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Critical dynamics of elastic phase transitions

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The critical dynamics at elastic phase transitions of second order is studied by the renormalization-group theory. The dynamical theory is based on a stochastic equation of motion for damped phonons. For systems with one-dimensional soft sectors usual dynamical scaling holds with the critical dynamical exponent $z = 2$. For two-dimensional sectors logarithmic corrections appear. Novel features are found for an isotropic-phonon model. The dynamical susceptibility and the characteristic frequency depend singularly on an irrelevant parameter, which, however, diverges at the fixed point. Consequently dynamical scaling breaks down; e.g., the characteristic frequency is no longer a homogeneous function of the wave number k and the inverse correlation length ξ^{-1}. Instead it is a homogeneous function of these variables and an irrelevant parameter. Consequently, the critical dynamics can be characterized by the relaxational exponent $z = 2 + c\eta$ at T_c, while in the hydrodynamic region the sound frequency is characterized by $z = 2 - \frac{1}{2}\eta$ and the damping by yet another exponent. This breakdown of scaling is also reflected by the fact that different fixing conditions, i.e., different choices of the frequency, lead to different values of z. All of these apparently different transformations lead to the same modified dynamical scaling relations.

I. INTRODUCTION

Elastic phase transitions of second order are accompanied by the softening of an acoustic phonon. The anisotropy of crystals implies that the softening is restricted to subsectors of wave-vector space, which are one- or two-dimensional in three-dimensional crystals.[1] As a consequence the critical dimensionality above which the critical behavior is classical is lowered in comparison to isotropic systems and the critical behavior is a function of the dimensionality m of the soft-wave-vector subspace.

The acoustic phonon which gets soft at an elastic phase transition is a hydrodynamic mode and its velocity of sound is proportional to the square root of a (combination of) elastic constant and thus $c_s \propto (T - T_c)^{(1-\eta/2)\nu}$.

Taking into account the linear dependence on the wave number one is led to conclude from the hydrodynamic limit that the dynamic critical exponent is given by $z = 2 - \frac{1}{2}\eta$. Given the structure (wave-number dependence) of the dynamics and assuming the validity of dynamical scaling, predictions about the temperature dependence of the transport coefficients can be made. For these predictions to be valid it is crucial that the critical frequency is a homogeneous function of the wave number k and the inverse correlation length ξ^{-1} but does not depend singularly on some irrelevant variable.

Our renormalization-group analysis for three-dimensional crystals shows that standard dynamical scaling is valid for $m = 1$ and logarithmic corrections appear for $m = 2$. On the other hand for isotropic systems, $m = d = 4 - \epsilon$, we find a singular dependence on an irrelevant variable, which invalidates dynamical scaling in its simple form. At the critical temperature T_c the soft acoustic phonons are overdamped and the wave-number dependence of their frequency is characterized by the Halperin, Hohenberg, and Ma[2] exponent $z = 2 + c\eta$. The apparent contradiction with the hydrodynamic result is resolved by noting that the sound velocity c_s depends singularly on irrelevant quantities like the mass density, which reconciles these differing dynamical exponents.

Our dynamical theory is based on semiphenomenological stochastic equations of motion for the phonon displacement field. As in the case of Langevin equations[3] it is convenient to introduce a path integral formulation also for the second-order phonon equations of motion. Starting from this path integral representation the elastic systems are treated by the renormalization-group theory. The dynamical recursion relations are computed explicitly for the isotropic-phonon model. A general discussion of the fixed points and modification of dynamical scaling[4] is given.

In Sec. II we present the equations of motion and introduce the corresponding Onsager-Machlup functional. After discussion of some general properties of the correlation functions the perturbation theory is introduced. In Sec. III we introduce the renormalization transformation and consider the consequences for the Gaussian region, which is relevant for systems with one-dimensional soft sectors. We also give a general discussion of the non-Gaussian case. Section IV is devoted to the isotropic-phonon model and to logarithmic corrections in this model and elastic transitions with two-dimensional soft sectors. In Sec. V the results are summarized. In Appendix A the equilibrium distribution function is derived and in Appendix B the fluctuation dissipation theorem is derived. In Appendix C we consider the relation to the relaxational model and in Appendix D the second-order contribution to the recursion relation for the damping constant is computed.

II. PHONON DYNAMICS

In this section we introduce the basic dynamic equations of motion and represent them in terms of an Onsager-Machlup functional.[5] After the discussion of some general properties of the correlation functions the perturbation expansion for the probability distribution will be introduced.

1. Equation of motion

Let us consider a d-dimensional system and denote the phonon normal coordinates by $Q_{\vec{k}}$ and the Landau-Ginzburg-Wilson free-energy functional (divided by kT) by $\mathcal{H}[Q_{\vec{k}}(t)]$. We will base our dynamic theory on the following equations of motion:

$$M\ddot{Q}_{\vec{k}} = -\frac{\delta\mathcal{H}}{\delta Q_{-\vec{k}}} - M\Gamma_{\vec{k}}\dot{Q}_{\vec{k}} + r_{\vec{k}} . \tag{2.1}$$

Here M is an effective mass and $\Gamma_{\vec{k}}$ is the damping coefficient. The random force $r_{\vec{k}}$ contains the effect of other noncritical phonons. We assume that its fluctuations are related to the damping coefficient by

$$\langle r_{\vec{k}}(t) r_{\vec{k}'}(t')\rangle = 2\Gamma_{\vec{k}} M\delta(\vec{k}+\vec{k}')\delta(t-t') , \tag{2.2}$$

with $\delta(\vec{k}) = (2\pi)^d\delta^d(\vec{k})$. This relation guarantees that the equilibrium distribution function for $Q_{\vec{k}}(t)$ is determined by $\exp(-K-\mathcal{H})$, where K is the kinetic energy (see Appendix A). In addition we shall assume in the following that the normalized probability distribution for $r_{\vec{k}}(t)$ is Gaussian, i.e.,

$$w_r[r] = \mathfrak{z}^{-1}\exp\left[-\int_{-\infty}^{\infty} dt \int \frac{d^dk}{(2\pi)^d} |r_{\vec{k}}(t)|^2/4\Gamma_{\vec{k}} M\right] . \tag{2.3}$$

We will be interested in situations where the Ginzburg-Landau free energy is even in $Q_{\vec{k}}$

$$\mathcal{H}[Q_{\vec{k}}] = \int \frac{d^dk}{(2\pi)^d} v_2(\vec{k}) Q_{\vec{k}} Q_{-\vec{k}} + u \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d} \times \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4) v_4(\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4) Q_{\vec{k}_1} Q_{\vec{k}_2} Q_{\vec{k}_3} Q_{\vec{k}_4} . \tag{2.4}$$

For optic phonons, which also can be described by Eqs. (2.1)–(2.4), v_2 and v_4 are finite for vanishing arguments.

For acoustic phonons, which are our concern in this paper, v_2 and v_4 vanish if one of the arguments approaches zero. In general the dispersion law of an acoustic phonon is anisotropic, which becomes particularly manifest near the elastic phase transition. As has been shown earlier[1] softening occurs in three-dimensional crystals only in one- or two-dimensional subsectors. To be general we assume the soft subsector to be m-dimensional and decompose the wave vector $\vec{k}$ into an m-dimensional "soft" component $\vec{p}$ and a $(d-m)$-dimensional "stiff" component $\vec{q}$; i.e., $\vec{k} = (\vec{p}, \vec{q})$. Then v_2 has the form

$$v_2(\vec{k}) = (rp^2 + q^2 + p^4) , \tag{2.5}$$

with $r \propto T - T_c^0$. All other contributions to the harmonic part are irrelevant for the critical behavior.[1] For the same reason we may restrict ourselves as in the statics[1] to

$$v_4(\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4) = \begin{cases} p_1p_2p_3p_4 \\ (\vec{p}_1\vec{p}_2)(\vec{p}_3\vec{p}_4) \end{cases} , \tag{2.6}$$

i.e., model I and II, respectively.

Of course also the damping coefficient will be anisotropic

$$\Gamma_{\vec{k}} = Dp^2 + \tilde{D}q^2 . \tag{2.7}$$

2. Onsager-Machlup functional and Lagrangean

A Langevin equation with Gaussian noise can be represented by an Onsager-Machlup functional.[3,5] In order to derive the Onsager-Machlup functional for the oscillator Eq. (2.1) we start from the proper definition of the path probability distribution for the random force $r(t)$

$$\exp\left[-\int_{-T}^{T} dt \frac{r^2(t)}{4\Gamma M}\right]\mathfrak{D}[r] \equiv \lim_{\tau\to 0}\prod_{\sigma=0}^{n}\left[\left(\frac{\tau}{4\Gamma M\pi}\right)^{1/2} dr_\sigma\right]\exp\left[-\tau\sum_{\sigma=0}^{n}\frac{r_\sigma^2}{4\Gamma M}\right] . \tag{2.8}$$

Here the total time interval $2T$ is divided into $n=2T/\tau$ subintervals of length τ and $r_\sigma = r(-T+\sigma\tau)$. For clarity we suppress the indices $\vec{k}$ in all intermediate steps. The discretized version of the equation of motion (2.1)

$$\frac{M}{\tau^2}(Q_\sigma - 2Q_{\sigma-1} + Q_{\sigma-2}) + \frac{\Gamma M}{\tau}(Q_\sigma - Q_{\sigma-1}) + \frac{\delta\mathcal{H}}{\delta Q_\sigma} = r_\sigma \tag{2.9}$$

can be used to eliminate r_σ from Eq. (2.8). Thus we get the probability distribution

$$w_Q[Q] = w_r[r(Q)]\left|\frac{\partial r}{\partial Q}\right|$$

for the order parameter $Q(t)$

$$w_Q[Q] = \mathfrak{z}^{-1}\exp\left\{-\int_{-T}^{T} dt\left[\left(M\ddot{Q} + \Gamma M\dot{Q} + \frac{\delta\mathcal{H}}{\delta Q}\right)^2\Big/4\Gamma M - \Gamma\right]\right\} . \tag{2.10}$$

The last term in the exponent comes from the Jacobian $|\partial r/\partial Q|$, which in contrast to the relaxational Langevin equation is a constant and thus will be included into $\mathfrak{z}$ in the following. Because of the second-order time derivative in Eq. (2.10) the differential

$$\mathfrak{D}[Q] = \lim_{\tau\to 0}\prod_{\sigma=0}^{n}[(M/4\Gamma\pi\tau^3)^{1/2}dQ_\sigma]$$

contains a normalization factor $\tau^{-3/2}$. To lower the nonlinearity and to get rid of the inverse damping coefficient in Eq. (2.10) we rewrite $w_Q[Q]$ by means of the auxiliary field $\tilde{Q}(t)$,

$$w_Q[Q] = \mathfrak{z}^{-1}\int \mathfrak{D}[i\tilde{Q}]\exp\mathcal{L}[\tilde{Q},Q] , \tag{2.11}$$

with

$$\mathfrak{D}[i\tilde{Q}] = \lim_{\tau\to 0}\prod_{\sigma=0}^{n}\left[\left(\frac{\tau\Gamma M}{\pi}\right)^{1/2} d(i\tilde{Q}_\sigma)\right] ,$$

and the "Lagrangean"

$$\mathcal{L}[\tilde{Q}_{\vec{k}}(t),Q_{\vec{k}}(t)] = \int\frac{d^dk}{(2\pi)^d}\int dt\left[\tilde{Q}_{\vec{k}}(t)\Gamma_{\vec{k}}M\tilde{Q}_{-\vec{k}}(t) - \tilde{Q}_{\vec{k}}(t)\left(M\ddot{Q}_{-\vec{k}}(t) + \frac{\delta\mathcal{H}}{\delta Q_{\vec{k}}(t)} + \Gamma_{\vec{k}}M\dot{Q}_{-\vec{k}}(t)\right)\right] . \tag{2.12}$$

Since all correlation functions can be represented as path integrals weighted with the density $w_Q[Q]$ this formulation puts dynamics in a form analogous to statics.[3]

From Eq. (2.12) it is easily seen that an additional external field $\tilde{h}_{\vec{k}}(t)$ in the equation of motion gives a contribution $\tilde{h}_{-\vec{k}}(t)\tilde{Q}_{\vec{k}}(t)$ in $\mathcal{L}$. To generate all the correlation functions of interest we include also a field $h_{\vec{k}}(t)$ and define the generating functional

$$f(\{\tilde{h}\},\{h\}) \equiv \mathfrak{z}^{-1}\int\mathfrak{D}[i\tilde{Q}]\mathfrak{D}[Q]\exp\left[\mathcal{L}[\tilde{Q},Q] + \int\frac{d^dk}{(2\pi)^d}dt[\tilde{h}_{-\vec{k}}(t)\tilde{Q}_{\vec{k}}(t) + h_{-\vec{k}}(t)Q_{\vec{k}}(t)]\right] , \tag{2.13}$$

with a normalization factor $\mathfrak{z}$. The correlation functions of $Q_{\vec{k}}(t)$ can now be written as functional derivatives of f

$$\langle Q_{\vec{k}_1}(t_1)\ldots Q_{\vec{k}_n}(t_n)\rangle = \left.\frac{\delta^n f(\{\tilde{h}\},\{h\})}{\delta h_{-\vec{k}_1}(t_1)\ldots\delta h_{-\vec{k}_n}(t_n)}\right|_{\tilde{h}=h=0} \tag{2.14}$$

and in addition the response functions

$$\langle Q_{\vec{k}_1}(t_1)\ldots Q_{\vec{k}_n}(t_n)\tilde{Q}_{\vec{k}_1'}(t_1')\ldots\tilde{Q}_{\vec{k}_m'}(t_m')\rangle = \left.\frac{\delta^{n+m} f(\{\tilde{h}\},\{h\})}{\delta h_{-\vec{k}_1}(t_1)\ldots\delta h_{-\vec{k}_n}(t_n)\delta\tilde{h}_{-\vec{k}_1'}(t_1')\ldots\delta\tilde{h}_{-\vec{k}_m'}(t_m')}\right|_{\tilde{h}=h=0} \quad (2.15)$$

can be defined. The following basic properties of these functions

$$\langle \tilde{Q}_{\vec{k}_1}(t_1)\ldots\tilde{Q}_{\vec{k}_n}(t_n)\rangle = 0\ , \quad (2.16a)$$

$$\langle r_{\vec{k}_1}(t)\tilde{Q}_{\vec{k}_2}(t')\rangle = \delta(\vec{k}_1+\vec{k}_2)\delta(t-t')\ , \quad (2.16b)$$

$$\langle Q_{\vec{k}_1}(t)\tilde{Q}_{\vec{k}_2}(t')\rangle = \frac{1}{2M}\Gamma_{\vec{k}_2}^{-1}\langle Q_{\vec{k}_1}(t) r_{\vec{k}_2}(t')\rangle\ , \quad (2.16c)$$

are obtained readily from Eq. (2.13). Another important relationship is the fluctuation dissipation theorem

$$\langle Q_{\vec{k}_1}(t)\tilde{Q}_{\vec{k}_2}(t')\rangle = -\theta(t-t') \times\frac{\partial}{\partial t}\langle Q_{\vec{k}_1}(t) Q_{\vec{k}_2}(t')\rangle\ , \quad (2.17)$$

which is derived in Appendix B. The fluctuation dissipation theorem provides a relationship between the two point correlation function

$$c(\vec{k},\omega) \equiv \langle Q_{\vec{k}\omega} Q_{\vec{k}'\omega'}\rangle/\delta(\vec{k}+\vec{k}')\delta(\omega+\omega') \quad (2.18a)$$

and the response function

$$g(\vec{k},\omega) = \langle Q_{\vec{k}\omega}\tilde{Q}_{\vec{k}'\omega'}\rangle/\delta(\vec{k}+\vec{k}')\delta(\omega+\omega')\ , \quad (2.18b)$$

where

$$Q_{\vec{k}\omega} = \int dt\, e^{i\omega t} Q_{\vec{k}}(t)\ .$$

3. *Perturbation theory*

The Lagrangean (2.12) is decomposed into a harmonic part

$$\mathcal{L}_0[\tilde{Q},Q] = \int\frac{d^dk}{(2\pi)^d}\int\frac{d\omega}{2\pi}\{\tilde{Q}_{\vec{k}\omega}\Gamma_{\vec{k}} M\tilde{Q}_{-\vec{k}-\omega} - \tilde{Q}_{\vec{k}\omega}[-M\omega^2 + i\Gamma_{\vec{k}} M\omega + v_2(\vec{k})]Q_{-\vec{k}-\omega}\} \quad (2.19)$$

and the interaction

$$\mathcal{L}_1[\tilde{Q},Q] = -4u\int\frac{d^dk_1}{(2\pi)^d}\frac{d\omega_1}{2\pi}\cdots\frac{d^dk_4}{(2\pi)^d}\frac{d\omega_4}{2\pi}\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)2\pi \times\delta(\omega_1+\omega_2+\omega_3+\omega_4)v_4(\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4)Q_{\vec{k}_1\omega_1}Q_{\vec{k}_2\omega_2}Q_{\vec{k}_3\omega_3}\tilde{Q}_{\vec{k}_4\omega_4}\ . \quad (2.20)$$

Because of Eq. (2.16a) the only nonvanishing free propagators are

$$g_0(\vec{k},\omega) = [-M\omega^2 - i\Gamma_{\vec{k}} M\omega + v_2(\vec{k})]^{-1}\ , \quad (2.21a)$$

$$c_0(\vec{k},\omega) = 2\Gamma_{\vec{k}} M|g_0(\vec{k},\omega)|^2\ . \quad (2.21b)$$

Since $c(\vec{k},\omega)$ follows from the fluctuation dissipation theorem (2.17) it suffices to consider the perturbation expansion for $g(\vec{k},\omega)$ which can be represented in terms of the self energy $\Sigma(\vec{k},\omega)$ by the Dyson equation

$$g(\vec{k},\omega) = [g_0^{-1}(\vec{k},\omega) - \Sigma(\vec{k},\omega)]^{-1}\ . \quad (2.22)$$

There are two properties which can be read off immediately in every order of perturbation theory. Firstly, Eq. (2.21a) and the structure of the perturbation series imply that M enters $g(\vec{k},\omega)$ only in the following combinations:

$$g(\vec{k},\omega) = G(\vec{k},(M)^{1/2}\omega,(M)^{1/2}D,(M)^{1/2}\tilde{D})\ . \quad (2.23)$$

Second, since $v_4(\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4)$ is proportional to $p_i (i=1,2,3,4)$ the self energy $\Sigma(\vec{k},\omega)$ is proportional to p^2. Hence, there are no contributions to $\Sigma(\vec{k},\omega)$ of the form $M\omega^2$, $i\omega M\tilde{D}q^2$, and q^2. Moreover the Hartree diagram contributes only to the rp^2 term.

III. CRITICAL BEHAVIOR

In this section we introduce the renormalization-group (RNG) transformation and discuss the fixed-point structure above and below the critical dimensionality.

1. *Renormalization procedure*

A particular advantage of the path integral representation lies in the similarity of the dynamical RNG to the static RNG.

The anisotropy of $v_2(\vec{k})$ suggests an RNG transformation where the soft and the stiff components are scaled differently. Restricting the wave vectors to a cylindrical Brillouin zone, $p \leq 1$ and $q \leq 1$, the RNG procedure is defined accordingly[1]:

(i) At each step of the renormalization we integrate out all $Q_{\vec{k}\omega}$ and $\tilde{Q}_{\vec{k}\omega}$ with momenta and frequencies in the region

$$b^{-1} < p < 1 \, , \quad b^{-x} < q < 1 \, , \quad -\infty < \omega < \infty \, . \tag{3.1}$$

(ii) We scale the variables and fields according to

$$p' = bp \, , \quad q' = b^x q \, , \quad \omega' = b^z \omega \, ,$$

$$Q_{\vec{k}\omega} = \zeta b^z Q'_{\vec{k}'\omega'} \, ,$$

and

$$\tilde{Q}_{\vec{k}\omega} = \tilde{\zeta} b^z \tilde{Q}'_{\vec{k}'\omega'} \, . \tag{3.2}$$

(iii) The definition of the RNG is completed by choosing suitable length scales in the two subspaces and a proper time scale. As in the static theory[1] we fix the coefficients of the p^4 and q^2 terms, which leads to

$$\zeta\tilde{\zeta} b^{2z} b^{-z} b^{-m} b^{-x(d-m)} b^{-4+\eta} = 1 \, , \tag{3.3}$$

$$\zeta\tilde{\zeta} b^{2z} b^{-z} b^{-m} b^{-x(d-m)} b^{-2x} = 1 \, , \tag{3.4}$$

where η has been introduced to take into account the $p^4 \ln b$ term. A third relation is obtained by applying the transformation (3.1) and (3.2) to the fluctuation dissipation theorem (2.17)

$$\zeta / \tilde{\zeta} = b^z \, . \tag{3.5}$$

It can be checked in second-order perturbation theory that Eq. (3.5) guarantees that the RNG transformations of the first and the third term in Eq. (2.19) lead to one and the same transformation for D.

Combining Eqs. (3.3) and (3.4) we recover the static result

$$x = 2 - \tfrac{1}{2}\eta \tag{3.6}$$

and use of Eq. (3.5) gives

$$\zeta^2 = b^{m + x(d-m) + 4 - \eta} \, . \tag{3.7}$$

By inspection of the recursion relation for the coupling coefficient u one finds for the critical dimension

$$d_c(m) = 2 + \tfrac{1}{2} m \, , \tag{3.8}$$

in accordance with that of the static theory.[1,6] Equation (3.8) implies that systems with one-dimensional soft sectors ($m = 1$) are Gaussian ($u^* = 0$) in three dimensions while logarithmic corrections to the classical behavior appear for two-dimensional soft sectors ($m = 2$). In isotropic elastic systems the nontrivial fixed point is stable for $d = 3$. The static results of the ϵ expansion are summarized for $d < d_c(m)$ in Table I. In the use of scaling laws one has to remember that the dimensionality d is to be replaced by $m + (2 - \frac{1}{2}\eta)\,(d - m)$, e.g.,

$$2 - \alpha = [m + (2 - \tfrac{1}{2}\eta)(d - m)]\nu \, ,$$

and that the exponents γ and η are defined in terms of the susceptibility of the strain, i.e.,

$$\lim_{p \to 0} \langle |pQ_p|^2 \rangle \sim \tau^{-\gamma}$$

with $\tau = (T/T_c) - 1$ and

$$\langle |pQ_p|^2 \rangle \sim p^{-2+\eta}$$

for $T = T_c$.

Now we turn to the recursion relations of the dynamic quantities which after use of Eqs. (3.5) and (3.7) read

$$M' = b^{4-\eta-2z} M \, , \tag{3.9}$$

$$M'D' = b^{2-\eta-z}[MD + u^2 E(D,M) \ln b] \, , \tag{3.10}$$

$$M'\tilde{D}' = b^{-z} M\tilde{D} \, . \tag{3.11}$$

TABLE I. Nontrivial fixed-point value for u and static critical exponents in order

$$\epsilon \equiv d_c(m) - d, \quad C(0) = \int_0^1 d^m p \int_{1/b^2}^1 d^{d_c - m} q \frac{p^4}{[(r + p^2)p^2 + q^2]^2} + \int_{1/b}^1 d^m p \int_0^{1/b^2} d^{d_c - m} q \frac{p^4}{[(r + p^2)p^2 + q^2]^2} \, .$$

The exponents β and δ have been obtained from scaling laws.

	$\frac{C(0)}{\ln b} \cdot u^*$	α	γ	ν	β	δ
Model I	$\frac{1}{18}\epsilon$	$\frac{1}{3}\epsilon$	$1 + \frac{1}{3}\epsilon$	$\nu = \frac{1}{2}\gamma$	$\frac{1}{2}(1 - \frac{1}{3}2\epsilon)$	$3 + 2\epsilon$
Model II	$\frac{m(m+2)}{2(m^2+6m+20)}\epsilon$	$\frac{2(m+8) - (m+2)^2}{2(m+8) + (m+2)^2}\epsilon$	$1 + \frac{(m+2)^2}{m^2+6m+20}\epsilon$	$\nu = \frac{1}{2}\gamma$	$\frac{1}{2}\left[1 - \frac{2(m+8)}{m^2+6m+20}\epsilon\right]$	$3 + 2\epsilon$

The second-order contribution to Eq. (3.10) $E(D,M)$ is represented diagrammatically in Fig. 1 and, for model I,[1] has to be extracted from

$$18(4u)^2p^2\int^{>}\frac{d^dk_1}{(2\pi)^d}\frac{d\omega_1}{2\pi}\frac{d^dk_2}{(2\pi)^d}\frac{d\omega_2}{2\pi}\frac{d^dk_3}{(2\pi)^d}\frac{d\omega_3}{2\pi}\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)2\pi\delta(\omega_1+\omega_2+\omega_3-\omega)p_1^2p_2^2p_3^2$$
$$\times c_0(\vec{k}_1,\omega_1)c_0(\vec{k}_2,\omega_2)g_0(\vec{k}_3,\omega_3)\ . \tag{3.12}$$

As emphasized after Eq. (2.23), the wave-vector dependence of the vertex excludes contributions from step (i) of the RNG to Eqs. (3.9) and (3.11). Since z is positive, Eq. (3.11) implies that $M\tilde{D}$ is irrelevant.

2. Gaussian region

Let us consider first the behavior above the critical dimension, i.e., $d > d_c(m)$. Then the trivial fixed point with u^* and η zero is stable and we find

$$z=2\ , \tag{3.13}$$

with M^* and D^* arbitrary.

This result is applicable to the experimentally relevant case of three-dimensional crystals with one-dimensional soft sectors, i.e., $m=1$. For the Gaussian fixed point one also computes readily the dynamic $Q_{\vec{k}\omega}$ response function

$$\chi(\vec{k},\omega)=[-M\omega^2-i\omega M(Dp^2+\tilde{D}q^2)+\xi^{-2}p^2+p^4+q^2]^{-1}\ . \tag{3.14}$$

The correlation length ξ behaves as

$$\xi^{-2}\propto T-T_c\ ,$$

where T_c is the actual transition temperature. In Eq. (3.14) we have retained the irrelevant $\tilde{D}$ term. $\tilde{D}$ is temperature independent and does not contain terms proportional to ξ^2 as one might have expected from Eq. (3.11) and as would be admissable according to scaling. In this sense one could say that dynamical scaling is contracted to the soft subsector.

In the hydrodynamic region, $p\xi \ll 1$, the sound velocity of the acoustic waves in the soft direction is

$$c_s\sim(T-T_c)^{1/2} \tag{3.15}$$

and the damping coefficient is temperature independent. At the critical point both damping and real part of the frequency are proportional to p^2.

For $m=2$ and $d=3$ we are at the border dimension and logarithmic corrections appear, which will be computed in Sec. IV.

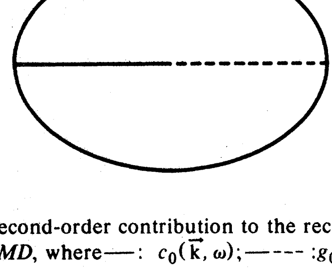

FIG. 1. Second-order contribution to the recursion relation for MD, where——: $c_0(\vec{k},\omega)$;- - - - :$g_0(\vec{k},\omega)$.

3. Non-Gaussian fixed point $d < d_c(m)$

For $d<d_c(m)$, η is finite and Eqs. (3.9) and (3.10) show that fixing M or MD will in general not lead to the same exponent z. We would like to consider now $d<d_c(m)$ in complete generality, including the case of isotropic systems which are non-Gaussian in three dimensions.

The fixed points of Eqs. (3.9) and (3.10) are determined best by considering first the parameter $X=\frac{1}{2}(M)^{1/2}D$, which transforms independently of z. Its transformation reads

$$X'=b^{-\eta/2}[X+u^2\bar{E}(X)\ln b]\ , \tag{3.16}$$

where because of Eq. (2.23),

$$\bar{E}(X)=E(D,M)/2(M)^{1/2} \tag{3.17}$$

has been introduced.

Equation (3.16) gives a finite fixed point X^* only if the function $\bar{E}(X)$ allows a solution of

$$\frac{\bar{E}(X^*)}{X^*}=\frac{\eta}{2u^{*2}}\ . \tag{3.18}$$

If such a finite fixed point exists one finds from both Eqs. (3.9) and (3.10)

$$z=2-\tfrac{1}{2}\eta \tag{3.19}$$

and M^* and D^* are finite and are related by $(M^*)^{1/2}D^*=2X^*$. Standard dynamical scaling[4] holds for such a fixed point.

However, for the isotropic model to be discussed in Sec. IV no solution of Eq. (3.18) exists, since $\bar{E}(X)/X > 3/16\pi^4$ and hence the only fixed point of Eq. (3.16) is $X^*=\infty$. [$X=0$ is not a fixed point since $\lim_{X\to 0}\bar{E}(X)/X=\infty$]. It is possible to determine $\bar{E}(X)$ for large X by general arguments without explicit computation of Eq. (3.12). According to Appendix C the phonon model degenerates into the relaxational model A of Halperin, Hohenberg, and Ma[2] in the limit $M\to 0$ (MD finite) and hence Eq. (3.9) has to coincide with the transformation of the inverse damping coefficient of the latter model in this limit, i.e.,

$$\lim_{\substack{M\to 0\\ MD=\text{fixed}}}\frac{E(D,M)}{DM}=\lim_{X\to\infty}\frac{\bar{E}(X)}{X}=(c+1)\frac{\eta}{u^{*2}}\ , \tag{3.20}$$

with c given in Eqs. (C3a) and (C3b) for model I. Inserting Eq. (3.20) into Eq. (3.16) we find for the exponent of X at the fixed point $X^*=\infty$

$$y_X=(c+\tfrac{1}{2})\eta\ . \tag{3.21}$$

If $X^*=\infty$ the recursion relations (3.9) and (3.10) admit more than one solution:

(i) If we require that M^* be finite we find

$$z=2-\tfrac{1}{2}\eta\ , \tag{3.22a}$$

which is the value one would anticipate from hydrodynamics. The scaling exponent of M in this case is of course

$$y_M=0 \tag{3.22b}$$

and $D^*M^*=\infty$.

(ii) On the other hand if we require D^*M^* to be finite we find a different exponent

$$z=2-\eta+u^{*2}\lim_{X\to\infty}\frac{\bar{E}(X)}{X}=2+c\eta \tag{3.23a}$$

and M approaches $M^*=0$ characterized by the scaling exponent

$$y_M=-(2c+1)\eta\ . \tag{3.23b}$$

It would be tempting to refer to the results of these differing fixing conditions as an oscillator and a relaxational fixed point. However despite the differing values of the individual exponents both representations describe the same physics, and lead to the same (modified) dynamical scaling law, as will be shown in Sec. IV. This might be anticipated from the fact that the mode is overdamped at criticality for both (i) and (ii).

Incidently we note that by fixing other combinations of D and M still other values of z and y_M could be generated. As will become clear later this is an artifact of a fixed point where one of the z-independent dynamical parameters (in our case X) is infinite or zero at the fixed point. The different values of z, resulting from different fixing conditions, reflect the fact that dynamical scaling in its usual form does not hold in such a situation.

While $X^*=\infty$ is the only fixed point of the isotropic model, it is conceivable that for other values of m the infinite fixed point and finite fixed point(s) exist simultaneously, with $X^*=\infty$ being stable for $c>-\frac{1}{2}$.

IV. ISOTROPIC MODEL, LOGARITHMIC CORRECTIONS

The isotropic-phonon model is an example where the RNG transformation can be computed explicitly to second order. The modifications of dynamical scaling implied by the infinite fixed point are studied in Sec. IV 2. Finally we will investigate possible logarithmic corrections at the border dimensionality of the isotropic model and of the $m=2$ system.

1. Isotropic acoustic phonons

In order to have a concrete example where the RNG transformation Eqs. (3.10) and (3.11) can be computed explicitly we study now the isotropic-phonon model, which is characterized by the Hamiltonian

$$\mathcal{H}=\tfrac{1}{2}\int\frac{d^dk}{(2\pi)^d}(rk^2+k^4)Q_{\vec{k}}Q_{-\vec{k}}+u\int\frac{d^dk_1}{(2\pi)^d}\cdots\frac{d^dk_4}{(2\pi)^d}\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)k_1k_2k_3k_4Q_{\vec{k}_1}Q_{\vec{k}_2}Q_{\vec{k}_3}Q_{\vec{k}_4} \tag{4.1}$$

and the corresponding Lagrangean

$$\begin{aligned}\mathcal{L}=&\int\frac{d^dk}{(2\pi)^d}\frac{d\omega}{2\pi}[\tilde{Q}_{\vec{k}\omega}MDk^2\tilde{Q}_{-\vec{k}-\omega}-\tilde{Q}_{\vec{k}\omega}(-M\omega^2+iMDk^2\omega+rk^2+k^4)Q_{-\vec{k}-\omega}]\\&-4u\int\frac{d^dk_1}{(2\pi)^d}\frac{d\omega_1}{2\pi}\cdots\frac{d^dk_4}{(2\pi)^d}\frac{d\omega_4}{2\pi}\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\delta(\omega_1+\omega_2+\omega_3+\omega_4)2\pi\cdot k_1k_2k_3k_4\tilde{Q}_{\vec{k}_1\omega_1}Q_{\vec{k}_2\omega_2}Q_{\vec{k}_3\omega_3}Q_{\vec{k}_4\omega_4}\end{aligned} \tag{4.2}$$

Concerning the physical relevance of the model we mention that there exist isotropic elastic systems in nature, for instance, polymers in the amorphous state. However, unfortunately we are not aware of any example

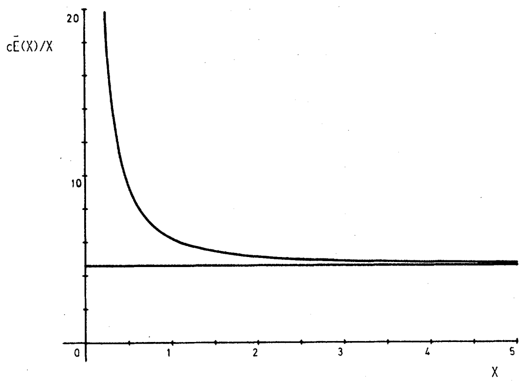

FIG. 2. Plot of the function $\bar{E}(X)/X$, which for convenience is multiplied by a factor $c=64\pi^4/9$. For the relaxational model the function would be the straight line $16\ln\frac{4}{3}$.

which would undergo an elastic phase transition. Nevertheless we should like to point out that Eq. (4.1) is merely a model and serves only as an oversimplified representation of reality. In particular it contains only one soft acoustic phonon whereas, for instance, in three dimensions one has two degenerate transverse acoustic phonons.

The statics of the Hamiltonian (4.1) is equivalent to the spatially isotropic-spin model as follows by changing to the variables $kQ_{\vec{k}}$. Thus the critical dimensionality is $d_c=4$ and

$$\eta=\frac{3}{8\pi^4}u^{*2}=\frac{1}{54}\epsilon^2\ , \tag{4.3}$$

for $m=d=4-\epsilon$.

The computation of the second-order contribution to the damping, Eq. (3.12), is deferred to Appendix D. The result is shown in Fig. 2, with $\bar{E}(X)$ being defined in Eq. (3.17). In the limit of large X we find in accordance with Eq. (3.20)

$$\lim_{X\to\infty}\bar{E}(X)=(c+1)\frac{3}{8\pi^4}X\ , \tag{4.4}$$

where now $c=6\ln\frac{4}{3}-1$. Obviously $X^*=\infty$ is the only fixed point of the isotropic model since $\bar{E}(X)/X$ is larger than $3/16\pi^4$.

2. *Dynamical scaling*

We will now investigate the physical consequences of the infinite fixed point $X^*=\infty$ and clarify the significance of the differing exponents obtained by the RNG transformations (i) and (ii). We have anticipated already that these different transformations and exponents describe one and the same critical point. In this section it will become clear how the exponent $z=2+c\eta$ obtained from (ii) can possibly describe an elastic phase transition. Without doubt, the sound velocity has to obey the hydrodynamic relation

$$c=\left(\frac{\text{elastic constant}}{\text{density}}\right)^{1/2}\sim\xi^{-(1-\eta/2)}\ ,$$

which would lead to $z=2-\frac{1}{2}\eta$. This contradiction with standard dynamical scaling is in fact a characteristic feature of this model. Since dynamical quantities depend on X, which diverges at the fixed point, usual dynamical scaling does not hold. Although X is an irrelevant parameter, in the sense that it approaches $X^*=\infty$ at the fixed point, it is necessary to keep track of X in the derivation of scaling laws, because it is infinite at the fixed point. For the fixing condition (ii) the same is true, since $M\propto X^{-2}$.

Recalling Eq. (2.23), the RNG transformation of the response function reads

$$G(\vec{k},(M)^{1/2}\omega,\xi,X)=b^{4-\eta}G[\vec{k}b,(M)^{1/2}\omega b^{z+y_M/2},\xi/b,Xb^{y_X}]\ . \tag{4.5}$$

First of all it is obvious that $y_X=(c+\frac{1}{2})\eta$ and the combination of exponents

$$z+\tfrac{1}{2}y_M=2-\tfrac{1}{2}\eta \tag{4.6}$$

are independent of the transformation, not only for the transformations (i) and (ii), but in general as can be seen directly from Eq. (3.16) and Eq. (3.9). Hence, one and the same scaling law is obtained for all of these different transformations. However, because of the dependence on X the correlation function is a homogeneous function of four variables and thus can be written in the form

$$G(\vec{k},(M)^{1/2}\omega,\xi,X)=k^{-4+\eta}G[1,(M)^{1/2}\omega k^{-2+\eta/2},k\xi,Xk^{-y_X}]\ . \tag{4.5$'$}$$

In dynamical problems like the time-dependent Ginzburg-Landau model[2] or the isotropic antiferromagnet[7] the dynamical susceptibility is a homogeneous function of the wave number, the frequency, and the correlation length and consequently the resonance frequency $\omega_c(k,\xi)$ scales as

$$\omega_c(k,\xi) = b^{-z}\omega_c(kb,\xi/b) \ .$$

In contrast, we find from Eq. (4.5) for the poles of the response function

$$\omega_c(k,\xi,X) = b^{(-2-\eta/2)}\omega_c(kb,\xi/b,Xb^{y_X}) \ , \tag{4.7a}$$

which allows us to represent ω_c in the following equivalent forms:

$$\omega_c(k,\xi,X) = k^{(2-\eta/2)}\omega_c(1,\xi k,Xk^{-y_X}) \ , \tag{4.7b}$$

$$\omega_c(k,\xi,X) = \xi^{(-2-\eta/2)}\omega_c(k\xi,1,X\xi^{y_X}) \ . \tag{4.7c}$$

Since the real part and imaginary part of the frequency will depend on X, dynamical scaling in its usual form does not hold. In order to determine the wave number and temperature dependence of the critical frequency we need to find the dependence on X of the dynamical susceptibility G. To achieve this in every order of perturbation theory it is convenient to compute the right-hand side of Eq. (4.5) for a sufficient large b, such that $Xb^{y_X} \gg 1$. Since the oscillator model approaches the relaxational model according to Appendix A in this limit, it can be shown in every order of perturbation theory that the self-energy part linear in ω is proportional to Xb^{y_X}. Of course b cannot be chosen arbitrarely large, but has to satisfy the inequalities $bk \leq 1$ and $b\xi^{-1} \leq 1$ in order for scaling law (4.5) to be valid. Thus the linear dependence on the initial damping coefficient $X = \frac{1}{2}(M)^{1/2}D$ is restricted to the domain $k \ll X^{1/y_X}$, $\xi^{-1} \ll X^{1/y_X}$. Recalling that the self-energy contains no ω^2 contribution and that G equals the static susceptibility $\chi(k,\xi)$ for $\omega = 0$ we get

$$G(k,(M)^{1/2}\omega,\xi,X) = [-M\omega^2 + \chi^{-1}(k,\xi) - i\omega Mk^2Xf(k,\xi) + O(\omega^3)]^{-1} \ . \tag{4.8}$$

Here $f(k,\xi)$ obeys the scaling law

$$f(kb,\xi/b) = b^{-y_X-\eta/2}f(k,\xi)$$

as is apparent from Eq. (4.5). Because $f(0,1)$ and $f(1,\infty)$ are finite, the damping term is proportional to $k^2\xi^{y_X+\eta/2}$ in the hydrodynamic region and to $k^{2-y_X-\eta/2}$ in the critical region.

Thus we find at the critical temperature ($k\xi \gg 1$),

$$\omega_c \sim -ik^{2+c\eta} \ . \tag{4.9}$$

In the hydrodynamic region , $k\xi \ll 1$, we find

$$\omega_c = \pm ck - \tfrac{1}{2}i\mathfrak{D}k^2 \ , \tag{4.10}$$

where the velocity and the damping coefficient behave as

$$c \sim (T-T_c)^{\nu(1-\eta/2)} \ ,$$
$$\mathfrak{D} \sim (T-T_c)^{-\nu(c+1)\eta} \ . \tag{4.11}$$

Equation (4.10) holds only if the damping is much smaller than the real part of the frequency, i.e., for $k \ll \xi^{-(1+y_X)}$. In the intermediate region $\xi^{-y_X}k\xi \ll k\xi \ll 1$, the mode found from Eq. (4.8) is overdamped (see Fig. 3).

At the critical temperature, the dynamics is characterized by the exponent of the relaxational model A,[2] $z = 2 + c\eta$. On the other hand the correct hydrodynamic sound velocity results for $k\xi \ll 1$, corresponding to $z = 2 - \frac{1}{2}\eta$. Still another exponent characterizes the damping coefficient in the hydrodynamic region. The breakdown of dynamical scaling can be traced to the dependence on the parameter X, which is irrelevant but diverges at the fixed point. The modification of dynamical scaling was not yet realized in Refs. 8 and 11, where the damping coefficient was inferred from $z = 2 - \frac{1}{2}\eta$ under the assumption of dynamical scaling.

3. *Logarithmic corrections*

The general theory of logarithmic corrections to static phenomena has been developed by Wegner.[9] Such corrections have to show up also in the dynamic response function to be consistent with the limit $\omega \to 0$. Of course one has to check whether there are additional logarithmic corrections from purely dynamical quantities (we use the notation of Refs. 9 and 10). The dynamical recursion relations can be rewrit-

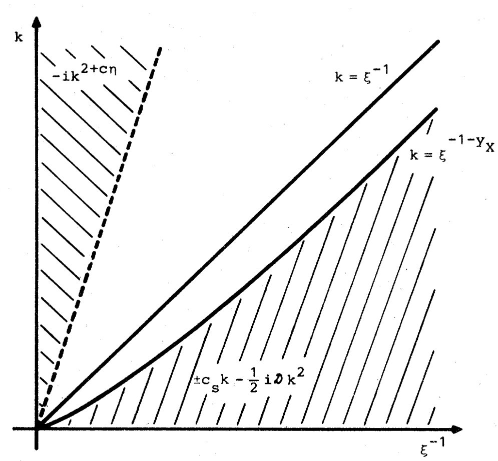

FIG. 3. Schematic representation of the critical and hydrodynamic region. To show the reduction of the hydrodynamic region, we have chosen a very large value of η.

ten in the form

$$\frac{dM_l}{dl} = (4 - \eta_l - 2z_l) M_l \ , \qquad (4.12a)$$

$$\frac{dD_l}{dl} = -(2 - z_l) D_l + \frac{u_l^2 E(D_l, M_l)}{M_l} \ . \qquad (4.12b)$$

For the four-dimensional isotropic elastic model we have $\eta_l = (3/8\pi^4) u_l^2$ with $u_l = u_0 l_0/(l + l_0)$ and $l_0 = 1/36\bar{C}u_0$. With

$$z_l = 2 - \tfrac{1}{2}\eta_l = 2 - (3/16\pi^4) u_l^2$$

we may set $M_l = 1$ and without loss of generality we find from Eq. (4.12b)

$$\int_{D_0}^{D_l} \frac{dt}{E(t,1) - (3/16\pi^4)t} = \frac{u_0^2 l_0 l}{l + l_0} \ . \qquad (4.13)$$

Since the integrand is always positive and behaves as $1/t$ for large t according to Eq. (4.4) we conclude from Eq. (4.13)

$$\lim_{l \to \infty} D_l = D^*(D_0) < \infty, \quad D^* > D_0 \ . \qquad (4.14)$$

Hence there is no logarithmic correction from D, which would require a power-law dependence of D on $(l + l_0)$. Fixing the damping coefficient MD leads of course to the same result. The absence of additional logarithmic corrections has also been proven for the uniaxial dipolar magnet with relaxational dynamics and not conserved energy.[10]

More interesting is the case of *two-dimensional* soft sectors, where logarithmic corrections are found in three dimensions. Although we have not computed $E(D,M)$ for this case we can nevertheless show that there are no logarithmic corrections beyond the static ones. From Eqs. (4.12b) or (4.13) we see that D_l is a function of $(1/(l + l_0) + \beta)$, where β is a finite constant of integration. Hence the logarithmic correction due to the first term in the parenthesis is negligible in leading order.

Therefore the dynamic susceptibility is given by

$$\chi(\vec{k}, \omega) = \left[-M\omega^2 + \tau \left(\frac{1}{4l_0} \log \tau \right)^{-1/3} p^2 + p^4 + q^2 - iM\omega(D^* p^2 + \tilde{D} q^2) \right]^{-1} \ . \qquad (4.15)$$

Here p is the component parallel to the soft plane and q perpendicular to it. The exponent of the logarithm is $-\frac{1}{3}$ for model I, while it is $-\frac{4}{9}$ for model II.[1]

V. SUMMARY AND DISCUSSION

In Secs. I–IV we have investigated the dynamics of elastic phase transitions based on a stochastic equation of motion for the soft acoustic phonon. Other phonons entered only as a stochastic force in this equation of motion. In view of the spatial anisotropy of phonon frequencies we had to decompose the d-dimensional k space into a "soft" subspace for which the sound velocity vanishes at T_c and a "stiff" subspace. Assuming the former to be m dimensional, we obtained for the critical dimension $d_c = 2 + \frac{1}{2}m$, which coincides with the static critical dimension as in the majority of dynamical models.

In three-dimensional crystals m can be 1 or 2. For situations where softening occurs only in selected directions, i.e., $m = 1$, the behavior is classical; in particular the sound velocity vanishes with the square root of $(T - T_c)$ along these directions and the damping is temperature independent. At T_c the dispersion is proportional to the square of the wave vector.

For two-dimensional soft sectors ($m = 2$) the classical critical behavior is modified by logarithmic corrections, as given in Eq. (4.15).

In Table II we have collected substances undergoing elastic phase transitions. The references are by no means complete but should allow the reader to trace the rest of the relevant literature about these systems. Except for KCN, NaCN, and s-triazine, which are examples for $m = 2$, these systems show softening only in certain directions and hence belong to $m = 1$. In the application of our theory to these systems the following limitations have to be kept in mind, which have already been discussed to some extent in Refs. 1 and 11.

First we have disregarded all third-order terms. The third-order elastic constants allowed by symmetry have been tabulated by Brugger[12] for the different Laue groups. From Table III in Ref. 12 one easily reads off that in cubic systems for $c_{44} \to 0$ there is only one third-order term of the form $\epsilon_{13}\epsilon_{23}\epsilon_{12}$, whereas for $c_{11} - c_{12} \to 0$ in cubic systems and for $c_{66} \to 0$ in hexagonal systems several third-order terms are allowed. In the other cases symmetry does not permit such terms. For instance in the orthorhombic system there are no terms of the form $(\epsilon_{13})^3$ and hence there are no third-order terms at transitions where $c_{55} \to 0$. If third-order terms are present the transition will be of the first order. Our theory is applicable in situations where these third-order terms are not present at all for symmetry reasons or where they are so small that the first-order character will appear only close to T_c and hence the transition can be considered as nearly second order.

Second, we have disregarded interactions of the soft acoustic phonon with other phonons. For example there will be interactions of the form $\epsilon_{ii} \times (\text{shear})^2$, which lead to a coupling of the soft transverse phonon to longitudinal phonons. The form of this interaction is similar in structure to the magnetostrictive interaction in compressible magnets.[13] The elimination of the noncritical longitudinal phonons leads to additional (negative) fourth-order interactions between the soft modes. One may anticipate that in analogy to the magnetostrictive case the transition becomes of first order for free boundaries and a positive exponent of the specific heat. For a clamped crystal the transition remains second order in magnets, however, clamping would interfere with the shear deformation, which is characteristic of our elastic systems. Again as in magnetic transitions and structural phase transitions accompanied by soft optic phonons, we have to assume that the first-order character is small enough such that the transitions can be considered to be almost of second order.

Third we have not considered the conservation of energy or alternatively the coupling to heat diffusion. It is well known that the sound velocities are determined not by the isothermal but by the adiabatic elastic constants if heat conduction is taken into account. For transverse acoustic waves in high-symmetry directions of orthorhombic, tetragonal, cubic, and hexagonal crystals isothermal and adiabatic elastic constants are identical. Modifications might occur for $T = T_c$ in the isotropic model for $d < d_c$.

Aside from KCN, NaCN, and *s*-triazine there is a conspicuous absence of elastic phase transitions with $m = 2$. In systems with $m = 2$ the local fluctuations and hence the Debye-Waller factor[1] would diverge logarithmically at a second-order transition. Thus one expects that the actual transition in such systems will be of first order, leaving aside other causes. Also KCN and NaCN undergo a first-order transition. It would be interesting to investigate if hydrostatic pressure would reduce the first-order character.

As discussed in some detail in the context of the static theory, there is in general more than one soft sector in three-dimensional crystals. There we concluded that the interactions between modes in different sectors are irrelevant near the fixed point, which justified the investigation of single sectors. These arguments can be generalized to the dynamics.

The basic fields in our theory are the components of the displacement vector. Compared to the use of the strain tensor, the essential advantage is that the compatibility conditions[14] are automatically contained in our theory.

We turn now to the discussion of isotropic elastic phase transitions. We are not aware of any isotropic elastic system undergoing an elastic transition, how-

TABLE II. Examples of elastic phase transitions.

Crystal	Transition Temperature in K	Symmetry below	Symmetry above	soft elastic constant	Refs.
$LiNH_4C_4H_4O_6 \cdot H_2O$	97,5	monoclinic	orthorhombic	c_{55}	a
$KH_3\ (SeO_3)_2$	211	monoclinic	orthorhombic	c_{44}	b
$KD_3\ (SeO_3)_2$	296	monoclinic	orthorhombic	c_{44}	b
$PrAlO_3$	118,5		monoclinic		c
InTl Alloys 25% Tl	196	tetragonal	cubic	$c_{11}-c_{12}$	d
InTl Alloys 27% Tl	127	tetragonal	cubic	$c_{11}-c_{12}$	d
NdP_5O_{14}	419	monoclinic	orthorhombic	c_{55}	e
LaP_5O_{14}	391	monoclinic	orthorhombic	c_{55}	e,f
TeO_2	P_c 9 kbar	tetragonal	orthorhombic	$c_{11}-c_{12}$	g
NaOH	513	orthorhombic	monoclinic	c_{55}	h
KH_2PO_4	122	orthorhombic	tetragonal	c_{66}	i
KD_2PO_4	220	orthorhombic	tetragonal	c_{66}	j
$TmVO_4$	2,1	orthorhombic	tetragonal	c_{66}	k
$DyVO_4$	14,8	orthorhombic	tetragonal	$c_{11}-c_{12}$	l,m
Nb_3Sn	49	tetragonal	cubic	$c_{11}-c_{12}$	n
γ-Mn alloy	175	tetragonal	cubic	$c_{11}-c_{12}$	o
$TbVO_4$	34	orthorhombic	tetragonal	c_{66}	m
Hg_2Cl_2	185	orthorhombic	tetragonal	c_{66}	p
NiF_2	P_c 18,3 kbar	tetragonal	orthorhombic	$c_{11}-c_{12}$	q
K_2CrO_4	893		orthorhombic		r
Rb_2CrO_4	923		orthorhombic		r
Cs_2CrO_4	953		orthorhombic		r
$PrAlO_3$	151	monoclinic	orthorhombic		s
DySb	9,5		cubic	$c_{11}-c_{12}$	t
TmCd	3.16	tetragonal	cubic	$c_{11}-c_{12}$	u
V_3Si	21	tetragonal	cubic	$c_{11}-c_{12}$	v

TABLE II. (Con't.)

Crystal	Transition Temperature in K	Symmetry below	Symmetry above	soft elastic constant	Refs.
UO_2	30,8			c_{44}	w
$PrCu_2$	≤ 10		orthorhombic	c_{66}	x
$NiCr_2O_4$	300	tetragonal	cubic	$c_{11}-c_{12}$	y
KCN	168	orthorhombic	cubic	c_{44}	z
NaCN	288	orthorhombic	cubic		aa
s-triazine	198	monoclinic	hexagonal		ab

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ever, it is conceivable that the decrease of shear rigidity could trigger the melting transition in a system. The critical exponents of isotropic elastic phase transitions are nonclassical in three dimensions. The most interesting and unexpected property of the isotropic elastic model investigated in Sec. IV, however, is the breakdown of dynamical scaling for $d < d_c$. The dynamical susceptibility and hence the characteristic frequency depend on an irrelevant parameter $[D(M)^{1/2}]$ which diverges at the fixed point. The characteristic frequency is a homogeneous function not only of the wave number and the correlation length but also of this parameter. Consequently different exponents z characterize the dynamics in different regions in the $k - \xi^{-1}$ plane. This breakdown of scaling manifests itself also by the fact, that different fixing conditions (different choices of the frequency scale) lead to different dynamical exponents z. However, all physical quantities are independent of the transformation.

This breakdown of scaling is reminiscent of other cases in statics and dynamics. For isotropic-spin systems at $d > d_c$[15] for instance the exponent β is modified, because the magnetization is proportional to the inverse square root of the coupling coefficient. A similar situation is found for the superfluid model[16] and the rotational symmetric n-component phonon model[17] in certain domains of d and n, the number of components, as found by De Dominicis and Peliti[16] and investigated further by Dohm and Ferrell.[18] In the field-theoretic language a divergent (or vanishing) fixed point of a z-independent combination of dynamical parameters is accompanied by different wave-function renormalizations of the order parameter and the generator of the symmetry operation (entropy in helium, angular momentum in the n-component phonon model). In the elastic model Q and $P=\dot{Q}$ would obtain different wave-function renormalizations. The above-mentioned authors prefer to discuss the situation in the models studied by them in terms of different exponents z for the field and for the generator, starting from scaling relations, where the dynamical parameters including the vanishing and divergent ones are set at their fixed-point values. We feel that including explicitly the diverging or vanishing parameters along the whole trajectory in the scaling laws Eqs. (4.5) and (4.7a)–(4.7c) is important and allows a description of the whole $k-\xi^{-1}$ plane.

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APPENDIX A: EQUILIBRIUM DISTRIBUTION FUNCTION

In order to find the equilibrium distribution function we explicitly solve the Fokker-Planck equation derived from the Langevin Eq. (2.1). To this end we define the probability distribution

$$g(\{\xi\},\{\pi\};t)=\prod_{\vec{k}}\delta[\pi_{\vec{k}}-P_{\vec{k}}(t)]\times\delta[\xi_{\vec{k}}-Q_{\vec{k}}(t)]\ , \tag{A1}$$

where the momentum is defined by

$$P_{\vec{k}}=M\dot{Q}_{\vec{k}}\ . \tag{A2}$$

Introducing $d\xi\equiv\prod_{\vec{k}}d\xi_{\vec{k}}$ and $d\pi\equiv\prod_{\vec{k}}d\pi_{\vec{k}}$ we can represent arbitrary equal-time correlation functions in the form

$$\langle Q_{\vec{k}_1}(t)\cdots Q_{\vec{k}_m}(t)P_{\vec{k}_{m+1}}(t)\cdots P_{\vec{k}_n}(t)\rangle=\int d\xi\, d\pi\,\langle g(\{\xi\},\{\pi\};t)\rangle\,\xi_{\vec{k}_1}\cdots\xi_{\vec{k}_m}\pi_{\vec{k}_{m+1}}\cdots\pi_{\vec{k}_n}\ . \tag{A3}$$

Equation (A3) contains as a weight function the expectation value of g which will be derived from its equation of motion. Using Eq. (2.1) and (A2) we find for the time derivative of $g(\{\xi\},\{\pi\};t)$

$$\frac{\partial}{\partial t}g(\{\xi\},\{\pi\};t)=-\sum_{\vec{k}}\left\{\frac{\delta}{\delta\xi_{\vec{k}}}\left[\frac{g(\{\xi\},\{\pi\};t)\pi_{\vec{k}}}{M}\right]+\frac{\delta}{\delta\pi_{\vec{k}}}\left[g(\{\xi\},\{\pi\};t)\left(-\frac{\delta\mathcal{H}}{\delta\xi_{-\vec{k}}}-\Gamma_{\vec{k}}\pi_{\vec{k}}+r_{\vec{k}}(t)\right)\right]\right\}\ . \tag{A4}$$

Taking the expectation value we obtain the Fokker-Planck equation

$$\frac{\partial}{\partial t}\langle g(\{\xi\},\{\pi\};t)\rangle=-\sum_{\vec{k}}\left[\pi_{\vec{k}}/M\frac{\delta}{\delta\xi_{\vec{k}}}-\frac{\delta}{\delta\pi_{\vec{k}}}\frac{\delta\mathcal{H}}{\delta\xi_{-\vec{k}}}-\Gamma_{\vec{k}}\frac{\delta}{\delta\pi_{\vec{k}}}\pi_{\vec{k}}\right]\langle g(\{\xi\},\{\pi\};t)\rangle-\frac{\delta}{\delta\pi_{\vec{k}}}\langle g(\{\xi\},\{\pi\};t)r_{\vec{k}}(t)\rangle\ . \tag{A5}$$

Using the relations

$$\langle g(\{\xi\},\{\pi\};t)r_{\vec{k}}(t)\rangle=2\Gamma_{\vec{k}}M\left\langle\frac{\delta}{\delta r_{-\vec{k}}(t)}g(\{\xi\},\{\pi\};t)\right\rangle \tag{A6}$$

and

$$\frac{\delta P_{\vec{k}}(t)}{\delta r_{\vec{k}'}(t')}=\begin{cases}0, & t<t'\\ \frac{1}{2}\delta(\vec{k}-\vec{k}'), & t=t'\end{cases}\ , \tag{A7}$$

we get

$$\langle g(\{\xi\},\{\pi\};t)r_{\vec{k}}(t)\rangle=-\Gamma_{\vec{k}}M\frac{\delta}{\delta\pi_{-\vec{k}}}\langle g(\{\xi\},\{\pi\};t)\rangle\ . \tag{A8}$$

If the free energy $\mathcal{H}$ does not contain any time-dependent external field for the equilibrium distribution function the left-hand side of Eq. (A5) vanishes. Then we eliminate the last term of Eq. (A5) by means of Eq. (A6) and find

$$\sum_{\vec{k}}\left[\pi_{\vec{k}}/M\frac{\delta}{\delta\xi_{\vec{k}}}-\frac{\delta\mathcal{H}}{\delta\xi_{-\vec{k}}}\frac{\delta}{\delta\pi_{\vec{k}}}-\Gamma_{\vec{k}}\left(\frac{\delta}{\delta\pi_{\vec{k}}}\pi_{\vec{k}}+M\frac{\delta^2}{\delta\pi_{\vec{k}}\delta\pi_{-\vec{k}}}\right)\right]\langle g(\{\xi\},\{\pi\};t)\rangle=0\ . \tag{A9}$$

The last equation has as a solution

$$\langle g(\{\xi\},\{\pi\};t)\rangle = Z^{-1}\exp\left(\sum_{\vec{k}}\frac{|\pi_{\vec{k}}|^2}{2M} - \mathcal{H}\right) \quad (A10)$$

the exponent of which is the sum of the kinetic energy and the equilibrium free-energy functional. This completes the proof that the Langevin equation (2.1) is compatible with the static distribution function.

APPENDIX B: FLUCTUATION DISSIPATION THEOREM

From Eq. (2.16c) and the requirement of causality, which implies that $\langle Q_{\vec{k}}(t) r_{\vec{k}'}(t')\rangle$ differs from zero only for $t > t'$, we obtain

$$\langle Q_{\vec{k}}(t)\tilde{Q}_{\vec{k}'}(t')\rangle = \theta(t-t')\langle Q_{\vec{k}}(t)\tilde{Q}_{\vec{k}'}(t')\rangle \ . \quad (B1)$$

Then we decompose the forces in the equation of motion (2.1) into a reversible and an irreversible part

$$M\ddot{Q}_{\vec{k}} = K^{\rm irr}_{\vec{k}} + K^{\rm rev}_{\vec{k}} + r_{\vec{k}} \ , \quad (B2)$$

with

$$K^{\rm irr}_{\vec{k}} = -M\Gamma_{\vec{k}}\dot{Q}_{\vec{k}}, \quad K^{\rm rev}_{\vec{k}} = -\frac{\delta\mathcal{H}}{\delta Q_{-\vec{k}}} \ . \quad (B3)$$

The probability distribution (2.11) gives rise to the time-reversal operation

$$Q_{\vec{k}}(t) \to Q_{\vec{k}}(-t), \quad \tilde{Q}_{\vec{k}}(t) \to \tilde{Q}_{\vec{k}}(-t) - \dot{Q}_{\vec{k}}(-t) \ ,$$
$$K^{\rm rev}_{\vec{k}}(t) \to K^{\rm rev}_{\vec{k}}(-t), K^{\rm irr}_{\vec{k}}(t) \to -K^{\rm irr}_{\vec{k}}(-t) \ . \quad (B4)$$

Using Eqs. (2.16c) and (B2) in the relation (B1) we find

$$\langle Q_{\vec{k}}(t)\tilde{Q}_{\vec{k}'}(t')\rangle = \theta(t-t')\langle Q_{\vec{k}}(t)\frac{1}{2M\Gamma_{\vec{k}'}}[M\ddot{Q}_{\vec{k}'}(t') - K^{\rm rev}_{\vec{k}'}(t') - K^{\rm irr}_{\vec{k}'}(t')]\rangle \ . \quad (B5)$$

Applying the invariance properties

$$\langle Q_{\vec{k}}(t)\ddot{Q}_{\vec{k}'}(t')\rangle = \langle \ddot{Q}_{\vec{k}'}(t)Q_{\vec{k}}(t')\rangle, \quad \langle Q_{\vec{k}}(t)K^{\rm rev}_{\vec{k}'}(t')\rangle = \langle K^{\rm rev}_{\vec{k}'}(t)Q_{\vec{k}}(t')\rangle$$

and using again the equation of motion we find

$$\langle Q_{\vec{k}}(t)\tilde{Q}_{\vec{k}'}(t')\rangle = \theta(t-t')\frac{1}{2M\Gamma_{\vec{k}'}}\{\langle [K^{\rm irr}_{\vec{k}'}(t) + r_{\vec{k}'}(t)]Q_{\vec{k}}(t')\rangle - \langle Q_{\vec{k}}(t)K^{\rm irr}_{\vec{k}'}(t')\rangle\} \ . \quad (B6)$$

By causality this reduces to

$$\langle Q_{\vec{k}}(t)\tilde{Q}_{\vec{k}'}(t')\rangle = \theta(t-t')\frac{1}{2M\Gamma_{\vec{k}'}}\langle K^{\rm irr}_{\vec{k}'}(t)Q_{\vec{k}}(t') - Q_{\vec{k}}(t)K^{\rm irr}_{\vec{k}'}(t')\rangle \ . \quad (B7)$$

Inserting the definition of $K^{\rm irr}_{\vec{k}}(t)$ we finally get the fluctuation dissipation theorem

$$\langle Q_{\vec{k}}(t)\tilde{Q}_{\vec{k}'}(t')\rangle = -\theta(t-t') \times \frac{\partial}{\partial t}\langle Q_{\vec{k}}(t)Q_{\vec{k}'}(t')\rangle \ . \quad (B8)$$

APPENDIX C: CONNECTION TO RELAXATIONAL MODELS

Since Eq. (3.11) shows that the damping constant $\tilde{D}$ in Eq. (2.2) is irrelevant, we restrict ourselves in the following to the model defined by Eqs. (2.1), (2.4), (2.5), and (2.8) with $\Gamma_{\vec{k}} = Dp^2$. In the limit $M \to 0$ with $DM = 1/\bar{\Gamma}$ kept fixed and $\bar{r}_{\vec{k}}(t) = \bar{\Gamma} r_{\vec{k}}(t)$ we recover, after the substitution $S_{\vec{k}} = pQ_{\vec{k}}$, the relaxational model

$$\dot{S}_{\vec{k}}(t) = -\bar{\Gamma}\frac{\delta\bar{\mathcal{H}}[S_{\vec{k}}]}{\delta S_{-\vec{k}}(t)} + \bar{r}_{\vec{k}}(t) \quad (C1)$$

and

$$\langle \bar{r}_{\vec{k}}(t)\bar{r}_{\vec{k}'}(t')\rangle = 2\bar{\Gamma}\delta(\vec{k}+\vec{k}')\delta(t-t') \ . \quad (C2)$$

For this model the critical exponent z is given by $z = 2 + c\eta[d < d_c(m)]$. Especially, performing this limit for the absolute value model[1] (model I) the constant c is given by

$$c = 0.92 \text{ for } m=2, \quad d = 3-\epsilon \text{ (Ref.10)} \ , \quad (C3a)$$

$$c = 6\ln\tfrac{4}{3} - 1 = 0.73 \text{ for } m=d=4-\epsilon \text{ (Ref.2)} \ . \quad (C3b)$$

APPENDIX D: CALCULATION OF $E(D,M)$ FOR $m = d$

We start with the second-order contribution to the harmonic part in $\mathcal{L}_0$ Eq. (2.19) (Fig. 1).

$$18(4u)^2k^2\int^{>}\frac{d^4k_1}{(2\pi)^4}\frac{d\omega_1}{2\pi}\cdots\frac{d^4k_3}{(2\pi)^4}\frac{d\omega_3}{2\pi}\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)2\pi\delta(\omega_1+\omega_2+\omega_3-\omega)k_1^2k_2^2k_3^2c_0(\vec{k}_1,\omega_1)c_0(\vec{k}_2,\omega_2)g_0(\vec{k}_3,\omega_3)\ , \tag{D1}$$

where $g_0(\vec{k},\omega)$ and $c_0(\vec{k},\omega)$ are given by the isotropic version of Eqs. (2.21a) and (2.21b) with $r=0$. The integration has to be carried out over the region defined in Eq. (3.1). To extract from Eq. (C1) the $i\omega\ln b$ terms, the condition on k_3 is relaxed and the k_3 integration is extended over the whole Brillouin zone. Performing the ω integrations in Eq. (D1) we arrive at

$$36u^2k^2(1-X^2)^{-3/2}\cdot\{[-iX+(1-X^2)^{1/2}]^{-2}\cdot[(-1,-1,-1)-(-1,-1,1)]+[iX+(1-X^2)^{1/2}]^{-2}$$
$$\times[(1,1,-1)-(1,1,1)]+[(1,-1,-1)-(1,-1,1)+(-1,1,-1)-(-1,1,1)]\}\ , \tag{D2}$$

where the definitions $X=\frac{1}{2}D(M)^{1/2}$ and

$$(\epsilon_1,\epsilon_2,\epsilon_3)=\int^{>}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{1}{k_1^2k_2^2}\{\omega(M)^{1/2}+iX[k_1^2+k_2^2+(\vec{k}_1+\vec{k}_2)^2]+(1-X^2)^{1/2}[\epsilon_1k_1^2+\epsilon_2k_2^2+\epsilon_3(\vec{k}_1+\vec{k}_2)^2]\}^{-1} \tag{D3}$$

have been introduced. The integration over the angle between $\vec{k}_1$ and $\vec{k}_2$ can be done easily. Retaining the linear term of the Taylor expansion in ω and putting $Y=X/(1-X^2)^{1/2}$ we get

$$(\epsilon_1,\epsilon_2,\epsilon_3)=\omega(M)^{1/2}\frac{K_3K_4}{8(1-X^2)(iY+\epsilon_3)^2}$$
$$\times\int_{1/b}^{1}\frac{dk_1}{k_1}\int_{1/b}^{1}\frac{dk_2}{k_2}\frac{2iY(k_1^2+k_2^2)+(\epsilon_1+\epsilon_3)k_1^2+(\epsilon_2+\epsilon_3)k_2^2}{\{[2iY(k_1^2+k_2^2)+(\epsilon_1+\epsilon_3)k_1^2+(\epsilon_2+\epsilon_3)k_2^2]^2-4k_1^2k_2^2(iY+\epsilon_3)^2\}^{1/2}}\ , \tag{D4}$$

with $K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(\frac{1}{2}d)$. Keeping now only the $\ln b$ terms we finally obtain

$$(\epsilon_1,\epsilon_2,\epsilon_3)=\omega\ln b\tfrac{1}{2}K_4^2\frac{(M)^{1/2}}{1-X^2}(iY+\epsilon_3)^2\ln\frac{(2iY+\epsilon_1+\epsilon_3)(2iY+\epsilon_2+\epsilon_3)}{(iY+\epsilon_1)(iY+\epsilon_2)+(iY+\epsilon_1)(iY+\epsilon_3)+(iY+\epsilon_2)(iY+\epsilon_3)}\ . \tag{D5}$$

Inserting this result into Eq. (D2) we find

$$E(D,M)=2(M)^{1/2}\bar{E}(X) \tag{D6}$$

with the function $\bar{E}(X)$ given by

$$\bar{E}(X)\frac{64\pi^4}{9}=16X\ln\tfrac{4}{3}-\frac{4X}{1-X^2}\ln\frac{8X^2+1}{9X^2}+\frac{2(1-4X^2)}{(1-X^2)^{3/2}}$$
$$\times\arctan\frac{(1-X^2)^{1/2}}{3X}+\frac{2}{(1-X^2)^{3/2}}\arctan\frac{(1-X^2)^{1/2}}{X},\quad\text{for } X<1\ , \tag{D7a}$$

$$\bar{E}(X)\frac{64\pi^4}{9}=16X\ln\tfrac{4}{3}-(X^2-1)^{-3/2}\ln\frac{X+(X^2-1)^{1/2}}{X-(X^2-1)^{1/2}}+\frac{4X^2-1}{(X^2-1)^{3/2}}$$
$$\times\ln\frac{3X+(X^2-1)^{1/2}}{3X-(X^2-1)^{1/2}}-\frac{4X}{X^2-1}\ln\frac{9X^2}{1+8X^2},\quad\text{for } X\geq 1\ . \tag{D7b}$$

The function $\bar{E}(X)/X$ is plotted in Fig. 2.

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