Spin-spin correlation functions in the frustrated two-dimensional planar model

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The low-temperature limit of the frustrated 2-D x-y model is studied via the generalized Villain model. The spin-spin correlation functions are calculated for two limits of the density of frustrated plaquettes "x_f." In the dilute case the model is described in terms of the usual spin waves, vortex and symmetry-breaking excitations, plus the "curved" or frustrated half-integer static charges. In this limit, to lowest order in x_f, the infinite susceptibility phase of the ferromagnetic model remains. The stability of this result is shown to hold including higher excited states for the static vortices although there seems to be a lower critical temperature. In the special limit $x_f \sim \frac{1}{2}$ a model with only vortex, symmetry-breaking, and frustrated plaquettes is studied. The spin-spin correlation functions are calculated for large separation distances. There, the susceptibility has an exponential behavior to lowest order in an expansion in the density of thermal vortices. We discuss under which conditions this result would lead us to the nonexistence of spin-glass ordering for the planar model in two dimensions.

I. INTRODUCTION

An important long-standing problem in the theory of phase transitions is to study the effects of disorder on the critical properties of model systems. In the past few years progress has been achieved towards this end mainly by using the renormalization-group theory. One type of disorder is to have antiferromagnetic impurities in a ferromagnetic host or vice versa. When the density of impurities is very small, it was found that the transitions remained sharp and second order with the same critical exponents independent of impurity concentration for $\alpha < 0$.[1-4] When the number of antiferromagnetic bonds increases, in addition to the paramagnetic and (anti)ferromagnetic phases, the onset of a new type of ordering known as the spin-glass (SG) may appear.[5]

The SG problem has drawn a lot of attention recently.[5,6] Several theories have been put forth to explain the experimental data, at least qualitatively, but full agreement has yet to be achieved. If, indeed, the SG phase posseses the attributes of a new phase in matter, it is tempting to try to achieve an understanding of its properties in analogy with the unified picture that has emerged recently in the studies of more conventional types of phase transitions. To carry out this program detailed analysis of different model systems should be performed. Progress in this direction indicates that the upper critical dimensionality, above which mean-field theory is exact, seems to be 6.[5,7] The lower critical dimensionality, below which SG ordering disappears, is suggested to be 4 by Fisch and Harris[8] and conjectured as 3 for vector spin systems by Anderson and Morgan.[9] For the Ising model it seems that the lower critical dimensionality is 2,[10] but this result is controversial.[11] In this paper we give an explicit calculation of the spin-spin correlation function of the two-dimensional frustrated classical x-y model that leads to an exponential decay of the susceptibility at low temperatures.

The SG phase presents a different type of ordering compared to the ones usually encountered in normal or nonpathological phase transitions. In an effort to pin down the essential feature of the SG phase, Toulouse,[12] following an idea attributed to Anderson, suggested that the concept of frustration is at the core of the SG properties (see also Kirkpatrick[13] and Vannimenus and Toulouse[14] for early discussions of frustration). In this paper we wish to study the disordered x-y model in two dimensions in terms of frustration. This will be done in the context of ideas and techniques recently developed by José, Kadanoff, Kirkpatrick, and Nelson[15] (JKKN hereafter), in their study of the low-temperature properties of the ferromagnetic case. An early qualitative analysis of the meaning of frustration in this model has been given by Villain.[16] Here the problem is recasted and Villain's results are rederived and used as the starting point in our analysis. The main result of this paper is the evaluation of the spin-spin correlation functions when the model has a finite density of frustrated squares. The goal is to determine how the behavior of the pure system changes due to the presence of disorder. The type of disorder is special and

corresponds to putting antiferromagnetic bonds randomly in a ferromagnetic configuration. The dilution by bonds of strength zero is not considered.

The key point in our analysis is the recognition that a frustrated square is equivalent to a half-integer vortex at a dual lattice site $\vec{R}$.[16] This type of vortex differs in character from the ones encountered in the ferromagnetic case; whereas the former ones appear and disappear as we change the temperature, the other ones are fixed in their positions and their number changes only when the density of antiferromagnetic bonds changes. For a given configuration of antiferromagnetic bonds we have a configuration of frustrated squares. Thus, we have to perform an average over the different configurations of frustrated plaquettes. Half-integer charges have appeared in another not completely unrelated set of problems.[17] Here we will use the nomenclature "fractional charge" instead of "half-integer static vortex" and leave the term vortex for the thermally excited ones.

In the disordered case at finite temperatures, both thermal vortices and fractional charges are present. Vortices interact via a logarithmic potential between themselves and with the fractional charges (see Sec. II). Thus, we will perform thermodynamic averages over the spin waves and vortices interacting among themselves and with the static background field created by the fractional charges. Afterwards, we shall perform the configurational average over the fractional-charge degrees of freedom. The fractional charge at the dual lattice site $\vec{R}$ is denoted by $\frac{1}{2}F(\vec{R})$. $F(\vec{R})$ is a random function that indicates whether a plaquette is frustrated or not and plays a role analogous to the frustration function defined by Toulouse. It can take values of 0 for nonfrustrated and of ± 1 for frustrated plaquettes.

The dilute frustration limit and the $x_f \sim \frac{1}{2}$ limit will be considered in this paper. In the dilute case ($x \ll 1$) most of the fractional-charge pairs will have $L \sim 1$ where L denotes the separation distance between the fractional charges forming the pair, and every fractional-charge dipole can be replaced by its effective field. If the total charge of the pair is zero, its effective field will be of a dipolar type and decays like $1/r^2$ (r is the distance at which we measure the field from the pair). At low temperatures, the vortex density is also very small and the effect of these fractional charges on the vortex pair as we change T will be negligible. Proving that indeed the correlation functions remain unchanged when $x_f \ll 1$ and for total fractional-charge dipole charge zero is straightforward. To show the stability of this result for higher excited states is, however, nontrivial. In the ferromagnetic case the stability of the results obtained in Ref. 15 were established from the recursion relations. Here it will be shown in Sec. III that states with ± 1 total fractional-charge dipole charge do not modify the pure result, although there is some evidence of a lower temperature below T_c for which the calculation seems to break down. The difficulty of extending the calculation to higher orders in x_f is discussed at the end of Sec. III.

When x is large, the density of fractional charges is large and the problem is rather complicated. However, because the fractional charge appear only in pairs forming large and small *fixed* dipoles and because we are interested only in the long-range properties of the spin-spin correlation functions, $\bar{g}_p$, we are able to calculate $\bar{g}_p$ when $x_f \sim \frac{1}{2}$.

The main result of this paper, given in Sec. IV, is that in this limit the thermal susceptibility of the model is finite. It decays exponentially with temperature as T goes to zero. The calculation is done in the strong-coupling limit to lowest order in y^2. In contrast with the pure case, there is not explicit appearance of a critical temperature in the model. Although, of course, we expect a Curie-like behavior at high temperatures. The derivation and arguments leading to this result are given in Sec. IV.

The outline of the paper is the following. In Sec. II we follow Toulouse and extract the frustration effect in the x-y model. This is done using the generalized Villain model Hamiltonian introduced in Ref. 15 and valid for large K. In terms of the dual representation techniques used by JKKN, the effective Hamiltonian is derived in terms of the dual thermal and disordered variables. In Sec. III the dilute limit $x_f \ll 1$ is considered and the first-order term in an expansion in powers of x_f for small x_f is given. Section IV gives the analysis for $x_f \sim \frac{1}{2}$ based on the plausibility arguments given above. In Sec. V, further comments on the results obtained in Secs. III and IV are given.

The model studied in this paper is defined on a lattice. In Appendix A, the continuum limit of the results of Sec. II are shown to lead to models analogous to the heuristically motivated, quenched-gauge model recently studied by Hertz.[18] Indications are given in this appendix on how to go about constructing frustration models from their lattice counterparts. An early brief communication of some of the results derived in this paper was published elsewhere.[19]

II. FRUSTRATION IN THE PLANAR MODEL

In this section, we shall follow Toulouse in extracting the frustration effect in the x-y model (see also Villain[16] for an alternative analysis). Since we are interested in the low-temperature regime, we prefer to study the generalized Villain (GV) model introduced by JKKN, due to its close similarity in behavior to the planar model. The generalized Villain model has the advantage that it separates explicitly the spin waves and vortex excitations in the problem. Because, as we shall see below, a frustrated plaquette has associated a half-integer fixed vortex, it is con-

venient to consider the generalized Villain model in our analysis. In the pure case, the generalized Villain model is defined by the Hamiltonian[15] ($\theta \in [0, 2\pi]$, $m = 0, \pm 1, \pm 2 \cdots$)

$$H_{GV} = -\frac{1}{2}K \sum_{\langle \vec{r},\vec{r}' \rangle} [\theta(\vec{r}) - \theta(\vec{r}') - 2\pi m(\vec{r},\vec{r}')]^2 + \ln y_0 \sum_{\vec{R}} [M(\vec{R})]^2 \ , \tag{2.1}$$

where (see Fig. 1)

$$M(\vec{R}) = m(1,2) + m(2,3) + m(3,4) + m(4,1) \ . \tag{2.2}$$

The last term in Eq. (2) corresponds to the cost in free energy of adding a pair of vortices to the system. y_0 takes values in the interval [0,1]. When $y_0 = 0$, the probability of exciting a vortex pair is zero and for $y_0 = 1$ we recover the original Villain model.[20] $\theta(\vec{r})$ and $m(\vec{r},\vec{r}')$ are the statistical variables of the model.

Adding a negative bond between two spins located at nearest-neighboring lattice sites $\vec{r}$ and $\vec{r}'$ introduces a "twist" in the orientation of the spins with respect to each other. In terms of the angular variables this is equivalent to transforming

$$\theta(\vec{r}) - \theta(\vec{r}') \rightarrow \theta(\vec{r}) - \theta(\vec{r}') - \pi f(\vec{r},\vec{r}') \ , \tag{2.3}$$

where $f(\vec{r},\vec{r}')$ is an integer random variable with probability x (density of antiferromagnetic bonds). The f's play a role similar to the m's if they are chosen to satisfy $f(\vec{r},\vec{r}') = -f(\vec{r}',\vec{r})$. Note that this choice gives a sign to the orientation of the f's in much the same way as Villain did for the m's. Without any loss of generality we take $f(\vec{r},\vec{r}') = \pm 1$ letting the m's take care of larger values of the f's. In fact, the $f(\vec{r},\vec{r}')$'s are analogous to the J_{ij} exchange integral in conventional spin-glass treatments.

The generalized-Villain model in this case can be written

$$\hat{H}_{GV} = -\frac{1}{2}K \sum_{\langle \vec{r},\vec{r}' \rangle} [\theta(\vec{r}) - \theta(\vec{r}') - 2\pi \mathfrak{L}(\vec{r},\vec{r}')]^2 + \ln \tilde{y}_0 \sum_{\vec{R}} [L(\vec{R})]^2 \ , \tag{2.4}$$

with $\mathfrak{L}(\vec{r},\vec{r}') = m(\vec{r},\vec{r}') + \frac{1}{2}f(\vec{r},\vec{r}')$. The total charge $L(\vec{R})$ at point $\vec{R}$ is (see Fig. 2)

$$L(\vec{R}) = M(\vec{R}) + \frac{1}{2}F(\vec{R}) \ . \tag{2.5}$$

Equation (2.4) has been written symmetrically in analogy with Eq. (2.1). The extra term added to Eq. (2.4) will not contribute to any of the results given in this paper. This is because we are interested in the case where x is fixed and T changes. If we were to consider changes in the density of fractional charge an appropriate separation of $\tilde{y}_0$ in terms of $y_0(M)$ and $y_0(F)$ would be needed. Toulouse has given a criteria to decide if a plaquette is frustrated or not in terms of a "frustration function" ϕ. In the case at hand, it becomes

$$\phi_{\vec{R}} = \exp[i2\pi L(\vec{R})] \ . \tag{2.6a}$$

If $L(\vec{R})$ is an integer, $\phi_{\vec{R}} = 1$, then the plaquette is not frustrated. When $L(\vec{R}) = \text{integer} \pm \frac{1}{2}$, $\phi_{\vec{R}} = -1$ and the square is frustrated. In terms of $F(\vec{R})$, these conditions are that $F(\vec{R}) = 0$ when the plaquette at $\vec{R}$ is not frustrated and $F(\vec{R}) = \pm 1$ when it is frustrated. Note that $\phi_{\vec{R}}$ does not distinguish from a plaquette with $F(\vec{R}) = 1$ or -1. However, thermodynamically the two states have different vorticities and are, therefore, distinguishable. In our calculation, from symmetry reasons, we choose to take the values of 0, ± 1 for $F(\vec{R})$, and leave the M's to take care of their thermodynamic character.

Extending the criteria given above, consider an arbitrary region $\mathfrak{R}$ contained in the two-dimensional lat-

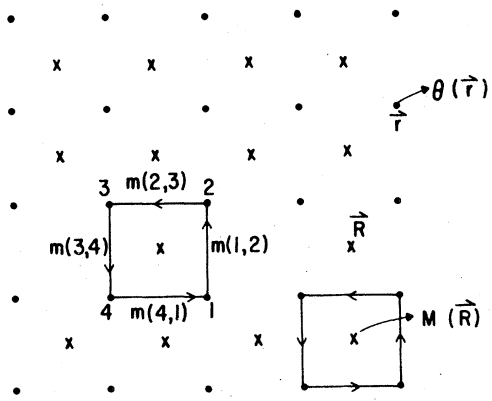

FIG. 1. Statistical variables in the planar model.

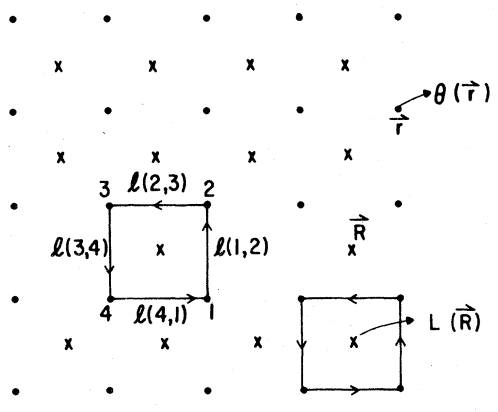

FIG. 2. Variables in the frustrated planar model.

tice. Take $\mathcal{R}$ to be simply connected. This means that we want to measure the total amount of frustration contained inside $\mathcal{R}$ without having to subtract any contributions from a subset region of $\mathcal{R}$. Because $F(\vec{R}) = \pm 1$, it follows that

$$\frac{1}{2}\sum_{\mathcal{R}} F(\vec{R}) = \frac{1}{2}{}^{\mathcal{R}}n \quad , \tag{2.6b}$$

where the integer ${}^{\mathcal{R}}n$ is the total number of positive minus negative fractional changes inside $\mathcal{R}$. The sum is taken over all the frustrated plaquettes inside $\mathcal{R}$ with $\vec{R} \in \mathcal{R}$, and ${}^{\mathcal{R}}N_f$ is equal to the number of frustrated squares contained in $\mathcal{R}$. If ${}^{\mathcal{R}}n$ is even, $\mathcal{R}$ is not frustrated and if odd, $\mathcal{R}$ is frustrated.

Following the general philosophy put forth by JKKN, we shall now find the dual representation of Eq. (2.4) in terms of the total charge $L(\vec{R})$. Thus we should eliminate the θ and $\mathcal{L}$ variables in favor of the dual variables. This operation is given by taking the trace

$$e^{H[L(\vec{R})]} = \prod_{\langle \vec{r},\vec{r}'\rangle,\mathcal{L}(\vec{r},\vec{r}')} \sum \int_0^{2\pi} \prod_{\vec{r}} \frac{d\theta(\vec{r})}{2\pi} e^{\tilde{H}_{GV}[\theta(\vec{r}),\mathcal{L}(\vec{r},\vec{r}')]} \quad , \tag{2.7}$$

with

$$\tilde{H}_{GV}[\theta,\mathcal{L}] = \hat{H}_{GV} + \sum_{\vec{R}} \ln V_{L(\vec{R})}(\vec{R}) \quad . \tag{2.8}$$

The last term in the Hamiltonian H_{GV} gives a constraint on the sums over $\mathcal{L}(\vec{r},\vec{r}')$,

$$V_{L(\vec{R})}(\vec{R}) = \delta_{L(\vec{R}),\mathcal{L}(1,2)+\mathcal{L}(2,3)+\mathcal{L}(3,4)+\mathcal{L}(4,1)} \quad . \tag{2.9}$$

This constraint is analogous to the one presented by Kirkpartrick[13] in his analysis of the Ising model and has a corresponding interpretation for the planar model. In the purely ferromagnetic case, $L(\vec{R}) = M(\vec{R})$ and the vorticity at $\vec{R}$ is generated thermally. Adding antiferromagnetic bonds to the system leads to plaquettes with a total number of negative bonds even or odd. In the first case, $L(\vec{R}) = M(\vec{R}) + M'(\vec{R})$, where $M'(\vec{R})$ is an integer charge, physically indistinguishable from $M(\vec{R})$ and which can be set equal to zero by a suitable gauge transformation. The behavior of the system is the same as in the ferromagnetic case, and thus we have a Mattis-like planar model.[21]

Note that we can perform all formal transformations for a given configuration of $\mathcal{L}$'s and that only at the end do we need to average over different configurations. A trace like that of Eq. (2.7) has been studied by Kadanoff recently.[22] Borrowing the result from his analysis we have

$$H[L(\vec{R})] = 2\pi K \sum_{\vec{R},\vec{R}'} L(\vec{R}) G(\vec{R}-\vec{R}') L(\vec{R}') + \ln y \sum_{\vec{R}} [L(\vec{R})]^2 \quad , \tag{2.10}$$

with

$$y = \tilde{y}_0 \exp(-\tfrac{1}{2}\pi^2 K) \tag{2.11}$$

and

$$G(\vec{R}-\vec{R}') \approx \ln|\vec{R}-\vec{R}'| \quad . \tag{2.12}$$

Equation (2.10) results after imposing the total charge neutrality condition

$$\sum_{\vec{R}} L(\vec{R}) = 0 \quad . \tag{2.13}$$

A similar result was found by Villain using a route different from ours. It should be stressed that the physical content of Eq. (2.10) is different from that of the usual two-dimensional (2-D) lattice Coulomb gas Hamiltonian. Although its expession is formally identical to the 2-D Coulomb gas model, the charge $L(\vec{R})$ is composed by two different kinds of vortices: the $M(\vec{R})$ vortices that are excited thermally and the static $F(\vec{R})$ vortices, or fractional charges, that act as quenched impurity charges in the system. The $F(\vec{R})$ are the vortices that are present due to the negative bonds in the system. They are present in the ground state and their number does not change as we change the temperature. The magnitude of the vorticity $F(\vec{R})$ may be taken, with no loss of generality, to be fixed as we increase T slightly. Therefore, we can subtract the ground-state energy associated with the $F(R)$'s and take the Hamiltonian

$$H = 2\pi K \sum_{\vec{R},\vec{R}'} M(\vec{R}) G(\vec{R}-\vec{R}') M(\vec{R}') + \pi K \sum_{\vec{R},\vec{R}'} F(\vec{R}) G(\vec{R}-\vec{R}') M(\vec{R}') + \ln y \sum_{\vec{R}} [M(\vec{R})]^2 \tag{2.14}$$

instead of Eq. (2.10). The vortices have a kind of "dynamical" behavior in the sense that as temperature is changed they may separate. In the pure case, the separation of vortices was an important element in the calculations of the correlation functions, particularly of the existence of T_c. On the other hand, the fractional charges are fixed in their positions and their distribution is related to the random distribution of antiferromagnetic bonds in the model. We believe that this is an important physical difference between

the two kinds of vortices that appear in the problem. The Hamiltonian in Eq. (2.14) has, in fact, the information that vortex-vortex and vortex-fractional-charge pairs may separate from each other. The last term of H corresponds to the energy required to excite a vortex pair, and justifies our not having taken a more explicit form for $\tilde{y}_0$ in terms of $y_0(F)$ in Eq. (2.4).

When the density of negative bonds is very small ($x \ll 1$) the density of frustrated squares, x_f, is also very small (roughly $\frac{1}{4}x$). The number of fractional-charges pairs is $\sim\frac{1}{2}N_f$ and the separation distance in a pair is, on the average, of the order of one lattice spacing. At large distances the pairs will look essentially like isolated charges. In this regime we can consider the total charge of a fractional-charge pair as the basic variable defined as

$$\tilde{F}(\vec{R}'') = [F(\vec{R}) + F(\vec{R}')]\delta_{\langle \vec{R}, \vec{R}' \rangle} \quad . \tag{2.15}$$

The bracket stands for nearest neighbors and $\vec{R}''$ is the center of mass position vector for the pair.

In Secs. III and IV, the vortex-vortex correlation function for a given configuration of fractional charges will be needed. As usual, we consider a vortex at $\vec{R}$ with charge +1 and another one at $\vec{R}'$ with charge of −1. The two point vortex correlation function is equal to the potential of mean force, e^{-W}, where W is the total field felt by the two vortices. The lowest order contribution to order y^2 is

$$\langle M(\vec{R})M(\vec{R}')\rangle_{F(\vec{R})} \sim -2y^2 \exp\left[-2\pi K G(\vec{R}-\vec{R}') + \pi K \sum_{\vec{R}''} F(\vec{R}'')[G(\vec{R}-\vec{R}'') - G(\vec{R}'-\vec{R}'')]\right] \quad . \tag{2.16}$$

The first term in the exponential comes from the bare interaction between vortices. The second term is the "mean field" felt by the two vortices due to the background field created by the fractional charges. Also, note that a configurational average should be taken with respect to $F(\vec{R})$ that takes values of 0, ± 1 in the manner described previously. The specific form of Eq. (2.16) follows from Eq. (2.14), which is appropriate in the two limits and approximations made in Secs. III and IV.

III. CORRELATION-FUNCTION CALCULATION

A. General form

In this section, we study the spin-spin correlation functions for the frustrated planar model. First we shall obtain the dual representation of the spin-correlation functions in general form. The dilute limit ($x_f \ll 1$) will be treated in this section, and the $x_f \sim \frac{1}{2}$ case will be examined in Sec. IV. Our goal is to determine how the correlations between well separated spins are modified by the combined effect of vortices and fractional charges. As was mentioned before, we shall average thermally over the vortex degrees of freedom and the resulting expression for the correlation functions will then be averaged configurationally over the fractional-charge degrees of freedom. There are two kinds of configurational averages, both of which have a different physical content: the "annealed" case that considers the impurities in thermal equilibrium with the normal elementary excitations in the system, and the quenched configurational average that considers the impurities frozen in their positions and far from equilibrium with the excitations in the model. Because in most experimental situations, the time it takes for the impurities to become in thermal equilibrium is much larger than the time it takes to carry out the experiment, it is the quenched state which is tested experimentally. In the present problem, taking the annealed average corresponds to assuming that there are no distinctions between the $M(\vec{R})$ and $F(\vec{R})$ excitations and implies that we should take the thermal average with respect to $L(\vec{R})$.

The quenched average entails taking averages over $F(\vec{R})$ instead of $\phi_{\vec{R}}$ as was the case in the Ising model.[13,16(b)] The explicit averaging procedures will be given below for $x_f \ll 1$ and in Sec. IV for $x_f \sim \frac{1}{2}$.

The configurational spin-spin correlation function of order p is defined as

$$g_p(\vec{r},\vec{r}';\{F\}) = \langle e^{ip[\theta(\vec{r})-\theta(\vec{r}')]}\rangle\{F\} \quad , \tag{3.1}$$

where $\langle\ \rangle$ stands for thermal average. The dual representation of g_p can be obtained following similar steps to those of Refs. 15 and 22. The result is

$$g_p(\vec{r},\vec{r}';\{F\}) = \exp\left[i\sum_{\vec{r}}\sum_{\vec{R}}\tfrac{1}{2}n(\vec{r})\Theta(\vec{r}-\vec{R})F(\vec{R})\right] \times \tilde{g}_p(\vec{r},\vec{r}';\{F\}) \quad , \tag{3.2}$$

with

$$\tilde{g}_p(\vec{r},\vec{r}';\{F\}) = \frac{\prod_{\vec{R}}\sum_{M(\vec{R})} e^{h[M,F,n]}}{\prod_{\vec{R}}\sum_{M(\vec{R})} e^{H[M,F]}} \quad . \tag{3.3}$$

The number $n(\vec{r})$ is zero everywhere, except at $\vec{r}$ and $\vec{r}'$, where it takes the values $n(\vec{r}) = -n(\vec{r}') = 1$. The Hamiltonian $h[M,F,n]$ is given by

$$h[M,F,n] = H[M,F] + ip\sum_{\vec{r}}\sum_{\vec{R}} n(\vec{r})\Theta(\vec{R}-\vec{r})M(\vec{R}) \quad . \tag{3.4}$$

The function $\Theta(\vec{R})$ was defined in Ref. 15 and, for large $\vec{R}$, it is given by

$$i\Theta(z) = \ln z - G(|z|) \ , \tag{3.5}$$

with $z = R_1 + iR_2$ and $\vec{R} = (R_1, R_2)$. At the end of this section we shall use a representation of Θ more convenient for our calculations.

Note that there is an extra weighting factor appearing in the correlation functions given in Eq. (3.2). To understand the meaning of this weighting factor, recall that $n(\vec{r})$ is different from zero only at the points where we have the spins being correlated. $F(\vec{R})$, on the other hand, is different from zero only in the case where the plaquette at $\vec{R}$ is frustrated. This means that the weighting factor will be different from one for a restricted number of $\vec{r}$ and $\vec{R}$ points. The angular potential $i\Theta(\vec{r}-\vec{R})$ couples the spins at $\vec{r}$ with the fractional charges at $\vec{R}$. Thus, this factor weights the contributions to g_p coming from different trajectories going between the different spins being correlated. This coefficient appears because of the gauge invariant calculation of the correlation functions.[16(b)] With our assumption that the total fractional charge and vortex charge are zero independently, it will be shown that, when $\mathcal{R}$ is equal to the area of the system, this extra factor does not contribute to our final result and is equal to one. $\tilde{g}_p$ is identical to the ferromagnetic correlation function studied by JKKN in the case where $F(\vec{R}) = 0$. In this section, we specialize to the dilute limit ($x_f << 1$) and obtain the correlation functions to leading order in an expansion in powers of x_f.

B. Dilute-limit calculation

Here we shall concentrate on the two-point spin-correlation functions. In this case Eq. (3.2) becomes

$$g_p(\vec{r},\vec{r}';\{F(\vec{R})\}) = \exp\left[i\left(\tfrac{1}{2}p\right)\sum_{\vec{R}} F(\vec{R})\,U(\vec{R})\right] \times \tilde{g}_p(\vec{r},\vec{r}';\{F(\vec{R})\}) \ . \tag{3.6}$$

$U(\vec{R})$ was defined in Ref. 15 as

$$U(\vec{R}) = \Theta(\vec{R}-\vec{r}) - \Theta(\vec{R}-\vec{r}') \pm 2\pi \ . \tag{3.7}$$

The extra $\pm 2\pi$ term comes from the discontinuity of the Θ function when crossing, say, the x axis. The value of the discontinuity depends on the ways of defining the path connecting the points $\vec{r}, \vec{r}'$ or, restated, on how we agree to measure the angles Θ with respect to that path. In the pure case, the vorticity is integral and this fact does not modify the equations. However, in the frustrated case, because the value of the fractional charges is $\pm\frac{1}{2}$, an extra factor

$$(-1)^{\sum_R pF(\vec{R})} = (-1)^{\mathcal{R}np} \tag{3.8}$$

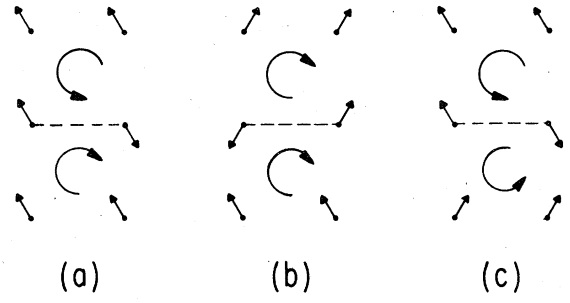

FIG. 3. Fractional-charge pairs with $\tilde{F}(\vec{R}) = 0$ [(a)] and with $\tilde{F}(\vec{R}) = \pm 1$ [(b), (c)].

appears in front of Eq. (3.2). This fact has been discussed with clarity in Ref. 16(b). In that work it is found that this extra factor appears from the gauge invariant evaluation of the spin-spin correlation functions. As mentioned in Sec. II, a region $\mathcal{R}$ is frustrated when $\mathcal{R}n$ is odd. In this case, $\mathcal{R}$ is the region enclosed by two arbitrary paths going from $\vec{r}$ to $\vec{r}'$. In the case when $\mathcal{R}$ is frustrated, the case of interest here, depending on the values of p we can have a $+1$ or -1 factor in Eq. (3.2); -1 for p odd and $+1$ for p even (see Fig. 3).

In order to obtain the physical spin-spin correlation function from Eq. (3.2) we shall average configurationally over $F(\vec{R})$. The averaging must incorporate the fact that fractional charges appear and disappear in pairs. As was mentioned in Sec. II, when the density of negative bonds is very small, the members of a fractional-charge pair are tightly bounded. The total charge of a fractional-charge pair that is considered here is equal to 0 or to ± 1 (see Fig. 4). This means that in the lowest order of approximation (in x_f) we can consider the fractional-charge pairs as individual entities. Thus we can take the total charge of a pair, $\tilde{F}(\vec{R})$ (see Eq. 2.15), as our basic variable instead of $F(\vec{R})$. This entails only changes of factors of $\frac{1}{2}$ in Eq. (3.2). We wish to expand $\tilde{g}_p$ in powers of x_f. The lowest order contribution requires averaging both with respect to the 0, ± 1 values of $\tilde{F}(\vec{R})$ and the volume average. This last average appears because although $\tilde{F}(\vec{R})$ can take only 0, ± 1 values for the frustrated plaquettes, their locations in $\vec{R}$ must be averaged too. The volume and $F(\vec{R})$ averages are given by

$$(1/N)\sum_{\vec{R}}\sum_{\tilde{F}(\vec{R})=0,\pm 1} \ . \tag{3.9}$$

N is the total number of dual lattice points and the sum over $\vec{R}$ is taken over the frustrated plaquettes.

To lowest order in x_f, g_p becomes

$$\tilde{g}_p(|\vec{r}-\vec{r}'|) \cong \tilde{g}_p(|\vec{r}-\vec{r}'|)(1-\tfrac{1}{2}x_f) + \tfrac{1}{2}x_f \prod_{\vec{R}} \sum_{\tilde{F}(\vec{R})=\pm 1} \exp\left(ip \sum \tilde{F}(\vec{R})\, U(\vec{R})\right) \tilde{g}_p(|\vec{r}-\vec{r}'|;\{\tilde{F}\}) + O(x_f^2) \quad . \quad (3.10)$$

The subscript $\tilde{F}(\vec{R})=0$ in the first term of Eq. (3.10) has been omitted. $\tilde{g}_p$ corresponds to the $x_f=0$ term in the expansion and is given by the correlation functions of the pure case. Because $x_f << 1$, the second term in the parenthesis can be neglected to lowest order of approximation. The form of the higher order contributions to $\tilde{g}_p$ will be discussed after evaluating Eq. (3.10) at low temperatures. The purely ferromagnetic $\tilde{g}_p$ was calculated in Ref. 15. It was found that below T_c, $\tilde{g}_p$ behaves like

$$g_p(r) \sim r^{-\eta(T,y)} \quad (3.11)$$

to lowest order in an expansion in powers of y^2. η is given by[15]

$$\eta = \left(\frac{p^2}{2\pi}\right)\left(T + 2\pi^2 y^2 \sum_{r_0} r_0^{2-2\pi K}\right) \quad , \quad (3.12)$$

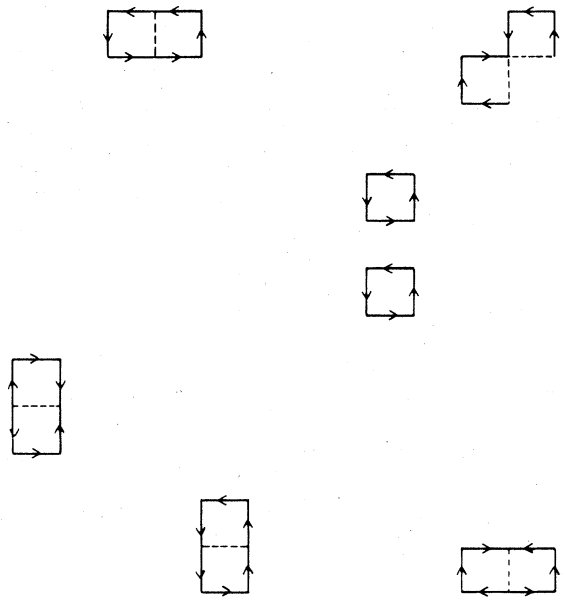

FIG. 4. Fractional-charge pairs and vortex pairs. The dotted line stands for an antiferromagnetic bond.

where the first term comes from the spin waves and the second from the vortex contribution good to order y^2. The critical temperature is defined by the condition $2\pi K = 4$, above which η blows up and the small y expansion breaks down. In this section, we are interested in finding the effects of the second term in Eq. (3.10) on the purely ferromagnetic result. We turn now to the explicit evaluation of the second term in Eq. (3.10) in the strong-coupling limit.

The total charge neutrality condition $\sum_{\vec{R}} L(\vec{R}) = 0$ allows for the following type of complexions: a fractional-charge–fractional-charge pair, a vortex-vortex pair, and a fractional-charge–vortex pair. The last two possibilities, as the temperature is changed, are qualitatively the same. Therefore, we will further assume that the total charge neutrality condition splits into two contraints $\sum_{\vec{R}} M(\vec{R}) = 0$ and $\sum_{\vec{R}} F(\vec{R}) = 0$.

The explicit evaluation of $\tilde{g}_p(\vec{r}_1, \vec{r}';\{\tilde{F}\})$ can be done by following the steps analogous to those in JKKN. The lowest nontrivial contribution to a cumulant expansion of $\tilde{g}_p$ is

$$\tilde{g}_p(\vec{r},\vec{r}';\{\tilde{F}\}) \cong \exp\left[i(\tfrac{1}{2}p) \sum_{\vec{R},\vec{R}'} \langle M(\vec{R})M(\vec{R}')\rangle_F U(\vec{R})\, U(\vec{R}')\right] \quad . \quad (3.13)$$

This expression is formally similar to the expression for $\tilde{g}_p$ in the pure case, except that now the vortices are in the presence of the background field generated by the $\tilde{F}$'s. This will change the explicit expression of $\tilde{g}_p(\tilde{F})$ with respect to its ferromagnetic counterpart. In the limit when vortices and fractional charges are very few in number (dilute regime), their average separation distances are large. As in the pure case, we shall assume that the main contribution to Eq. (3.13) comes from tightly bounded vortex pairs. This corresponds to taking $\vec{R} = \vec{R}_0 + \frac{1}{2}\vec{r}_0$ and $\vec{R}' = \vec{R}_0 - \frac{1}{2}\vec{r}_0$, with $|\vec{R}-\vec{r}| >> r_0$ and $|\vec{R}'-\vec{r}| >> r_0$, and the same condition is satisfied for the spin located at $\vec{r}'$. Carrying out an expansion around $\vec{R}_0$ [see Eq. (4.28) in Ref. 15] we obtain for the exponent in Eq. (3.13)

$$\frac{1}{2}p^2y^2\sum_{\vec{R}_0}\sum_{\vec{r}_0} r_0^{-2\pi K}\exp\left[\pi K\sum_{\vec{R}''}\tilde{F}(\vec{R}'')\vec{r}_0\cdot\nabla_{\vec{R}_0-\vec{R}''}G(\vec{R}_0-\vec{R}'')\right][\vec{r}_0\cdot\nabla_{\vec{R}_0}U(\vec{R})]^2 \ , \tag{3.14}$$

where we have used Eq. (2.16) in deriving Eq. (3.14). Equation (3.14) can be re-expressed as

$$\frac{1}{4}p^2y^2\sum_{\vec{R}_0}\sum_{\vec{r}_0} r_0^{-2\pi K}\exp\left[\pi K\sum_{\vec{R}''}F(\vec{R}'')\vec{r}_0\cdot\nabla_{\vec{R}_0-\vec{R}''}G(\vec{R}_0-\vec{R}'')\right]\{\vec{r}_0\cdot\nabla_{\vec{R}_0}[G(\vec{R}_0-\vec{r})-G(\vec{R}_0-\vec{r}')]\}^2 \ , \tag{3.15}$$

in which the Cauchy-Riemann conditions

$$\frac{\partial G(\vec{r})}{\partial x}=\frac{\partial\Theta(\vec{r})}{\partial y};\quad\frac{\partial G(r)}{\partial y}=-\frac{\partial\Theta(\vec{r})}{\partial x} \tag{3.16}$$

were used. $\vec{R}_0$ denotes the center of a vortex pair and $\vec{R}''$ the location of the center of mass for a fractional-charge dipole. In the dilute regime ($x_f << 1$), $|\vec{R}_0-\vec{R}''| >> 1$ and Eq. (3.15) can be approximated by

$$\frac{1}{4}p^2y^2\sum_{\vec{R}_0}\sum_{\vec{r}_0} r_0^{-2\pi K}\left[1+\pi K\sum_{\vec{R}''}F(\vec{R}'')\vec{r}_0\cdot\nabla_{\vec{R}_0}G(\vec{R}_0-\vec{R}'')\{\vec{r}_0\cdot\nabla_{\vec{R}_0}[G(\vec{R}_0-\vec{r})-G(\vec{R}_0-\vec{r}')]\}^2\right] \ . \tag{3.17}$$

The justification for this approximation is given in Appendix B. The first term in Eq. (3.17) when summed over $\vec{R}_0$ gives the same contribution as that of Eq. (4.32) in Ref. 15, i.e.,

$$\pi p^2y^2G'(|\vec{r}-\vec{r}'|)\sum_{r_0} r_0^{2-2\pi K} \ , \tag{3.18}$$

in which $G'(|\vec{r}-\vec{r}'|)$ is defined by

$$2\pi[G(\vec{r}-\vec{r}')-G(\vec{0})]=-G'(|\vec{r}-\vec{r}'|) \ . \tag{3.19}$$

The summation over $\vec{R}_0$ in the second term of Eq. (3.17) can be done by integrating by parts using Eq. (3.19) and

$$\nabla^2_{\vec{R}}G(\vec{R})=-2\pi\delta_{\vec{R},\vec{0}} \ , \tag{3.20}$$

The resulting expression for Eq. (3.17) takes the form

$$\pi p^2y^2G'(|\vec{r}-\vec{r}'|)\sum_{r_0} r_0^{2-2\pi K}\left[1-K\sum_{\vec{R}''}F(\vec{R}'')\vec{r}_0\cdot[\nabla_{\vec{r}}G(\vec{R}''-\vec{r})-\nabla_{\vec{r}'}G(\vec{R}''-\vec{r}')]\right] \ . \tag{3.21}$$

Substituting Eq. (3.21) in Eq. (3.13) and then into Eq. (3.10) we get

$$\bar{g}_p(|\vec{r}-\vec{r}'|)\cong\tilde{g}_p(|\vec{r}-\vec{r}'|)[1+\tfrac{1}{2}x_f(-1)^{\mathcal{R}_{np}}\Lambda(\vec{r},\vec{r}')]+O(x_f^2) \ , \tag{3.22}$$

where

$$\Lambda(\vec{r},\vec{r}')=\prod_{\vec{R}}\sum_{F(\vec{R})=\pm1}\delta\left[\sum_{\vec{R}}F(\vec{R})\right]\exp\left[i\sum_{\vec{R}}F(\vec{R})S(\vec{R})\right] \ . \tag{3.23}$$

The total fractional-charge neutrality condition is given in terms of the δ function. $S(\vec{R})$ has been defined as

$$S(\vec{R})=pU(\vec{R})-i\pi^2p^2y^2KG'(|\vec{r}-\vec{r}'|)\sum_{r_0} r_0^{2-2\pi K}\vec{r}_0\cdot[\nabla_{\vec{r}}G(\vec{R}-\vec{r})-\nabla_{\vec{r}'}G(\vec{R}-\vec{r}')] \ . \tag{3.24}$$

Carrying out the sum over F in Eq. (3.23) we find

$$\Lambda(\vec{r},\vec{r}')=\int_{-\infty}^{\infty}dt\exp\left[-i\sum_{\vec{R}}[t+S(\vec{R})]\right]\prod_{\vec{R}}(1+e^{2i[t+S(\vec{R})]}) \ . \tag{3.25}$$

Expanding the bracket and integrating over t Eq. (3.25) yields

$$\Lambda(\vec{r},\vec{r}')=\exp\left[-i\sum_{\vec{R}}S(\vec{R})\right]\left[\delta_{N_f,0}+\delta_{N_f-2,0}\sum_{\vec{R}_1}e^{i2S(\vec{R}_1)}+\delta_{N_f-4,0}\sum_{\vec{R}_1,\vec{R}_2}e^{i2[S(\vec{R}_1)+S(\vec{R}_2)]}+\cdots\right] \ . \tag{3.26}$$

In the limit of $|\vec{r}-\vec{r}'| >> 1$ we can keep only the first term in Eq. (3.26). The evaluation of the sum $\sum_{\vec{R}} S(\vec{R})$ will now be carried out. The sum over the first term in Eq. (3.25) can be obtained using the representation[22]

$$\Theta(\vec{r}-\vec{R}) = \lim_{\epsilon\to 0} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} K_\epsilon^{-1}(q) e^{i\vec{q}\cdot(\vec{r}-\vec{R})} \tag{3.27}$$

with

$$K_\epsilon^{-1}(q) = i\csc(\tfrac{1}{2}q_y)(2\cos q_x + 2\cos q_y - 4)(e^{-\epsilon+iq_x/2} - e^{iq_x/2}) \ . \tag{3.28}$$

For large separations ($|\vec{R}-\vec{r}|, |\vec{R}-\vec{r}'| >> 1$), $U(\vec{R})$ and $K_\epsilon(q)$ can be expanded in powers of q for small q. The evaluation of the integrals is straightforward, but the limit $\epsilon \to 0$ should be taken at the end. The summation of the second term in Eq. (3.24) is evaluated again integrating by parts and using Eq. (3.20). Therefore, to lowest order of approximation, Eq. (3.26) reads

$$\Lambda(\vec{r},\vec{r}') = \exp\left[-(\tfrac{1}{4}N_f p\pi^2)|\vec{r}-\vec{r}'| - \tilde{A}\ln|\vec{r}-\vec{r}'| \sum_{\vec{r}_0} r_0^{2-2\pi K}\vec{r}_0\cdot(\vec{r}-\vec{r}')\right] \tag{3.29}$$

with $\tilde{A} = 2\pi^2 y^2 K p^2$. Substituting Eq. (3.29) into Eq. (3.22) the final result of this section takes the form

$$\bar{g}_p(|\vec{r}-\vec{r}|) \cong \tilde{g}_p(|\vec{r}-\vec{r}'|)\left[1 + \tfrac{1}{2}x_f \exp\left[-\tilde{A}\ln|\vec{r}-\vec{r}'|\cdot|\vec{r}-\vec{r}'| \sum_{\vec{r}_0} r_0^{3-2\pi K}\right] + \cdots\right] \ . \tag{3.30}$$

The factor $(-1)^{\mathcal{R}np}$ is not present in Eq. (3.30) because of the fractional-charge neutrality condition, and $\mathcal{R}$ has been taken equal to the total area of the system. The first thing to note about Eq. (3.30) is that in the limit $x_f \to 0$, it reduces, as it should, to the purely ferromagnetic case result, i.e., zero magnetization for all $T \neq 0$ and infinite susceptibility for $T \leq \frac{1}{2}\pi$. Adding frustration has the effect of adding a term that is exponentially small as $|\vec{r}-\vec{r}'| \to \infty$ and therefore does not modify the infinite susceptibility phase of the ferromagnetic model at sufficiently low temperature. Recall, however, that in the limit of $x_f << 1$, the fractional-charge pairs are tightly bounded and the correction term has come from the fractional-charge pairs that have total charge of ± 1. These states have a lower probability of being excited in the system compared with the fractional-charge pairs with total charge of zero. The result for $\tilde{F}(\vec{R}) = 0$ can be easily understood by recalling that in the pure case the role of the vortex pairs does not become important at low temperatures. Furthermore, the vortex pairs become relevant only when the size of the vortex pair is of the order of the separation distance between the spins being correlated. Our result Eq. (3.30) shows that the higher-excited fractional-charge pair configurations do not change the result for the correlation functions obtained with the $\tilde{F}(\vec{R}) = 0$ charge. As it was mentioned before, in the pure case, the critical temperature ($y = 0$) was determined through the divergence of the exponent at $2\pi K = 4$. Of course, if we were to take higher-order contributions in the expansion around $\vec{R}_0$, lower critical temperatures would be expected because higher powers of $\vec{r}_0$ would appear in the expansions. However, it can be shown[23] that including higher-order terms in powers of $\vec{r}_0$ does not move the KT fixed point. Moreover, the fixed point obtained from the recursion relations is not affected. Note that the exponent in the second term in Eq. (3.30) seems to diverge at a lower temperature given by $T' = \frac{2}{5}\pi$. This divergence is of a different nature to the one found at $2\pi K = 4$. The reason is that it appears in a term that is exponentially small as $r \to \infty$. If we follow the arguments given by Kosterlitz[24] we see that there is no singular contribution from this term to the free energy. On the other hand, because of the behavior of the susceptibility when $x_f \sim \frac{1}{2}$ (see Sec. IV), this may be the first hint that the region in which the susceptibility is infinite has been shrunk from $[0, \frac{1}{2}\pi]$ to $[0, \frac{2}{5}\pi]$. However, we do not know how to connect these two facts conclusively because of the different nature of the model in the two regimes.

Finally, we turn to the way of calculating higher-order terms in the perturbation expansions in terms of x_f. This seems to be a rather difficult problem. In obtaining Eq. (3.30) it has been assumed that the system consists of a set of decoupled spin waves from vortex and frustrated squares. For $x << 1$, the fractional dipole size is ~ 1 and, in the dilute vortex regime, this is a good approximation. When the density of fractional charges increases, so do their sizes, and the decoupling of spin waves from the fractional charges is not obvious. In fact, the mere existence of spin waves at $x = \frac{1}{2}$ in equilibrium is questionable.[25,26] In order to extend the calculation given in this section, the fractional-charge spin-wave interactions may have to be included. This problem will be the subject matter of another communication. A

second problem is to include a configurational averaging procedure that incorporates the fractional-charge pairs formation properly. For instance, if the decoupling is valid,

$$\left[\frac{1}{N^2}\right]\prod_{\vec{R}}\sum_{F(\vec{R})=0,\pm1}\prod_{\vec{R}'}\sum_{F(\vec{R}')=0,\pm1}\delta_{|\vec{R}-\vec{R}'|,1+\delta(x)}$$

would account for the configurational average of the next-order correction of Eq. (3.30). $\delta(x)$ stands for the increment of the fractional-charge size (in units of the lattice spacing) as a function of x. To this order, we do not see why to expect changes from the $O(x_f)$ results. It is not obvious how the higher-order contributions to the expansion in powers of x_f will give the desired result of depressing T_c. Perhaps a double expansion in powers of x_f and $\sim\exp(-1/x_f)$ contain this information.[27]

In Sec. IV, we will calculate the vortex contribution to the spin-spin correlation function when x is not small. This will be based on a Hamiltonian that contains the vortex-vortex and vortex–fractional-charge logarithmic interactions. If the spin-waves are screened and the short-wavelength cooperative excitations of the spins serve the purpose of just renormalizing the Coulomb interactions, the calculation given in Sec. IV will acount for the properties of the frustrated (x finite) planar model in two dimensions when $x_f\sim\frac{1}{2}$.

IV. CORRELATION FUNCTIONS FOR $x_f\sim\frac{1}{2}$

In this section we investigate the behavior of the correlation functions for $x_f\sim\frac{1}{2}$. This limit corresponds to having less than half of the plaquettes frustrated and slightly more than half not. This statement follows from assuming $x_f=4[x^3(1-x)+(1-x)^3x]$ (see Ref. 13). Note, that $x_f\sim\frac{1}{2}$ for a range of values of $x\in[0.2, 0.5]$.[13] As pointed out by Villain[16] the Edwards-Anderson (EA) point ($x=\frac{1}{2}$)[5] is special. For this density of negative bonds, the plaquettes are distributed randomly with probability $\frac{1}{2}$ of being frustrated and $\frac{1}{2}$ of not. The probability of finding a plaquette at $\vec{R}$ which is frustrated takes the form

$$P(F(\vec{R}))=(1-x_f)\delta(F(\vec{R}))+x_f\delta(F^2(\vec{R})-1)\ . \tag{4.1}$$

Strictly speaking Eq. (4.1) makes sense physically only at $x_f=\frac{1}{2}$ ($x=\frac{1}{2}$). Here we assume, however, that Eq. (4.1) makes sense for x_f very close to $\frac{1}{2}$ but not for values that differ appreciably from $\frac{1}{2}$. Note that Eq. (4.1) refers to isolated fractional charges. Nevertheless, as we have noted above, the fractional-charge number can only be even, since they come in pairs. These facts should be implemented in the calculation. It will be argued below that within our approximation this can be done at the level of $P(F(\vec{R}))$. At finite densities of antiferromagnetic bonds, the system contains thermally excited vortices and an array of quenched fractional-charge dipoles of different sizes (see Fig. 5).

Formally speaking the problem should be defined in terms of a Hamiltonian that includes all possible interactions: the dipole-dipole and dipole-charge, that decay like $1/r^2$ and $1/r$, respectively, plus the charge-charge that grows logarithmically. To lowest order of approximation we shall take into account only the logarithmic interactions between fractional charges and vortices. Note that very few fractional charges will "look" isolated to the dilute thermally excited vortices. Recall that this argument is true for an element of the ensemble of fractional-charge configurations. The average spin-spin correlation function takes the form

$$\bar{g}_p(\vec{r},\vec{r}')=\prod_{\vec{R}}\sum_{F(\vec{R})}a^{-1}P(F(\vec{R}))\times\exp\left[i\left(\tfrac{1}{2}p\right)\sum_{\vec{R}}F(\vec{R})U(\vec{R})\right]\times g_p'(\vec{r},\vec{r}';\{F(\vec{R})\})\ , \tag{4.2}$$

where the normalization factor is chosen so that

$$a=\prod_{\vec{R}}\sum_{F(\vec{R})}P(F(\vec{R}))\ . \tag{4.3}$$

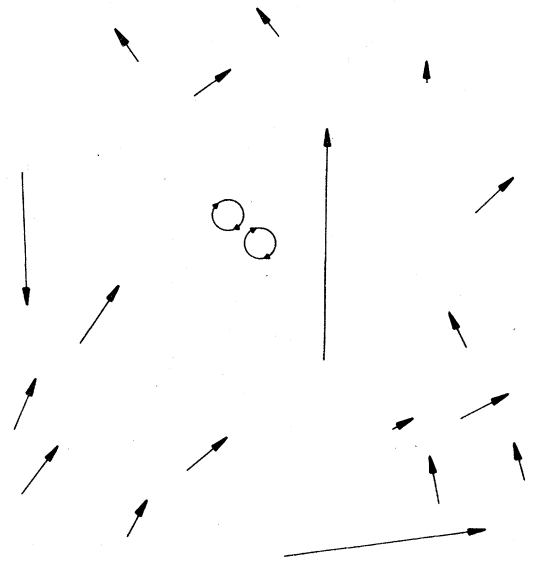

FIG. 5. Fractional-charge dipoles of different sizes (arrows) and a vortex pair (circles).

In general $g_p'(\vec{r},\vec{r}';\{F(\vec{R})\})$ is the configurational spin-spin correlation function obtained from the effective Hamiltonian. Within our approximation, g_p' is obtained from the Hamiltonian with logarithmic interactions. Care must be taken to not overcount the degrees of freedom in the system. If the sum is over N_f frustrated squares, we must subtract the ones that form part of the small fractional-charge dipoles. This is easily done at the level of $P(F(\vec{R}))$. A plaquette at $\vec{R}$ is either frustrated or not frustrated. However, there will be many frustrated plaquettes that will not contribute because they form part of a fractional-charge pair that acts like a small dipole. Therefore, we should add a term to Eq. (4.1) that measures the probability that a frustrated plaquette belongs to a pair that does not contribute, i.e.,

$$P(F(\vec{R}))=(1-x_f)\delta(F(\vec{R})) + x_f\delta(F^2(\vec{R})-1)-\Omega\delta(F(\vec{R})) \quad . \tag{4.4}$$

When x_f is very close to $\frac{1}{2}$, very few of the fractional charges will count as isolated and therefore Ω will also be close to $\frac{1}{2}$. At the particular point

$$x_f+\Omega=1 \tag{4.5}$$

the probability function (4.4) becomes

$$P(F(\vec{R}))\propto\delta(F^2(\vec{R})-1) \quad . \tag{4.6}$$

This equation is in fact true for $x_f\sim\frac{1}{2}$, the region of validity of Eq. (4.1). At this point Eq. (4.6), the correlation functions (4.2) become

$$\bar{g}_p(\vec{r},\vec{r}')=\prod_{\vec{R}}\sum_{F(\vec{R})=\pm1}\exp\left[i(\tfrac{1}{2}p)\sum_{\vec{R}}F(\vec{R})U(\vec{R})\right]g'(\vec{r},\vec{r}';\{F(\vec{R})\}) \quad . \tag{4.7}$$

At first glance, the correlation function $\bar{g}_p$ seems to have the same form as the second term in Eq. (3.10) but the physics behind both of them is completely different. First, in the dilute limit, the decoupling between spin waves and vortex–fractional-charge configurations was fully justified. The second difference between Eqs. (3.10) and (4.7) is that we were able to expand the vortex-vortex configurational correlation function in powers of N_f/N because $N_f \ll N$. Here, we can not approximate the vortex-vortex correlation function and must resort to a different procedure.

A. Spin-spin correlation-function calculation

The expression to be calculated is

$$\bar{g}_p(\vec{r},\vec{r}')=\prod_{\vec{R}}\sum_{F(\vec{R})=\pm1}\delta\left[\sum_{\vec{R}}F(\vec{R})\right]\exp\left[i(\tfrac{1}{2}p)\sum_{\vec{R}}F(\vec{R})U(\vec{R})\right]\left\langle\exp\left[ip\sum_{\vec{R}}M(\vec{R})U(\vec{R})\right]\right\rangle . \tag{4.8}$$

As in Sec. III the strong-coupling limit of Eq. (4.8) will be obtained by considering that the main contribution to Eq. (4.8) comes from tightly bounded vortex pairs. A cumulant expansion of the thermal average leads to

$$\bar{g}_p(\vec{r},\vec{r}')=\prod_{\vec{R}}\sum_{F(\vec{R})=\pm1}\delta\left[\sum_{\vec{R}}F(\vec{R})\right]\exp\left[i(\tfrac{1}{2}p)\sum_{\vec{R}}F(\vec{R})U(\vec{R})+W(F(\vec{R});\vec{r},\vec{r}')\right] , \tag{4.9}$$

where W has been defined by

$$W(F(\vec{R});\vec{r},\vec{r}')=-\tfrac{1}{2}p^2y^2\sum_{\vec{R}_0}\sum_{\vec{r}_0}r_0^{-2\pi K}\exp\left[\pi K\sum_{\vec{R}''}F(\vec{R}'')\vec{r}_0\cdot\nabla_{\vec{R}_0-\vec{R}''}G(\vec{R}_0-\vec{R}'')\right][\vec{r}_0\cdot\nabla_{R_0}U(\vec{R})]^2 , \tag{4.10}$$

with $\vec{R}_0$, $\vec{r}_0$, and $\vec{R}''$ having the same meaning as in Sec. III. Using integration by parts repeatedly and Eqs. (3.19) and (3.20), Eq. (4.10) takes the form

$$W=p^2y^2G'(|\vec{r}-\vec{r}'|)\sum_{r_0}r_0^{2-2\pi K}\left[\exp\left[\pi K\sum_{\vec{R}''}F(\vec{R}'')\vec{r}_0\cdot\nabla_{\vec{r}}G(\vec{R}''-\vec{r})\right]+\exp\left[\pi K\sum_{\vec{R}''}F(\vec{R}'')\vec{r}_0\cdot\nabla_{\vec{r}'}G(\vec{R}''-\vec{r}')\right]\right] . \tag{4.11}$$

Expanding $\exp W$ in series, for large $|\vec{R}-\vec{r}|$ and $|\vec{R}''-\vec{r}'|$, we find in Appendix C that Eq. (4.9) becomes

$$\bar{g}_p(\vec{r},\vec{r}') \cong \sum_n \left(\frac{1}{n!}\right) \hat{A}^n(r_0) \prod_{\vec{R}} \sum_{F(\vec{R})=\pm 1} \delta\left(\sum_{\vec{R}} F(\vec{R})\right) \exp\left(\sum_{\vec{R}} F(\vec{R})\, T_n(\vec{R})\right) , \tag{4.12}$$

in which

$$\hat{A}(r_0) = -p^2 y^2 G'(|\vec{r}-\vec{r}'|) \sum_{r_0} r_o^{2-2\pi K} \tag{4.13}$$

and

$$T_n(\vec{R}) = i(\tfrac{1}{2}p)\, U(\vec{R}) + n\pi K\, \vec{r}_0 \cdot [\nabla_{\vec{r}} G(\vec{R}''-\vec{r}) - \nabla_{\vec{r}'} G(\vec{R}''-\vec{r}')] \ . \tag{4.14}$$

Carrying out the sum over F in Eq. (4.12) we find

$$\bar{g}_p(\vec{r},\vec{r}') \cong \sum_n \left(\frac{1}{n!}\right) \hat{A}^n(r_0) \int_{-\infty}^{\infty} dt \exp\left(\sum_{\vec{R}} T_n(\vec{R}) + iN_f t\right) \prod_{\vec{R}} [1 + e^{-i2[T_n(\vec{R})+t]}] \ . \tag{4.15}$$

If we now expand the result in square brackets and integrate over t, Eq. (4.15) becomes

$$\bar{g}_p(\vec{r},\vec{r}') \cong \sum_n \left(\frac{1}{n!}\right) \hat{A}^n(r_0) \exp\left(\sum_{\vec{R}} T_n(\vec{R})\right) \left(\delta_{N_f,0} + \delta_{N_f-2,0} \sum_{\vec{R}} e^{-2iT_n(\vec{R})} + \cdots\right) \ . \tag{4.16}$$

As in Sec. III we shall keep only the leading contribution to Eq. (4.16). This is again a good approximation in the limit $r \to \infty$. The sum over $\vec{R}$ in $T_n(\vec{R})$ can be carried out giving

$$\sum_{\vec{R}} T_n(\vec{R}) = -(\tfrac{1}{2}\pi^2) N_f p\, |\vec{r}-\vec{r}'| - 2n\pi^2 K\, \vec{r}_0 \cdot (\vec{r}-\vec{r}') \ , \tag{4.17}$$

where we have used the fact that

$$\sum_{\vec{R}''} [\nabla G(\vec{R}''-\vec{r}) - \nabla_{\vec{r}'} G(\vec{R}''-\vec{r}')] = -2\pi(\vec{r}-\vec{r}') \ , \tag{4.18}$$

which is easily shown using integration by parts and Eq. (3.20) plus the expression (3.27) for $\Theta(\vec{R}-\vec{r})$. Performing the sums over n, Eq. (4.16) takes the form

$$\bar{g}_p(|\vec{r}-\vec{r}'|) \cong (-1)^{\mathfrak{N} n p} \exp[-\ln|\vec{r}-\vec{r}'|\, \hat{A}(r_0) e^{-\pi^2 r_o K|\vec{r}-\vec{r}'|}] \ . \tag{4.19}$$

The extra factor of $(-1)^{\mathfrak{N} np}$ disappears when we take the two paths going from $\vec{r}$ to $\vec{r}'$, and enclosing $\mathfrak{N}$, covering the whole area of the system plus the total fractional-charge neutrality condition (see Fig. 6). The sum over r_0 can be transformed into the integral

$$(\pi^2 K|\vec{r}-\vec{r}'|)^{4-2\pi K} \int_1^{\infty} dr_0\, r_0^{3-2\pi K} e^{-\pi^2 K|\vec{r}-\vec{r}'| r_0} = \Gamma(4-2\pi K,\, \pi^2 K|\vec{r}-\vec{r}'|) \ , \tag{4.20}$$

where Γ is the conventional incomplete Γ function. In the limit when $|\vec{r}-\vec{r}'| >> 1$ the asymptotic form of Eq. (4.20) gives

$$\int_1^{\infty} dr_0 r_0^{3-2\pi K} e^{-\pi^2 K r_0|\vec{r}-\vec{r}'|} \approx \frac{e^{-\pi^2 K|\vec{r}-\vec{r}'|}}{\pi K|\vec{r}-\vec{r}'|} \left[1 + O\left(\frac{3-2\pi K}{2\pi^2 K|\vec{r}-\vec{r}'|}\right) + \cdots\right] \ . \tag{4.21}$$

Substituting this result back into Eq. (4.19) we obtain

$$\bar{g}_p(|\vec{r}-\vec{r}'|) \cong \exp\{-[2\pi^2 y^2 p^2 \ln|\vec{r}-\vec{r}'|\, (\pi^2 K|\vec{r}-\vec{r}|)^{-1} \exp(-\pi^2 K|\vec{r}-\vec{r}'|)]\} \ . \tag{4.22}$$

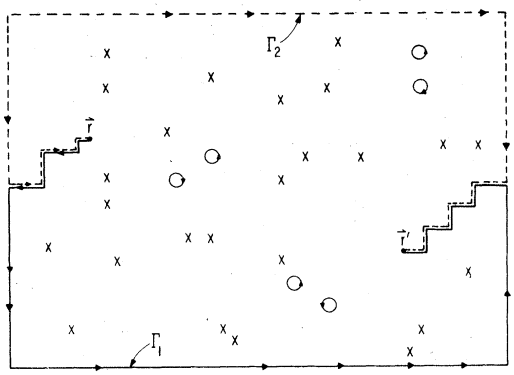

FIG. 6. Paths Γ_1 and Γ_2 going from $\vec{r}$ to $\vec{r}'$ encircling the region R. Crosses denote fractional charges and circles vortices.

This is our final result for the calculation of the spin-spin correlation functions. In Sec. IV B we will put together the result of Sec. IV B plus Eq. (4.22) and will discuss their consequences.

B. Finite susceptibility at low temperatures

We shall now derive and discuss the consequences from the result of Sec. IV A. The contributuion to the correlation functions coming from the pure frustration variables has so far been left out. This corresponds to the $T=0$ properties of the model, and therefore, to the possible degeneracies of the ground state. In the limit of sufficiently low temperatures, such that the SG phase is stable, the full spin-spin correlation functions can be written

$$\hat{G}_p(r) \cong \bar{g}_p^{(m)}(r)\bar{g}_p(r) \ , \tag{4.23}$$

where $\bar{g}_p^{(m)}$ is the pure frustration $T=0$ contribution to $\hat{G}_p(r)$. As we shall see below, the not knowing $\bar{g}_p^{(m)}(r)$ explicitly will not hamper our results for the behavior of the thermal susceptibility of the model. From Eq. (4.22), $\hat{G}_p(r)$ is equal to

$$\hat{G}_p(r) \cong \bar{g}_p^{(m)}(r)(\exp - \{2y^2p^2[\ln r \exp(-\pi^2 Kr)](Kr)^{-1}\}) \ . \tag{4.24}$$

In the large r limit, we obtain

$$\hat{G}_p(r) \cong \bar{g}_p^{(m)}(r)\left[1 - 2p^2y^2\frac{e^{-\pi^2 Kr}}{Kr}\right] \ , \tag{4.25}$$

where we have taken the leading contribution in the large r expansion, and have neglected the logarithmic contributions. The first term of Eq. (4.25) is a nonthermodynamic $T=0$ contribution to $\hat{G}_p(r)$. It is the second, temperature-dependent term in which we are interested. We can define

$$\overline{\Psi}_p(r) = \bar{g}_p^{(m)}(r)(2p^2y^2e^{-\pi^2 Kr}/Kr) \ , \tag{4.26}$$

as the thermal contribution to the spin-spin correlation functions. It is for this function that it makes sense to apply the fluctuation dissipation theorem. Before that can be done, we ought to know the form of $\bar{g}_p^{(m)}(r)$. Without having an explicit calculation for $\bar{g}_p^{(m)}(r)$, we can expect that for large r it should have the canonical behavior[28]

$$\bar{g}_p^{(m)}(r) \sim e^{-\alpha_F r}/r^{\beta_F} \ , \tag{4.27}$$

where α_F and β_F are temperature-independent functions of x_f. α_F should be positive definite. If $\alpha_F=0$, β_F is strictly bigger than zero. Because of the form of $\bar{g}_p(r)$, we can assume, without loss of generality that $\beta_F=0$ and $\alpha_F \geq 0$. The fluctuation dissipation theorem

$$\chi_p = \left(\frac{K}{N}\right)\sum_r \Psi_p(r) \ , \tag{4.28}$$

can now be applied. The evaluation of Eq. (4.28) is straightforward giving

$$\chi_p \cong 2p^2y^2/(\pi^2 K + \alpha_F) \ . \tag{4.29}$$

This is the main result of this section. It gives a susceptibility that is finite, and that goes to zero essentially exponentially with temperature. This result should be contrasted with the infinite susceptibility result of the pure and dilute cases. The calculation has been carried out in the strong-coupling limit. In contrast to the ferromagnetic case no evidence is found for the existence of a critical point. At high temperatures we expect that χ should have a Curie-Weiss behavior. The change of behavior of χ, going from the low to the high-temperature regions, can occur smoothly or abruptly. In the first case, χ would have a rounded maximum, in a similar way as a one-dimensional antiferromagnetic Heisenberg model.[29] We see, however, no evidence of a possible critical temperature in our calculation, and this may indeed be the case. This would agree with the speculations put forth by Anderson and Morgan[9] and a $2+\epsilon$ calculation done by Grest,[4] although their analyses did not take into account the specific properties of the frustrated planar model. On the other hand, we have no reasons to rule out a possible cusp or a sharper peak at a "freezing" temperature separating the low from the high-temperature behavior. If this is the case, a different kind of SG phase would exist at low temperatures. This possibility would be in disagreement with conjectures put forth by some authors.

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APPENDIX A: CONTINUUM FRUSTRATION MODELS

Here we wish to extend the analysis presented in Sec. II in two respects: First, continuous variations of $F(\vec{R})$ will be allowed. The continuum limit of the lattice Hamiltonian will be taken and, at the end, some comments will be made with respect to the connections of the resulting models to the continuous gauge model of Hertz.[18]

We start by recalling the gaugelike properties of the random field $f(\vec{r},\vec{r}')$. It has zero divergence and its curl is twice the fractional charge. In fact, $f(\vec{r},\vec{r}')$ plays an analogous role to the exchange coupling constant J_{ij} in the more conventional treatment of spin-glass models. Therefore, allowing continuous variations of $f(\vec{r},\vec{r}')$ is in some respects analogous to changing the magnitude of the coupling constants between the nearest-neighboring spins. For convenience we shall denote by $A(\vec{r},\vec{r}')$ the continuous version of $f(\vec{r},\vec{r}')$. $F(\vec{R})$ was defined in Sec. II as the circulation of $f(\vec{r},\vec{r}')$ around a plaquette. $F(\vec{R})$ can be rewritten (see Fig. 1)

$$F(\vec{R})=[f(1,2)-f(4,3)]-[f(3,2)-f(4,1)] \ . \tag{A1}$$

In order to differentiate an $f(\vec{r},\vec{r}')$ in the x direction from another in the y direction a Greek subindex will be added to A from here on. Hence Eq. (A1) in terms of A_ν takes the form

$$F_{\mu\gamma}(\vec{R})=[A_\gamma(\vec{r}+\hat{e}_\mu,\vec{r}+\hat{e}_\mu+\hat{e}_\gamma)-A_\gamma(\vec{r}_1,\vec{r}+\hat{e}_\gamma)] - [A_\mu(\vec{r}+\hat{e}_\mu,\vec{r})-A_\mu(\vec{r}+\hat{e}_\mu+\hat{e}_\gamma,\vec{r}+\hat{e}_\gamma)] \ , \tag{A2}$$

where $\hat{e}_\mu$ and $\hat{e}_\gamma$ are unitary vectors in the μ and γ directions, respectively. Written in this way the connection of $F_{\mu\gamma}(\vec{R})$ to the usual

$$F_{\mu\gamma}=\nabla_\mu A_\gamma-\nabla_\gamma A_\mu \tag{A3}$$

in the continuum limit is quite suggestive. However, we should warn the reader about the formal similarities among expressions like Eq. (A3) in the frustration problem and their electrodynamic counterparts. They have, of course, completely different physical meanings.

The full x-y model Hamiltonian can now be written coupled to the frustration gauge field, in the form

$$H[\Theta,A]=-K\sum_{\vec{r},\vec{r}'}\{1-\cos[\Theta(\vec{r})-\Theta(\vec{r}')-\pi A(\vec{r},\vec{r}')]\} \ . \tag{A4}$$

The prescription for evaluating thermodynamic quenched variables is: first, calculate the thermodynamic average with respect to the Boltzmann factor resulting from Eq. (A4) for a given configuration of A's and then, average over the A's. For any thermodynamic density, T, defined in terms of the dual variables $F_{\mu\gamma}$, the configurational average can be defined by

$$\bar{T}=\int_{-\infty}^{\infty}\prod_{\vec{R}}dF_{\mu\gamma}(\vec{R})P(F_{\mu\gamma}(\vec{R}))T(F_{\mu\gamma}(\vec{R})) \ , \tag{A5}$$

where $P(F(\vec{R}))$ is the probability weighting function for $F_{\mu\gamma}(\vec{R})$. As motivated in Sec. IV we can take

$$P(F_{\mu\gamma}(\vec{R}))\propto\delta(F_{\mu\gamma}^2(\vec{R})-1) \ , \tag{A6}$$

for the $x_f\sim\frac{1}{2}$. Equation (A6) forces a frustrated plaquette at the dual lattice site, $\vec{R}$, to have either $+\frac{1}{2}$ or $-\frac{1}{2}$ fractional charge. Considering smooth variations of $F_{\mu\gamma}(\vec{R})$ we can approximate Eq. (A6) by

$$P(F_{\mu\gamma}(\vec{R}))\cong e^{-(1/e^2)(F_{\mu\gamma})^2-\lambda(F_{\mu\gamma})^4} \tag{A7}$$

for a given point $\vec{R}$. If $\lambda=0$ and $e^2=\infty$ this forces the value of $F(\vec{R})$ to zero, or, equivalently, to the nonfrustrated case. When $\lambda=0$ and $e^2\neq 0$ the plaquette is "slightly" frustrated. In the limit when $-e^2\to 0$ and $\lambda\to\infty$ while $1/e^2=-2\lambda$ we recover Eq. (A6).

We proceed now to take the continuum limit of the lattice model. Clearly, the difference between a lattice point in the original lattice and its dual vanishes and $F(\vec{R})$ becomes a frustration density at point $\vec{R}$. Of course, an appropriate prescription must be given to obtain the averaged density of frustration at point $\vec{R}$. Here, this concept shall be used formally without any explicit definition because it seems to be of no consequence for the conclusions of this appendix. The Hamiltonian (A4) can now be expanded in powers of θ and the gradients of θ. For arbitrary dimensionality d we shall take θ ranging from $-\infty$ to ∞ and denoting it instead by φ. The gauge invariant gradient is

$$\nabla_\mu-\pi A_\mu \ .$$

Keeping the lowest order contribution in the gradient, Eq. (A4) becomes formally

$$H_c=\int d^dx\{[(\nabla_\mu-\pi A_\mu)\varphi]^2+V(\varphi)\} \ . \tag{A8}$$

x denotes a d-dimensional position vector. The second term of H_c is an even polynomial in φ that

can be of the Ginzberg-Landau form. In the continuum limit the integral in Eq. (A5) becomes a functional integral as usual. We can think now of various extensions of the model defined by Eqs. (A7) and (A8). First, instead of having φ a scalar, φ can be thought of as being an n-component vector. Different constraints on the magnitude of φ introduce extra terms in the Hamiltonian that can be incorporated in $V(\varphi)$ in the usual manner. A_μ can be taken as an Abelian "gauge frustration" variable[2] or as a non-Abelian one.[30] Therefore, in two dimensions we have point frustration defects, in three they form open or closed frustration rings, and surfaces in four dimensions.

APPENDIX B: LOWER BOUND FOR THE VORTEX-VORTEX CORRELATION FUNCTION IN THE DILUTE LIMIT

We wish to show that

$$\exp\left[\pi K r_0 \sum_{\vec{R}''} F(\vec{R}'')\nabla_{\vec{R}_0} G(\vec{R}_0-\vec{R}'')\right] \leq 1+\pi K r_o \sum_{\vec{R}''} F(\vec{R}'')\nabla_{\vec{R}_0} G(\vec{R}_0-\vec{R}'') \quad . \tag{B1}$$

Expanding the left hand side of Eq. (B1), with $a=\pi K r$ gives

$$\exp\left[a\sum_{\vec{R}''} F(\vec{R}'')\nabla_{\vec{R}_0} G(\vec{R}_0-\vec{R}'')\right] = \left[\sum_{n}^{(1)}+\sum_{n}^{(2)}\right]\frac{a^n}{n!}\left[\sum_{\vec{R}''} F(\vec{R}'')\nabla_{\vec{R}_0} G(\vec{R}_0-\vec{R}'')\right]^n \quad ; \tag{B2}$$

where the sums (1) and (2) run over even and odd n respectively. For $n=2m$ (m an integer) the inequality

$$\left[\sum_{\vec{R}''} F(\vec{R}'')\nabla_{\vec{R}_0} G(\vec{R}_o-\vec{R}'')\right]^{2m} \leq N_f^m\left[\sum_{\vec{R}''}[\nabla_{\vec{R}_0} G(\vec{R}_0-\vec{R}'')]^2\right]^m \tag{B3}$$

follows from Cauchy's inequality

$$\left[\sum_K a_K b_K\right]^2 \leq \sum_K a_K^2 \sum_K b_K^2 \tag{B4}$$

and, because $F=\pm 1$, implies $\sum F^2 = N_f$. When $n=2m+1$, using Cauchy's inequality again, it results

$$\left[\sum_{\vec{R}''} F(\vec{R}'')\nabla_{\vec{R}_0} G(\vec{R}_0-\vec{R}'')\right]^{2m+1} \leq \sum_{\vec{R}''} F(\vec{R}'')\nabla_{\vec{R}_0} G(\vec{R}_0-\vec{R}'')N_f^m\left[\sum_{\vec{R}''}[\nabla_{\vec{R}_0} G(\vec{R}_0-\vec{R}'')]^2\right]^m \quad . \tag{B5}$$

Evaluating the $\vec{R}''$ sum gives

$$\sum_{\vec{R}''}[\nabla_{\vec{R}_0-\vec{R}''} G(\vec{R}-\vec{R}'')]^2 = \left[\sum_{\vec{R}_0}+\sum_{\vec{R}''-R_0}\right][\nabla_{\vec{R}''-\vec{R}_0} G(\vec{R}''-\vec{R}_0)]^2 = \sum_{\vec{R}}[\nabla_{\vec{R}} G(R)]^2 \quad , \tag{B6}$$

where an integration by parts has been carried out and Eq. (3.20) was used.

Changing the sum in Eq. (B6) to a continuous integral it takes the form

$$\sum_{\vec{R}}[\nabla_{\vec{R}} G(R)]^2 = 2\pi x_f \int dR\ R^{-1} = \pi\left(\frac{N_f}{N}\right)(\ln N - \ln 2\pi) \quad . \tag{B7}$$

Note that in our units $N=2\pi R^2=A$ (area of the system). Therefore, in the dilute limit, $N_f << N$, Eq. (B7) is very small and to lowest order of approximation Eq. (B1) becomes the equality quoted in Sec. III.

APPENDIX C: DERIVATION OF EQ. (4.12)

Start by rewriting e^W as

$$e^W = \exp\left[\hat{A}\left(\prod_{\vec{R}''} f(\vec{R}''-\vec{R}) + \prod_{\vec{R}''} f(\vec{R}''-\vec{R})\right)\right] \quad , \tag{C1}$$

with

$$\hat{A} = p^2 y^2 G'(|\vec{r}-\vec{r}'|) \sum_{r_0} r_0^{2-2\pi K} \tag{C2}$$

and

$$f(\vec{r}) = \exp[(r_0 \pi K) F(\vec{R}'') \nabla_{\vec{r}} G(\vec{R}''-\vec{r})] \ . \tag{C3}$$

As mentioned in the text an isolated fractional charge corresponds to having a fractional-charge pair in which one of the members is very far away from its partner. Or, in other words, separated by a large number of antiferromagnetic bonds. So, even in the $x = \frac{1}{2}$ regime there will not be too many of these isolated fractional-charges in the system. In this limit, the condition $\ln f << 1$ will be satisfied. Therefore, we can approximate

$$\prod_{\vec{R}''} f(\vec{R}''-\vec{r}) + \prod_{\vec{R}''} f(\vec{R}''-\vec{r}') \approx \prod_{\vec{R}''} [f(\vec{R}''-\vec{r}) + f(\vec{R}''-\vec{r}')] - 2^{N_f} O\left(\sum_{\vec{R}''} (\ln f)^{N_f}\right) \ . \tag{C4}$$

The last term on the right-hand side of Eq. (C4) is, when integrated, very small. Thus, Eq. (C1) can be written

$$e^W \sim e^{-2^{N_f}} \sum_n \left(\frac{1}{n!}\right) \hat{A}^n(r_0) \prod_{\vec{R}''} [f(\vec{R}''-\vec{r}) + f(\vec{R}''-\vec{r}')]^n \ . \tag{C5}$$

Expanding and rearranging terms the previous equation takes the form

$$e^W - e^{-2^{N_f}} \sum_n \left(\frac{1}{n!}\right) \hat{A}^n(r_0) \prod_{\vec{R}''} \left(\frac{f(\vec{R}''-\vec{r})}{f(\vec{R}''-\vec{r}')}\right)^n \prod_{\vec{R}''} \left\{2^n + n 2^{n-1} \left[\ln\left(\frac{f^2(\vec{R}''-\vec{r}')}{f(\vec{R}''-\vec{r})}\right) + \cdots\right]\right\} \ . \tag{C6}$$

Neglecting the last term in Eq. (C6) and factorizing the 2^n factor we obtain Eq. (4.8). Note that the factor $\exp(-2^{N_f})$ was not written in Eq. (4.8) because, when the sum over n is carried out, it cancels out with the 2^n factor in Eq. (C6).

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