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Renormalization-group theory of the λ transition

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The renormalization-group transformation is performed on the microscopic quantum Hamiltonian of a boson system. The transformation involves a step in which the occupation number of each plane-wave state is rescaled. This is equivalent to a transformation of the quantum amplitude and leads to the same fixed point that Singh found previously. The quantum effect again disappears, but however, for a different reason.

In earlier papers,[1] Singh has studied the static critical phenomena of the λ transition in liquid helium-4 using the method of the renormalization-group transformation[2] (RGT). It is interesting that the RGT is performed in this work directly on the microscopic quantum Hamiltonian instead of the Landau-Ginzburg model. It was shown that the critical properties of the quantum system of bosons are the same as those of the classical system of spins. The quantum effect disappears because in Singh's RGT scheme the mass of each helium atom is rescaled at each iteration with a small scaling factor so that after many iterations the rescaled mass becomes very large and consequently the quantum effect is completely obliterated. (This is not an artifact since the choice of the small scaling factor leads to a fixed point.)

Singh's RGT scheme is as follows. First the creation and annihilation operators corresponding to the modes of large momenta are eliminated from the Hamiltonian. This is done by taking the trace over the subspace of those modes. Then length is rescaled as usual, and finally the mass of the helium atom is rescaled. This leads to fixed-point critical properties which are the same as those of a two-component classical spin system.

We wish to point out in this note that the same fixed point may be obtained by rescaling the occupation number of each plane-wave state instead of the mass. This is equivalent to a transformation of the quantum amplitude. Of course the basis vectors of the Hilbert space must also be changed properly at the same time. Since the mass is not related to the order parameter and furthermore since the Hamiltonian is written in terms of the quantum amplitude, we feel that this modification would bring the RGT scheme a step closer to the block picture of Kadanoff. As the temperature is lowered toward the critical temperature, Bose-Einstein condensation will prematurely take place here and there forming islands of patches. At the critical temperatue, these patches come in all sizes. So if we lower the resolution of our "microscope" and shrink the system and then lower the "contrast" by counting the number of particles (thus density) in a properly increased scale, we would not see any difference.

The occupation number n_i may be rescaled by writing $n_i = n_i'\zeta$, where ζ is the scale factor. Then we have for the state vector $| \cdots n_i' \cdots \rangle$ instead of $| \cdots n_i \cdots \rangle$. Next we need to find an appropriate set of "creation" and "annihilation" operators $b_i^\dagger$ and b_i for the primed vector. The action of the original creation operator $a_i^\dagger$ on the unprimed vector is

$$a_i^\dagger | \cdots n_i \cdots \rangle = (n_i+1)^{1/2} | \cdots n_i+1 \cdots \rangle \ . \qquad (1)$$

Since this may be rewritten

$$\sqrt{\zeta}(n_i'+\zeta^{-1})^{1/2} | \cdots n_i'+\zeta^{-1} \cdots \rangle \ , \qquad (2)$$

we define $b_i^\dagger$ and b_i by

$$b_i^\dagger | \cdots n_i' \cdots \rangle = (n_i'+\zeta^{-1})^{1/2} | \cdots n_i'+\zeta^{-1} \cdots \rangle \ ,$$
$$b_i | \cdots n_i' \cdots \rangle = (n_i')^{1/2} | \cdots n_i'-\zeta^{-1} \cdots \rangle \ , \qquad (3)$$

and replace $a_i^\dagger$ and a_i by

$$a_i^\dagger \to \sqrt{\zeta} b_i^\dagger \ , \quad a_i \to \sqrt{\zeta} b_i \ . \qquad (4)$$

It is obvious that $b_i^\dagger b_i$ gives the occupation number of the i th state as usual, i.e.,

$$b_i^\dagger b_i | \cdots n_i' \cdots \rangle = n_i' | \cdots n_i' \cdots \rangle \ . \qquad (5)$$

The new operators satisfy the commutation rules given by

$$[b_i, b_j^\dagger] = \zeta^{-1}\delta_{i,j} \ , \quad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0 \ . \qquad (6)$$

This is simply the usual Bose commutation rule expressed in the rescaled occupation number representation.

So one can rescale the Bose amplitude without violating any rule of Bose statistics thereby bringing the theory in line with the usual approach followed for the Landau-Ginzburg model. However, there is one subtle difference. The rescaling has made it necessary to use fractional (noninteger) occupation numbers along with the "creation" and "annihilation" operators which change the occupation number by the fraction $1/\zeta$, not by unity. The scale factor ζ is changed after each RGT, and consequently the basis vectors of the Hilbert space are also changed. Therefore, the new Hamiltonian after performing RGT cannot be considered as the Hamiltonian of a physical system which differs from the original system merely in having different values for various parameters of the Hamiltonian.

Note that, since the particle number has been rescaled, all extensive quantities such as energy and momentum should also be rescaled with the same scale factor. So each of the n_k' particles in the primed state vector carries an amount of energy ϵ_k, not $\zeta\epsilon_k$, where ϵ_k is the energy of the free-particle state. Note also that the scale factor ζ in Eqs. (2)–(6) will be replaced by ζ^2 in the second iteration of RGT and by ζ^3 in the third and so on. Since $\zeta > 1$, it is quite clear from Eq. (6) that the quantum effect will disappear. The only question is whether or not such a scale change will allow for a fixed point.

How does the statistical mechanics work in this rescaled occupation number representation? Everything works in the same way as in the original representation except that the creation and annihilation operators in the interaction picture are given by

$$b_k(\tau) = b_k e^{-\epsilon_k \zeta^{-1}\tau/\hbar},$$
$$b_k^\dagger(\tau) = b_k^\dagger e^{\epsilon_k \zeta^{-1}\tau/\hbar}, \tag{7}$$

and the contractions in the Wick theorem for $b_k^\dagger b_k$ and $b_k b_k^\dagger$ are respectively given by

$$\frac{\zeta^{-1}}{e^{\beta\epsilon_k\zeta^{-1}}-1}, \quad -\frac{\zeta^{-1}}{e^{-\beta\epsilon_k\zeta^{-1}}-1}. \tag{8}$$

We proceed with the Hamiltonian

$$\mathcal{H} = \sum_{q<p_c} \frac{\beta}{m}(q^2+r_o)b_q^\dagger b_q + \frac{u_0\beta}{4V}\sum_{q,p,k<p_c} b_{p-k}^\dagger b_{q+k}^\dagger b_q b_p, \tag{9}$$

where we have set $\hbar = 1$, m is twice the mass of ^{4}He atom, r_0 is related to the chemical potential μ by $r_o = -\mu m$, V is the d-dimensional volume, u_0 is the coupling constant, and p_c is the cutoff momentum. Following Singh, we rewrite Eq. (9),

$$\mathcal{H} = \sum_{q<p_c} s(q^2+r)b_q^\dagger b_q + \frac{s^2 v p_c^{-d}}{4V}\sum_{q,p,k<p_c} b_{p-k}^\dagger b_{q+k}^\dagger b_q b_p, \tag{10}$$

where the dimensionless constants are defined by

$$s = \beta p_c^2/m, \quad r = r_0 p_c^{-2}, \quad v = m^2 u_0 p_c^{d-4}/\beta. \tag{11}$$

The second term in Eq. (10), which represents the interaction, is treated as a small perturbation.

The three steps of our RGT are: (i) the elimination of the large momentum modes in the shell $p_c\zeta^{-1} < p < p_c$, (ii) scale change of length $q \to q\zeta$, (iii) scale change of the Bose amplitude for those modes that survive the elimination process of (i),

$$b^\dagger \to \sqrt{\zeta}b^\dagger, \quad b \to \sqrt{\zeta}b. \tag{12}$$

The three steps may be put together to give

$$e^{-\mathcal{H}'} = (\mathrm{Tr}_{p_c\zeta^{-1}<q<p_c} e^{-\mathcal{H}})_{b_k^\dagger(b_k)\to\sqrt{\zeta}b_{\zeta k}^\dagger(b_{\zeta k})}. \tag{13}$$

The trace may be taken using the thermal Wick theorem and Eq. (8). Since our algebra of this step differs from that of Singh only in that we use Eq. (8), we will not present the detail. Step (ii) is trivial and step (iii) has been thoroughly explained. The final result for one transformation is

$$s' = s\zeta^{-1},$$
$$r' = \zeta^2[r + svf_1(s,r,\zeta)], \tag{14}$$
$$v' = \zeta^{4-d}[v - \tfrac{5}{2}sv^2 f_2(s,r,\zeta)],$$

where

$$f_1(s,r,\zeta) = \int_{\zeta^{-1}<|k|<1} \frac{d^dk}{(2\pi)^d}\frac{\zeta^{-1}}{e^\epsilon - 1},$$
$$f_2(s,r,\zeta) = \frac{s}{5}\int_{\zeta^{-1}<|k|<1}\frac{d^dk}{(2\pi)^d} \times \left[\frac{1}{\epsilon}\left(1+\frac{1}{2\epsilon}(e^{-2\epsilon}-1)\right)\left(\frac{\zeta^{-1}}{e^{-\epsilon}-1}\right)^2 - 4\frac{\zeta^{-1}}{e^\epsilon-1}\frac{\zeta^{-1}}{e^{-\epsilon}-1}\right];$$
$$\epsilon = s(k^2+r)\zeta^{-1}. \tag{15}$$

This result differs from that of Singh in two ways. In Singh's scheme, the mass parameter s is set to transform as

$$s' = s\zeta^{-2}. \tag{16}$$

The parameters r and v transform like ours but with different f_1 and f_2: to get his f_1 and f_2 from ours, set $\zeta = 1$. These differences are, however, inconsequential: s remains an irrelevant variable and the difference in the f's disappear in the limit $\zeta \to \infty$. We refer the reader to Ref. 1 for the fixed point and the scaling properties of the free energy and correlation function.

Note that the transformation of the mass parameter given in Eq. (14) is a *result* of the three steps of RGT and none of these steps requires a scale change of mass. It is, however, possible to choose the scale factor of the Bose amplitude in step (iii) of RGT in such a way that the mass parameter remains a constant. The appropriate scale change is

$$b^{\dagger} \rightarrow \sqrt{\zeta^2}\, b^{\dagger}\ , \quad b \rightarrow \sqrt{\zeta^2}\, b\ . \tag{17}$$

In Singh's RGT scheme, the irrelevant nature of the parameter s plays a crucial role. So it is interesting to adopt Eq. (17) and see if it makes any difference. It is easy to see that it does not make any difference. This is so because taking the limit $\zeta \rightarrow \infty$ with a fixed s for all the diagrams resulting from Eq. (17) gives the same result as taking the limit $s \rightarrow 0$ for the corresponding diagrams in Singh's RGT [and our RGT adopting Eq. (12)]. This is no surprise. In our RGT as $\zeta \rightarrow \infty$ the operator $b^{\dagger}(b)$ adds (takes off) less and less amount of energy. In Singh's scheme, this is accomplished by increasing the mass. With the choice of Eq. (12), we have to do both. With Eq. (17), one is enough; we do not need to do both. So the idea of infinite mass at the critical point is only a question of what and how one chooses to rescale.

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[1]K. K. Singh, Phys. Rev. 12, 2819 (1975); 13, 3192 (1976).

[2]K. G. Wilson and J. B. Kogut, Phys. Rep. 12 C, 75 (1974).