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XY model and the superfluid density in two dimensions

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By computing the incremental free energy of the two-dimensional XY model in a state with a long-wavelength twist of the local short-range order, we evaluate for all $T < T_c$ the superfluid density ρ_s for superfluid systems. Spin-wave excitations in the XY model imply a substantial (nonuniversal) depletion of ρ_s at low temperatures. We find that these contributions do not affect the asymptotic universality; as $T \rightarrow T_{c}-$, ρ_s/T becomes universal and we recover the previous results of Nelson and Kosterlitz. In the Appendix we derive the Kosterlitz recursion relations by transforming to a sine-Gordon system and using conventional momentum shell integration techniques.

I. INTRODUCTION

Recently Nelson and Kosterlitz[1] presented a calculation of the superfluid density for two-dimensional superfluid systems. The significant prediction is that

$$\lim_{T\rightarrow T_c-} \rho_s(T)/T$$

is a *universal* quantity. This means that experiments on real ^{4}He films can be compared quantitatively with predictions based on models believed to be in the same universality class. Rudnick's reanalysis[2] of third sound data[3] supports the theoretical prediction[1]

$$m^2 k_B T_c/\rho_s(T_c)\hbar^2 = \tfrac{1}{2}\pi \ , \tag{1.1}$$

as do the more recent experimental results of Bishop and Reppy.[4] We note, however, that the interpretation of the experimental data is not without some controversy.[5]

In this paper we present an alternative derivation of the superfluid density in two dimensions which we feel has several advantages. In the first place we work completely within the framework of the XY model. This eliminates the need for the detailed phenomenology of two-dimensional hydrodynamics. Second, for the same effort one obtains the leading low-temperature "spin-wave" depletion of ρ_s. Such contributions are nonuniversal, and one explicitly demonstrates the *irrelevance* of such terms. Hence we recover, as $T \rightarrow T_c-$, the universality of ρ_s/T and reproduce the Nelson-Kosterlitz[1] result (1.1). The presence of the nonuniversal terms does not affect the asymptotic proportionality of the critical exponent η and ρ_s/T as $T \rightarrow T_c-$.

The method employed is based entirely on ideas and computational techniques introduced previously.[6–8] Briefly one imagines that a long-wavelength "twist" of the local short-range order is imposed on the system. If the twist has wavelength λ_0 or pitch $k_0 \equiv 2\pi/\lambda_0$, one expects the free energy to rise by $O(k_0^2)$ over the value for the uniform untwisted state. The stiffness or helicity modulus[6] Υ is defined according to[7,8]

$$\Upsilon \equiv \frac{\partial^2 F(T;k_0)}{\partial k_0^2}\Big|_{k_0\rightarrow 0} \tag{1.2}$$

$$= \left(\frac{\hbar}{m}\right)^2 \rho_s \ , \tag{1.3}$$

where $F(T;k_0)$ is the free energy per unit volume of such a state. The relation (1.3) expresses the free-energy increment in terms of ρ_s for the case of a superfluid.[6]

To use Eq. (1.2) we begin with the standard XY model defined by

$$\bar{\mathcal{H}} = -\frac{\mathcal{H}}{k_B T} = \beta J \sum_{\langle \vec{r},\vec{r}\,'\rangle} \cos[\theta'(\vec{r}) - \theta'(\vec{r}\,')] \ , \tag{1.4}$$

where $\{\vec{r} = (x,y)\}$ denote the sites of a square lattice with lattice constant a, and the sum is over nearest-neighbor pairs. The angle variables $\theta'(\vec{r})$ are restricted to lie $-\pi < \theta' < \pi$. The twist is introduced by measuring each $\theta'(\vec{r})$ with respect to the local direction of (short-range) order, i.e., by defining

$$\theta'(\vec{r}) = \Phi(\vec{r}) + \theta(\vec{r}) \ , \text{ with } \Phi(\vec{r}) \equiv k_0 x \ . \tag{1.5}$$

Hence angles θ are measured with respect to a rotating reference frame.[7] To calculate ρ_s we merely must evaluate the free energy, $-k_B T \ln Z$, to $O(k_0^2)$, where the partition function Z is given by

$$Z=\int_{-\pi}^{\pi}\frac{d\{\theta\}}{2\pi}\exp\left\{\beta J\sum_{\langle\vec{r},\vec{r}'\rangle}\cos[\theta(\vec{r})-\theta(\vec{r}')+\Phi(\vec{r})-\Phi(\vec{r}')]\right\}. \quad (1.6)$$

Fortunately the duality transformations used for $\Phi\equiv 0$ can still be applied to Eq. (1.6).

The layout of the remainder of this paper is as follows. In Sec. II we apply the duality transformation in order to evaluate Eq. (1.6)) to $O(k_0^2)$. We isolate the "spin wave" and vortex contributions to ρ_s. In Sec. III we evaluate ρ_s by combining the perturbative results of Sec. II with scaling and the structure of the Kosterlitz recursion relations.[9] Section IV is devoted to a brief summary and discussion.

In the analysis of Sec. III we shall require the appropriate renormalization-group recursion relations. Various derivations have been presented,[9–11] most of which rely on eliminating vortex (or charge) pairs at greater and greater separations. In the Appendix we present a derivation of the recursion relations based on ordinary momentum-shell integration techniques which have proven so useful in higher dimensions. Our derivation is based on earlier work on the sine-Gordon system by Ohta[12] and has some similarities with a derivation by Wiegmann.[13] Further discussion is reserved for the Appendix.

II. DUALITY TRANSFORMATION

We are concerned with the superfluid density or more generally, the helicity modulus, at low temperatures and at temperatures near the critical point. In these regions there are two types of excitations, spin waves and vortices, which make their respective contributions to the helicity modulus and effectively deplete ρ_s. The exact duality transformation (to be discussed below) extracts the vortex configurations automatically, but spin-wave effects are implicitly contained in the expressions. In fact if we use an approximate version such as the Villain model,[14] the interactions among spin waves are completely discarded. After the duality transformation we calculate Υ or ρ_s at low temperatures so that we can separate out the respective contributions.

The duality transformation has been discussed by José *et al.*[10] and Savit.[15] We indicate the steps involved in the transformation of Eq. (1.6). First the representation

$$\exp(\beta J\cos\omega)\equiv\sum_{n=-\infty}^{\infty}I_n(\beta J)\exp(in\omega) \quad (2.1)$$

is inserted for the integrand in Eq. (1.6). The functions I_n are modified Bessel functions of the first kind. This introduces an integer $n(\vec{r},\vec{r}')$ for each nearest-neighbor bond of the real lattice, yielding

$$Z=\int_{-\pi}^{\pi}\frac{d\{\theta\}}{2\pi}\prod_{\langle\vec{r},\vec{r}'\rangle}\sum_{n(\vec{r},\vec{r}')=-\infty}^{\infty}I_{n(\vec{r},\vec{r}')}(\beta J)\exp\{in(\vec{r},\vec{r}')[\theta(\vec{r})-\theta(\vec{r}')]+in(\vec{r},\vec{r}')[\Phi(\vec{r})-\Phi(\vec{r}')]\}\ . \quad (2.2)$$

The $\theta(\vec{r})$ variables can be integrated out yielding a Kronecker δ condition on the n variables for bonds emanating from a given site. As shown in Refs. 10 and 15 the constraints on the $n(\vec{r},\vec{r}')$ can automatically be satisfied by going to the dual lattice $\{\vec{R}\}$. We need not repeat the argument here; the only new feature is a redefinition for the Φ variables appropriate to the dual lattice. From Fig. 1 we define

$$\Phi(\vec{R})=\tfrac{1}{2}[\Phi(\vec{r})+\Phi(\vec{r}+a\vec{e}_x)]\ , \quad (2.3)$$

which is an exact interpolation in view of the linear dependence assumed in Eq. (1.5). Recalling[10] that $n(\vec{r},\vec{r}')$ can be represented as the difference of two integers $S(\vec{R})$ defined on the associated dual lattice *sites* $\{\vec{R}\}$, we have

$$n(\vec{r},\vec{r}')[\Phi(\vec{r})-\Phi(\vec{r}')]=[S(\vec{R}+a\vec{e}_y)-S(\vec{R})]\times[\Phi(\vec{R})-\Phi(\vec{R}+a\vec{e}_x)]\ , \quad (2.4)$$

where $\vec{e}_x$ and $\vec{e}_y$ are unit vectors.

The remaining steps of the dual transformation are identical to those of Savit.[15] The I_n in Eq. (2.2) must be exponentiated to produce a Hamiltonian for furth-

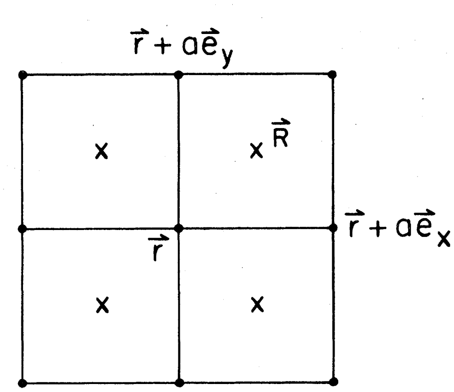

FIG. 1. Direct lattice sites $\{\vec{r}\}$ indicated by the dots and the associated dual lattice sites $\{\vec{R}\}$ indicated by the crosses.

er manipulations. One finds directly

$$Z = \sum_{\{S\}=-\infty}^{\infty} e^{\bar{\mathcal{H}}_D\{S\}} , \quad (2.5)$$

where

$$\bar{\mathcal{H}}_D = \sum_{\langle \vec{R},\vec{R}'\rangle} \sum_{p=1}^{\infty} \frac{(-1)^p}{(2p)!} D_{2p}(\beta J)[S(\vec{R}) - S(\vec{R}')]^{2p} + i \sum_{\vec{R}} [S(\vec{R}+a\vec{e}_y) - S(\vec{R})] \times [\Phi(\vec{R}) - \Phi(\vec{R}+a\vec{e}_x)] . \quad (2.6)$$

For $\beta J >> 1$ the coefficients D_{2p} are given by

$$D_2 = \frac{1}{\beta J}\left[1 + \frac{1}{2\beta J} + O[(\beta J)^{-2}]\right] ,$$
$$D_4 = (\beta J)^{-3} + O[(\beta J)^{-4}] ,$$
$$D_6 = O[(\beta J)^{-5}] \cdots . \quad (2.7)$$

The sum in Eq. (2.6) runs over dual lattice sites and $\langle \vec{R},\vec{R}'\rangle$ denotes nearest-neighbor pairs. Note the $S(\vec{R})$ are integers. A final transformation[10] removes this restriction and yields

$$Z = \int_{-\infty}^{\infty} d\{S\} \sum_{\{m\}=-\infty}^{\infty} \exp\{\bar{\mathcal{H}}_D + \bar{\mathcal{H}}_m\} , \quad (2.8)$$

where $\bar{\mathcal{H}}_D$ is given in Eq. (2.6) but with $S(\vec{R})$ lying in the range $-\infty < S(\vec{R}) < \infty$, and

$$\bar{\mathcal{H}}_m = 2\pi i \sum_{\vec{R}} m(\vec{R}) S(\vec{R}) , \quad (2.9)$$

with $\{m(\vec{R})\}$ integers. The term associated with the "external disturbance" $\Phi(\vec{R})$ is consistent with that of Wiegmann[13]; when $\Phi \equiv 0$, Eqs. (2.8) and (2.9) agree precisely with the form derived by Savit.[15]

The incremental free energy is to be calculated as an expansion in $k_B T/J$ and in the fugacity y for the vortex excitations, which are associated with the integer degrees of freedom $\{m(\vec{R})\}$. The terms in Eq. (2.6) involving D_2 and D_4 are required to obtain the incremental free energy (hence the helicity modulus and ρ_s) to order $k_B T/J$ and y^2 (see below). However a dimensional analysis of Eq. (2.6) suggests that the coupling constant D_4 is irrelevant near the critical point. Nonetheless this term cannot be neglected for a consistent description of the spin-wave contribution. We shall return to this point in our summary below.

If the $p=2$ term in Eq. (2.6) is treated as a perturbation, the integration over $S(\vec{R})$ is easily performed. One first eliminates all linear terms in $S(\vec{R})$ via the transformation (shift)

$$S_{\vec{k}} = \sigma_{\vec{k}} + \frac{i}{D_2 f(\vec{k})} \phi_{\vec{k}} , \quad (2.10)$$

where Fourier variables are defined in general according to

$$A_{\vec{k}} = \sum_{\vec{R}} A(\vec{R}) e^{i\vec{k}\cdot\vec{R}} . \quad (2.11)$$

The remaining variable σ_k lies in the range $-\infty < \sigma_{\vec{k}} < \infty$ and[16]

$$\phi_{\vec{k}} \equiv 2\pi m_{\vec{k}} - f_{xy}(\vec{k}) \Phi_{\vec{k}} , \quad (2.12)$$

with

$$f(\vec{k}) = \tfrac{1}{2} \sum_{\vec{\delta}} |1 - e^{i\vec{k}\cdot\vec{\delta}}|^2 ,$$
$$f_{xy}(\vec{k}) = (1 - e^{ik_x a})(1 - e^{-ik_y a}) . \quad (2.13)$$

To first order in D_4 we have

$$Z = Z_0 Z\{\Phi\} , \quad (2.14)$$

where Z_0 is independent of the Φ (or twist) variables and

$$Z\{\Phi\} \equiv \sum_{\{m\}} \exp(\bar{\mathcal{H}}\{m, \Phi\}) , \quad (2.15)$$

with

$$\bar{\mathcal{H}}\{m,\Phi\} = -\frac{1}{2D_2} \int_{\vec{k}} \frac{\phi_{\vec{k}}\phi_{-\vec{k}}}{f(\vec{k})} + \frac{D_4}{4!D_2^4} \int_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} \tilde{V}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4} - \frac{6D_4}{4!D_2^3} \int_{\vec{k}\vec{q}} \tilde{V}(\vec{k},-\vec{k},\vec{q},-\vec{q}) f(-\vec{k}) \phi_{\vec{q}}\phi_{-\vec{q}} . \quad (2.16)$$

The potential $\tilde{V}$ is defined by

$$\tilde{V}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = \frac{\tfrac{1}{2}\sum_{\vec{\delta}} \prod_{i=1}^{4} (1 - e^{i\vec{k}_i\cdot\vec{\delta}}) \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)}{\prod_{i=1}^{4} f(\vec{k}_i)} . \quad (2.17)$$

Now the partition function (2.15) is evaluated in a manner similar to that of José *et al.*[10] At sufficiently low temperatures only the terms with $m(\vec{R})=0, \pm 1$ are important; higher terms are shown to be irrelevant under renormalization transformations near the critical point (see also the Appendix). The vortex pair correlation function $\langle |m_{\vec{q}}|^2\rangle$ is proportional to y^2, where y is the fugacity. To leading order in $k_T T/J$ and y we have for the incremental free energy

$$\Delta \ln Z = \ln Z\{\Phi\}$$
$$= \int_{\vec{k}} |\Phi_{\vec{k}}|^2 \hat{f}(\vec{k}) \left[-\frac{1}{2D_2} - \frac{D_4}{4D_2^3}\int_{\vec{q}} V(\vec{k},\vec{q}) + \frac{2\pi^2}{D_2^2}\frac{\langle |m_{\vec{k}}|^2\rangle}{f(\vec{k})} + \frac{\pi^2 D_4}{D_2^4}\int_{\vec{q}} V(\vec{k},\vec{q})\frac{\langle |m_{\vec{q}}|^2\rangle}{f(\vec{q})}\right] , \quad (2.18)$$

where, for notational convenience we have redefined

$$V(\vec{k},\vec{q}) = f(-\vec{k})\,\tilde{V}(\vec{k}, -\vec{k}, \vec{q}, -\vec{q}) f(-\vec{q}) \quad (2.19)$$

and

$$\hat{f}(\vec{k}) = |f_{xy}(\vec{k})|^2 / f(\vec{k}) \ . \quad (2.20)$$

Furthermore, following the work of José *et al.*[10] we have

$$\langle |m_{\vec{q}}|^2\rangle = 4\pi \left[\frac{y}{a}\right]^2 \int_a^\infty \frac{dr}{a}[1 - J_0(qr)] \left[\frac{r}{a}\right]^{1-2\pi/D_2} . \quad (2.21)$$

Here $J_0(x)$ is the Bessel function of the first kind. The fugacity y is proportional to $\exp(-\pi^2/2D_2)$. In Eq. (2.18) we have kept terms of order $|\Phi|^2$ which contribute to Υ or ρ_s.

The expression in the brackets of Eq. (2.18) has a physical interpretation. The first term includes the incremental ground-state energy and the effect of spin-wave interactions {recall $D_2^{-1} = \beta J[1-(2\beta J)^{-1}]$}. The second term can also be interpreted as a spin-wave depletion of ρ_s, while the terms involving $\langle |m_{\vec{q}}|^2\rangle$ clearly imply the contribution of vortex excitations.

The expression (2.18) is applicable at low temperatures. In Sec. III we use scaling along with the recursion relations to evaluate ρ_s up to the critical temperature.

III. CALCULATION OF THE SUPERFLUID DENSITY

The final result (2.18) yields the superfluid density at low temperatures. Consistent with Eq. (1.2) we may write

$$\Delta F = \frac{1}{2(Na^2)} \int_{\vec{k}} k_x^2 |\Phi_{\vec{k}}|^2 \Upsilon(\vec{k}) \ , \quad (3.1)$$

where Na^2 is the total area and identify

$$\Upsilon = \lim_{k_y \to 0} \lim_{k_x \to 0} \Upsilon(\vec{k}) \ . \quad (3.2)$$

The final limit $k_y \to 0$ corresponds to our taking at the end a function Φ which varies in the X direction.[16] We note

$$\lim_{k_x \to 0} \frac{\hat{f}(\vec{k})}{k_x^2} = 1 \ ,$$
$$\lim_{k_x \to 0} \int_{\vec{q}} V(\vec{k},\vec{q}) = \tfrac{1}{2} \ . \quad (3.3)$$

The result is

$$\beta\Upsilon = \frac{1}{D_2} + \frac{D_4}{4D_2^3} - \frac{4\pi^2}{D_2^2}\lim_{\vec{k}\to 0}\frac{\langle |m_{\vec{k}}|^2\rangle}{f(\vec{k})} - \frac{2\pi^2 D_4}{D_2^4}\int_{\vec{q}} \frac{\langle |m_{\vec{q}}|^2\rangle}{f(\vec{q})} \ , \quad (3.4)$$

where D_2 and D_4 are given in Eq. (2.7), $f(\vec{k})$ is defined in Eq. (2.13) and $\langle |m_{\vec{q}}|^2\rangle$ is given in Eq. (2.21).

At sufficiently low temperatures we may neglect the vortex contributions as they carry a coefficient proportional to $y \sim \exp(-\beta c)$. We then obtain

$$\beta\Upsilon = \beta J - \tfrac{1}{4} + O(1/\beta J) \ , \quad (3.5)$$

which implies, according to Eq. (1.3), that the low-temperature superfluid density is expected to behave like

$$\frac{\rho_s(T)}{\rho_s(0)} = 1 - \frac{1}{4}\frac{k_B T m^2}{\hbar^2 \rho_s(0)} + \cdots \ . \quad (3.6)$$

Equation (3.6) shows that at low temperatures ρ_s has a linear dependence due to spin-wave excitations; the vortex contributions have been left out of Eq. (3.6).

The low-temperature result (3.6) applies to the lattice XY model defined in Eq. (1.4). We do not necessarily expect the linear T dependence in real superfluid films. We shall return to this point in the summary below.

When D_2 approaches $\frac{1}{2}\pi$ from below, the third term in Eq. (3.4) diverges. In order to avoid this difficulty we employ a renormalization-group procedure which is essentially the same as that of Nelson and Kosterlitz.[1] We sketch only the essential features.

First note that the helicity modulus scales according to

$$\beta \Upsilon \equiv \Gamma(D_2,D_4,y) = \Gamma[D_2(l),D_4e^{-2l},y(l)] \quad , \qquad (3.7)$$

where the transformation involves a scale change $a \rightarrow ae^l$. Relation (3.7) is equivalent to the Josephson relation[17] applied to the case at hand.[7] The equations for $D_2(l)$ and $y(l)$ follow from the Kosterlitz recursion relations; in the Appendix we suggest an alternative derivation based on the sine-Gordon model. The method of José *et al.* can be used to extract the recursion relations directly from Eq. (3.4).

From Eq. (A17) we obtain

$$[D_2(\infty)]^{-1} \simeq (2/\pi)[1+\tfrac{1}{2}\pi c(|t|)^{1/2}]$$

and $y(\infty)=0$, where c is a nonuniversal positive constant and $|t|=|T-T_c|/T_c$, T_c being the transition temperature. Thus the helicity modulus and superfluid density for $T \lesssim T_c$ are given by

$$\beta\Upsilon(T) \simeq [D_2(\infty)]^{-1} \simeq (2/\pi)[1+\tfrac{1}{2}\pi c(|t|)^{1/2}] \quad , \qquad (3.8)$$

$$\frac{\rho_s(T)}{k_B T} \simeq \left(\frac{2m^2}{\pi \hbar^2}\right)[1+\tfrac{1}{2}\pi c(|t|)^{1/2}] \quad , \quad |t| \rightarrow 0 \quad . \qquad (3.9)$$

Equation (3.9) agrees with the result of Nelson and Kosterlitz.[1]

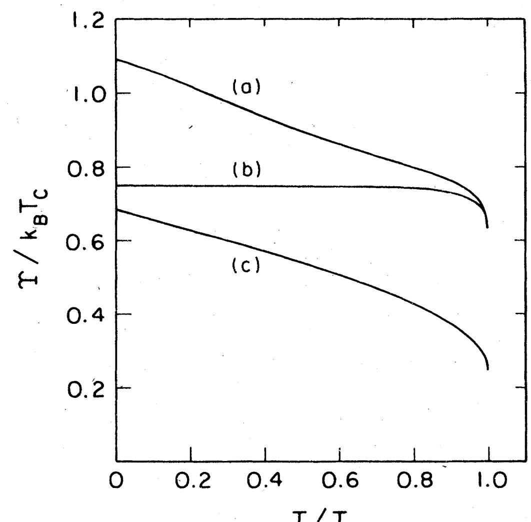

FIG. 2. Numerical evaluation of the helicity modulus $\Upsilon/k_BT_c=(\hbar/m)^2\rho_s/k_BT_c$. Curve (a) follows from the present work using the initial conditions $D_2(0)=(\beta J)^{-1}[1+(2\beta J)^{-1}]$ and $y(0)=\exp[-\pi^2/2D_2(0)]$. Curve (b) corresponds to the work of Nelson and Kosterlitz (Ref. 1) so that $D_2(0)=(\beta J)^{-1}$. Curve (c) represents the results of the self-consistent approach of Pokrovskii and Uimin (Ref. 19).

IV. SUMMARY AND FINAL COMMENTS

In Secs. I–III we have studied the behavior of the superfluid density in two dimensions via the evaluation of the helicity modulus for the *XY* spin system. Near the critical point, where vortex excitations are important, the present result agrees with that of Nelson and Kosterlitz[1]; Eq. (3.9) demonstrates the universality of ρ_s/T as $T \rightarrow T_c-$. Nelson and Kosterlitz derived Eq. (3.9) by an alternative method. They began with a model Hamiltonian for a two-dimensional superfluid which is equivalent to the generalized Villain model.[10,14] If we approximate D_2^{-1} by βJ and neglect D_4 completely in Eq. (3.4), Υ is just the renormalized coupling constant of the generalized Villain model. As has been noted above, D_4 is irrelevant and the difference between D_2^{-1} and βJ disappears after successive renormalization transformations. This reflects the fact that the vortex excitations dominate the spin-wave excitations near T_c. Hence our value of $\rho_s(T)/k_BT$ agrees with theirs at T_c.

On the other hand the spin-wave excitations cannot be neglected at sufficiently low temperatures. As noted by Banavar,[18] one expects such excitations to be nonuniversal and to provide, for classical spin models, a linear decrease of ρ_s with temperature. Pokrovskii and Uimin[19] computed ρ_s for the *XY* model using a self-consistent approach. Their result at low temperatures is consistent with the present results although they find a larger decrease near the critical point.

As noted above the linear decrease of ρ_s at low temperatures is characteristic of classical spin models (even in higher dimensionality). In three dimensions the quantal *XY* model yields the proper low-temperature behavior expected for bulk superfluid systems.[8] Hence it is doubtful that the classical *XY* model used here correctly predicts the low-temperature behavior of helium films. A proper (approximate) quantum mechanical calculation has not yet been done presumably because spin-wave ideas, in their simplest forms, require spontaneous order which is precluded.[20]

The results of the various computations are summarized in Fig. 2. The present results, labeled (a) in the figure, are for the "model" dependences

$$D_2(0)=(\beta J)^{-1}[1+(2\beta J)^{-1}]$$

and

$$y(0)=\exp[-\pi^2/2D_2] \ .$$

While the universal critical behavior is not modified, the numerical results will differ slightly at lower temperatures if the functional dependence of D_2 on βJ is modified.

We finally note that a renormalization scheme using a Migdal[21] approximation has been applied by Berker and Nelson[22] to evaluate ρ_s for ^{3}He-^{4}He films.

ACKNOWLEDGMENTS

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APPENDIX: RECURSION RELATIONS

In this Appendix we discuss the renormalization of the coupling constant D_2 and the fugacity y which characterize the Kosterlitz-Thouless phase transition and enter Eqs. (2.18) and (2.21). Kosterlitz[23] derived a set of recursion relations for D_2 and y for the Coulomb gas system in two dimensions using a position space renormalization-group method. Recently José *et al.*[10] obtained the same renormalization-group equations from the approximate calculation of the XY model spin-pair correlation function. If it were possible to describe the Kosterlitz-Thouless phase transition in terms of a model Hamiltonian with a continuous field, more conventional momentum shell integration schemes could be applied. Furthermore, additional insights into the structure of Kosterlitz-Thouless phase transitions might be obtained.

As has been noted by many authors,[24] the sine-Gordon system is related to the models which exhibit Kosterlitz-Thouless transitions. In a previous publication[12] one of us studied approximately renormalization of the sine-Gordon system using an infinitesimal recursion method. A portion of the computation was incomplete; here we re-examine the problem using a *finite* momentum shell recursion method.

Consider the partition function (2.8). We are only interest in uniform phase properties, so we set $\Phi\equiv 0$. If we further retain only terms with $m(\vec{R})=0,\ \pm 1$, the partition function is approximated by

$$Z=\int_{-\infty}^{\infty} d\{S\}\exp[\mathcal{H}_s\{S\}] \;, \tag{A1}$$

where $\mathcal{H}_s$ is of the sine-Gordon type:

$$\mathcal{H}_s\{S\}=-\tfrac{1}{2}D_2\int_{\vec{x}}(\nabla S)^2 + 2ya^{-2}\int_{\vec{x}}\cos[2\pi S(\vec{x})] \;. \tag{A2}$$

Here we have used a continuum notation and introduced a parameter y controlling the strength of the self-interaction. (See Refs. 10 and 13 for further details.)

We imagine that $S(\vec{x})$ is divided into two parts

$$S(\vec{x})=S^{(1)}(\vec{x})+S^{(2)}(\vec{x}) \;, \tag{A3}$$

where $S^{(1)}(\vec{x})$ contains the low wave-number information of $S(\vec{x})$ and $S^{(2)}(\vec{x})$ contains the high wave-number information. If μ is a wave number defined such that $\mu \ll \Lambda\equiv a^{-1}$, we have approximately

$$\begin{aligned} S_{\vec{k}} &\approx S^{(1)}_{\vec{k}} \;, \quad \text{for } |\vec{k}|\lesssim\mu \;,\\ S_{\vec{k}} &\approx S^{(2)}_{\vec{k}} \;, \quad \text{for } |\vec{k}|\gtrsim\mu \;. \end{aligned} \tag{A4}$$

The parameter μ will play the role of a lower momentum cutoff. We shall, according to Eq. (A2), define the zero-order correlation function by

$$\begin{aligned} G(\vec{x}-\vec{y}) &\equiv \langle S^{(2)}(\vec{x})S^{(2)}(\vec{y})\rangle \\ &= \frac{1}{D_2}\int\frac{d^2\vec{q}}{(2\pi)^2}\frac{e^{i\vec{q}\cdot(\vec{x}-\vec{y})}}{q^2}h\left(\frac{q}{\mu}\right) \;, \end{aligned} \tag{A5}$$

where $h(x)$ is a rather general smoothing function defined such that

$$\begin{aligned} h(x) &\approx 1 \;, \quad \text{for } x\gg 1 \;,\\ &\approx 0 \;, \quad \text{for } x \ll 1 \;. \end{aligned} \tag{A6}$$

For the two forms

$$\begin{aligned} h_1(x) &= x^2/(1+x^2) \;,\\ h_2(x) &= x^4/(1+x^4) \;, \end{aligned} \tag{A7}$$

many of the details can be worked out explicitly. For example, if $h(x)=h_1(x)$, we have from Eq. (A5)

$$\begin{aligned} \langle S^{(2)}(0)S^{(2)}(\vec{r})\rangle &= \frac{1}{2\pi D_2}K_0(r\mu) \quad \text{for } r\Lambda \gg 1 \\ &= \frac{1}{2\pi D_2}\ln\left(\frac{\Lambda}{\mu}\right) \quad \text{for } r\Lambda \ll 1 \;, \end{aligned} \tag{A8}$$

where $K_0(x)$ is the modified Bessel function of the second kind.

We now eliminate from Eq. (A2) the variable $S^{(2)}(\vec{x})$ and attempt to rewrite the Hamiltonian (A2) with renormalized constants. To order y^2 we have, after integration over $S^{(2)}(\vec{x})$, before any rescaling,

$$\mathcal{H}'\{S^{(1)}(\vec{x})\}=-\tfrac{1}{2}(D_2)\int_{\vec{x}}(\vec{\nabla}S^{(1)})^2+2ya^{-2}\int_{\vec{x}}\langle\cos 2\pi S(\vec{x})\rangle^{(2)}+\tfrac{1}{2}(2ya^{-2})^2\int_{\vec{x}}\int_{\vec{y}}\langle\cos 2\pi S(\vec{x})\cos 2\pi S(\vec{y})\rangle^{(2)}_{c} \;, \tag{A9}$$

where $\langle\,\rangle_{\mathcal{C}}$ implies a cumulant average and the superscript (2) implies an average over the $S^{(2)}$ field using the propagator in Eq. (A5). We have only Gaussian integrals with continuous fields so, for example,

$$\langle \cos 2\pi S(\vec{x})\rangle^{(2)} = [\cos 2\pi S^{(1)}(x)] e^{-2\pi^2 G(0)} \tag{A10a}$$

and

$$\langle \cos 2\pi S(\vec{x}) \cos 2\pi S(\vec{y})\rangle_{\mathcal{C}}^{(2)} = \tfrac{1}{2} e^{-4\pi^2 G(0)} \big((e^{-4\pi^2 G(x-y)} - 1) \cos\{2\pi[S^{(1)}(\vec{x}) + S^{(1)}(\vec{y})]\} + (e^{4\pi^2 G(x-y)} - 1) \cos\{2\pi[S^{(1)}(\vec{x}) - S^{(1)}(\vec{y})]\} \big) \ . \tag{A10b}$$

The result (A10a) yields the renormalization of y while a contribution to $\nabla S^{(1)}$ must be extracted from Eq. (A10b). Higher harmonics such as $\cos 4\pi S^{(1)}(\vec{x})$ are generated but can be shown to be irrelevant. Rescaling is accomplished be defining $S^{(1)}(\vec{x}) \rightarrow S[\vec{x}/(\Lambda/\mu)]$, which returns us to a field $S(\vec{x})$ with Fourier components out to $k = \Lambda$. (The spin rescaling factor is unity in the present calculation.)

The renormalization of y is straightforward. From Eq. (A9) we have

$$y^2(\mu) = y^2 e^{-4\pi^2 G(0)} \left(\frac{\Lambda}{\mu}\right)^4 \approx y^2 \exp\left(-\frac{2\pi c_1}{D_2}\right) \left(\frac{\Lambda}{\mu}\right)^{4-2\pi/D_2} , \tag{A11}$$

where we have approximated

$$G(0) = (2\pi D_2)^{-1} \int_0^\Lambda \frac{dk}{k} h\left(\frac{k}{\mu}\right) \approx (2\pi D_2)^{-1}(c_1 + \ln \Lambda/\mu) \ . \tag{A12}$$

The recursion relation for D_2 involves the integral

$$I = \int_0^\infty dr\, r^3 (e^{4\pi^2 G(r)} - 1) \ . \tag{A13}$$

We break the integral up into the ranges $0 < r < \Lambda^{-1} = a$ and $\Lambda^{-1} < r$. In the first range we approximate $G(r)$ by $G(0)$; the appearance of an integral in this range is an artifact of the continuum notation we have been using. In the outer range we use the fact that $G(r)$ is in fact a function $g(\mu r)$ as can be seen in Eq. (A8). Using these facts and approximations we find

$$D_2(\mu) = D_2 + 4\pi^3 y^2(\mu) \left[\tfrac{1}{4} \exp\left(\frac{2\pi c_1}{D_2}\right) \left(\frac{\mu}{\Lambda}\right)^{4-2\pi/D_2} - \tfrac{1}{4}\left(\frac{\mu}{\Lambda}\right)^4 + \int_{\mu/\Lambda}^\infty dx\, x^3 (e^{4\pi^2 g(x)} - 1) \right] \ . \tag{A14}$$

Equations (A11) and (A14) are the recursion relations to order y^2. Differentiating Eqs. (A11) and (A14) with respect to μ and putting $\Lambda = \mu$ we convert to a differential form with $dl = -d\mu/\mu$,

$$\frac{dy^2(l)}{dl} = \left(4 - \frac{2\pi}{D_2(l)}\right) y^2(l) + O(y^4) \ , \tag{A15}$$

$$\frac{dD_2(l)}{dl} = 4\pi^3 y^2(l) + O\left[y^2\left(4 - \frac{2\pi}{D_2}\right)\right] \ . \tag{A16}$$

In writing these equations we have neglected the constant c_1 appearing in Eq. (A12). For a wide class of smoothing functions such as $h_p(x) \equiv x^p/(1+x^p)$ $c_1 \equiv 0$, and the leading correction to the $\ln\Lambda/\mu$ term is $O[(\mu/\Lambda)^p]$. For alternate choices c_1 becomes smaller the sharper the function $h(x)$. In any event a simple redefinition

$$y \rightarrow y \exp(-\pi c_1/D_2) \approx y \exp(-2c_1)$$

retains the structure of the recursion relations.

The results (A15) and (A16) in the regime $D_2 \simeq \frac{1}{2}\pi$ are equivalent to the recursion relations derived by Kosterlitz[23] and José *et al.*[10] Note that our $D_2 = 2J$ as defined in Ref. 9 and $D_2 = K^{-1}$ as used in Ref. 10.

Finally, following Kosterlitz[23] we consider the conserved quantity,

$$[\tfrac{1}{2}\pi - D_2(l)]^2 - \pi^4 y^2(l) = C \tag{A17}$$

for $D_2(l) \approx \frac{1}{2}\pi$. The constant C (independent of l) is identified as being proportional to the reduced temperature, i.e., $C = c^2(T_c - T)/T_c = c^2|t|$, where c is nonuniversal. Although c and T_c are nonuniversal we estimate the values which follow from the intial conditions $D_2 = (1/\beta J)[1 + 1/(2\beta J)]$ and $y(0) = \exp[-\pi^2/2D_2]$ as $k_B T_c/J \simeq 0.917$ and $c \simeq 1.2$.

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