment).
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