Side-Jump Mechanism for the Hall Effect of Ferromagnets*

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The center of mass of a wave packet undergoes a discontinuous and finite sideways displacement on scattering by a central potential, in the presence of spin-orbit interaction. This is the main Hall-effect mechanism ($\rho_H \propto \rho^2$) for Fe, Ni, and their alloys above 100 K, while asymmetric scattering dominates below 100 K. Displacement Δy per actual collision is calculated by partial waves. In the case of Born expansion, the leading term of Δy or ρ_H/ρ^2 is of zero order in the scattering potential. The magnitude is predicted correctly ($\Delta y \approx 10^{-10}$–$10^{-11}$ m) when using the effective spin-orbit Hamiltonian derived by Fivaz from spin-orbit interband mixing. The calculation of ρ_H is extended to arbitrary $\omega_c\tau$ for compensated and uncompensated metals. Other nonclassical physical mechanisms proposed by Karplus and Luttinger and by Doniach and by Fivaz are spurious for the dc Hall effect.

I. INTRODUCTION

The various contributions to the Hall effect of ferromagnets can be classified as follows:

(a) Terms caused by the magnetic Lorentz force and dependent on the field **B** (ordinary Hall effect).

(b) Terms dependent on the orientation of the magnetization **M** (extraordinary Hall effect), and which can be explained by a classical Boltzmann equation (asymmetric scattering[1–4]). They are caused by spin-orbit interaction, and are important[4] at low temperatures.

(c) Terms similar to (b), but which require new terms in the Boltzmann equation. These nonclassical terms may be expected to become important[5,6] as soon as the dimensionless quantity $\hbar/\epsilon_F\tau$ is not very small, where ϵ_F is the Fermi energy and τ the electron relaxation time. This corresponds to high temperatures or concentrated alloys (small τ). The Hall resistivity ρ_H of ferromagnets seems to be more sensitive to these nonclassical effects than the (larger) Ohmic resistivity ρ. Kohler's rule, valid[4] for the asymmetric scattering, fails in the nonclassical case.

We show on Fig. 1 the data of Jellinghaus and DeAndres[7] for various dilute Fe alloys at room temperature. Data points for varying impurity concentrations are located on the same line $R_s \propto \rho^n$, where R_s is defined by the usual equation $\rho_H = R_0 B_z + R_s M_z$ for the Hall resistivity ρ_H, and where $n = 2.08$ over almost two decades. As shown by Kooi and by Jan,[7] the same line with the same slope $n \approx 2$ also holds for varying phonon concentration in pure iron. On the other hand, any classical asymmetric scattering theory[1–4] would predict $n = 1$ for impurity scattering in dilute alloys. Thus, the room-temperature data of Fig. 1 constitute a powerful objection against a theory based only on classical asymmetric scattering.

Existing nonclassical theories[6,8,9] are successful in predicting $n \approx 2$, but are usually very complicated. There still is a need for a relatively simple formalism which might more readily be applied to the explanation of various specific experiments, and which would provide an intuitive picture of the physical mechanisms responsible for the Hall effect of ferromagnets around room temperature. An important step ("physical bands") was made by Adams and Blount,[10] and even further by Doniach[11] and Fivaz.[12] Unfortunately, as we will show later, the physical mechanism ("dipole driving term") proposed by the last two authors is spurious.

We will propose a different mechanism,[13] namely, the "side jump" Δy which an electron undergoes at every scattering by impurities or phonons, and which leads to $n = 2$. The value of the displacement ($\Delta y \approx 10^{-11}$ m) may be calculated easily by considering the motion and scattering of a wave packet.

The present paper is dedicated to the proposition that the complexity of the problem of the Hall effect of ferromagnets is more apparent than real, and results mostly from the excessive use of momentum space and from not using "physical bands" [Eqs. (22) below].

II. SIDE JUMP Δy

According to quantum mechanics, a free electron (represented by a wave packet) moves on the average with a constant velocity,[13a] along a straight line (Fig. 2). Assume then the electron to be scattered at $t \approx 0$ by a central potential. Again, and for the same reasons, the average electron trajectory after scattering ($t \gg 0$) will be a straight line (Fig. 2). In the presence of spin-orbit interaction, the symmetry of the problem is low, and there is no reason why the two straight lines should coincide. Thus two new effects are expected: First, the two lines form an angle δ related to asymmetric scattering [Fig. 2(a)]. Secondly, the two lines do not meet at the center of the scattering potential; there is a small and abrupt side jump Δy [Fig. 2(b)]. We will find later $\Delta y \approx 10^{-11}$ m for band electrons.

Of course, the side jump is not instantaneous, but the details of electron motion inside the scattering potential are irrelevant for our purpose.

The side jump Δy is of no consequence in the case of conventional experiments where free particles are scat-

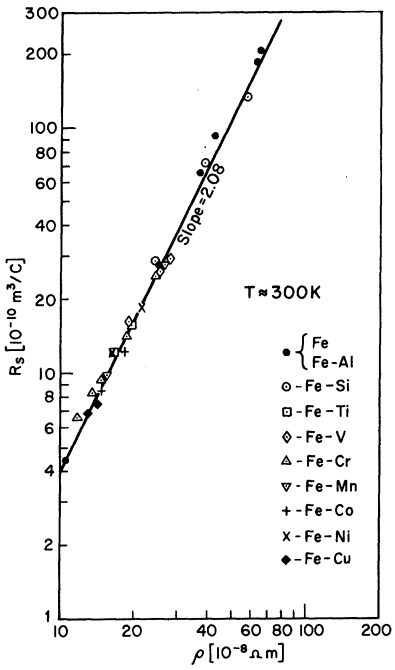

FIG. 1. Extraordinary Hall coefficient R_s of several Fe dilute alloys at room temperature, according to Jellinghaus and De Andres. Alloys of concentration larger than 0.05 have been excluded.

tered by atoms or nuclei. This is understandable, since the particle detectors are usually located at several cm or m from the target, distances much larger than Δy. But Δy is more important in ferromagnetic metals and alloys, where the mean free path Λ_e of a conduction electron after a collision may be as small as 10^{-9}–10^{-8} m.

While asymmetric scattering arises from the collision term of the classical Boltzmann equation, the side jump Δy is nonclassical. The physical nature of Δy is easily understood in terms of localized electrons or of wave packets, but not in terms of plane waves or of the momentum representation. Δy exists because the impurity distorts the wave function locally, and creates a local current density.

Since, in a disordered dilute alloy, the location of impurities is almost random, outgoing wavelets scattered by different impurities have almost no definite phase relationship. As a result, it is sufficient to consider the scattering of a wave packet by one impurity at a time, and to neglect interference phenomena involving two or more impurities. This is especially true when using the first Born approximation.

Note that the jump is not necessarily transverse to the incident $\mathbf{k}$. A longitudinal component Δx exists too, but it does not influence the Hall effect directly, and exists even in the absence of spin-orbit interaction. It is related[14] to the lifetime Δt of virtual bound states. As in the case of Δy, the nature of Δt is easily understood in terms of wave packets, but not in terms of plane waves.

III. SCATTERING OF PLANE WAVES

The scattering of a free-electron plane wave e^{ikx} by a short-range square-well potential has been solved by Smit[1] in the presence of spin-orbit interaction. We summarize here his results, as a preliminary step to the problem of wave packets.

The Hamiltonian is $H=H_v+H_{so}$, where

$$H_v=-(\hbar^2/2m)\nabla^2+V(r),$$

$$H_{so}=(1/2m^2c^2)(r^{-1}\partial V/\partial r)S_zL_z, \tag{1}$$

$$V(r)=0, \qquad r>R$$

$$V(r)=V_0, \qquad r<R.$$

Keeping only s- and p-wave phase shifts, the wave function may be written as $\psi_k=\psi_k{}^v+\psi_k{}^{so}$

$$\psi_k{}^v=\exp(i\mathbf{k}\cdot\mathbf{x})+\sum_{l=0}^{l=1} b_l h_l(kr)P_l(\cos\hat{\theta}), \qquad r>R$$

$$\psi_k{}^v=\sum_{l=0}^{l=1} a_l j_l(k_1 r)P_l(\cos\hat{\theta}), \qquad r<R$$

$$(\hbar^2/2m)(k^2-k_1{}^2)=V_0,$$

$$\psi_k{}^{so}=\sum_{l=0}^{l=1} q_l h_l(kr)\sin\hat{\varphi}(d/d\hat{\theta})P_l(\cos\hat{\theta}), \qquad r>R$$

$$\psi_k{}^{so}=\sum_{l=0}^{l=1} p_l j_l(kr)\sin\hat{\varphi}(d/d\hat{\theta})P_l(\cos\hat{\theta}), \qquad r<R$$

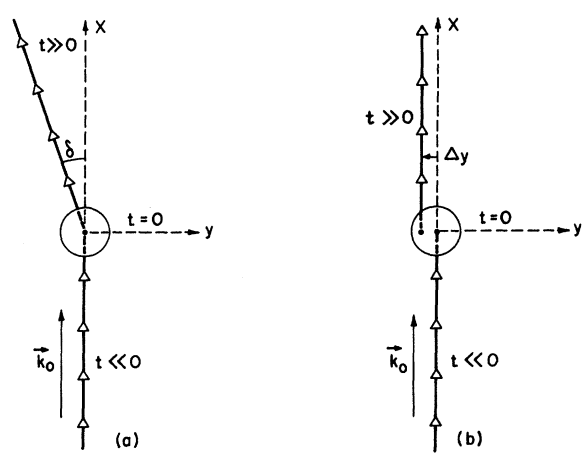

FIG. 2. Average motion of an electron before and after scattering by a central potential, in the presence of spin-orbit interaction. The spin is $\mathbf{S}\parallel z$. The incident direction is $\mathbf{k}_0\parallel x$. (a) Asymmetric scattering. (b) Side jump Δy. Actually, these two effects are usually superposed.

where[15]

$$h_0(kr)=-\frac{ie^{ikr}}{kr},$$

$$h_1(kr)=-e^{ikr}[(i/k^2r^2)+(1/kr)].$$

The polar axis $\hat{x}$ is chosen parallel to the incident direction $\mathbf{k}$. The $\hat{z}$ direction $\hat{\theta}=\frac{1}{2}\pi$, $\hat{\varphi}=0$ is parallel to $\mathbf{S}$ and $\mathbf{S}\perp\mathbf{k}$.

We have calculated the following exact expressions, to be used later:

$$b_0=\frac{(\sin\beta\cos\alpha/\beta)-(\sin\alpha\cos\beta/\alpha)}{(i\cos\alpha/\beta)+(\sin\alpha/\alpha)}e^{-i\beta}, \tag{2}$$

$$q_1=-i\left(\frac{\hbar}{2mcR}\right)^2(\alpha^2-\beta^2)\left(\frac{3}{\beta}\right)e^{-2i\beta}$$
$$\times\left(\frac{-\cos\alpha/\alpha+\sin\alpha/\alpha^2}{-i\cos\alpha/\alpha+i\sin\alpha/\alpha^2-\sin\alpha(i/\beta^2+1/\beta)}\right)^2, \tag{3}$$

where

$$\alpha=k_1R,\qquad \beta=kR.$$

IV. SCATTERING OF WAVE PACKET

A scattering state[14] for the case of an incident Gaussian wave packet of average wave vector $\mathbf{k}_0$ may be written

$$\psi=\sum_k c_k\psi_k.$$

Assuming $x\parallel\mathbf{k}_0$, $\mathbf{S}\parallel z\perp\mathbf{k}_0$, the new system x, y, z is fixed and related to $\mathbf{S}$ and $\mathbf{k}_0$, while the old system $\hat{x}$, $\hat{y}$, $\hat{z}$ is variable and related to the individual $\mathbf{k}$. The wave packet is assumed to have been launched at a time $t_0\ll 0$, at the location $x_0=v_0t_0\ll 0$, $y_0=z_0=0$. Scattering takes place at $t\approx 0$. Then, writing $\psi=\psi_v+\psi_{so}$, setting Ω for the volume of crystal, and assuming

$$c_k=[\pi^{3/4}8^{1/2}/\Omega(\Delta k)^{3/2}]$$
$$\times\exp[-i\omega_k(t-t_0)-\tfrac{1}{2}(\mathbf{k}-\mathbf{k}_0)^2/(\Delta k)^2-i\mathbf{k}\cdot\mathbf{x}_0],$$

we obtain

$$\psi_v=\sum_k c_k[\exp(i\mathbf{k}\cdot\mathbf{x})+b_0h_0(kr)+b_1h_1(kr)(\mathbf{k}\cdot\mathbf{x}/kr)],\qquad r>R$$

$$\psi_{so}=\sum_k c_k[-q_1h_1(kr)(\mathbf{k}\times\mathbf{x})\cdot\mathbf{S}/\tfrac{1}{2}\hbar kr],\qquad r>R.$$

The last term of ψ_v may be neglected if k_0R is small. Then we expand ω_k, $(\mathbf{k}-\mathbf{k}_0)^2$, $kr-k_xx_0$, $\ln(b_0/k)$, d^3k, and $\ln(q_1/k)$ in powers of $(k-k_0)$, k_y, and k_z, keeping terms up to second order in order to take into account packet dispersion correctly. After integration, if ψ_0 represents the unscattered wave packet, and if $t\gg 0$,

$$\psi_v=\psi_0+\Delta\psi_v,$$

$$\psi_0=\frac{\pi^{3/4}F^3}{(2\pi)^{3/2}(\Delta k)^{3/2}}\exp\left(-\frac{[(\mathbf{x}-\mathbf{x}_0)-(t-t_0)\mathbf{v}_0]^2}{2(\Delta x)^2}+i\frac{[(\mathbf{x}-\mathbf{x}_0)-\mathbf{v}_0(t-t_0)]^2(\hbar/m)(t-t_0)(\Delta k)^2}{2(\Delta x)^2}\right.$$
$$\left.+i\mathbf{k}_0\cdot(\mathbf{x}-\mathbf{x}_0)-i\omega_0(t-t_0)\right), \tag{4}$$

$$\Delta\psi_v=\frac{(\Delta k)^{1/2}}{\pi^{3/4}2^{1/2}}\frac{-iF}{k_0r[1-(\Delta k)^2ix_0/k_0]}\exp\left(\frac{\{-[A_1+r-x_0-v_0(t-t_0)]^2+B_1^2+2iB_1[A_1+r-x_0-v_0(t-t_0)]\}}{2(\Delta x)^2}\right.$$
$$\left.\times[1-2B_2+2iA_2-i(\hbar/m)(t-t_0)](\Delta k)^2+iA_0+B_0+ik_0(r-x_0)-i\omega_0(t-t_0)\right), \tag{5}$$

$$\psi_{so}=\frac{(\Delta k)^{1/2}}{\pi^{3/4}2^{1/2}}\frac{yF}{k_0r^2[1-(\Delta k)^2ix_0/k_0]}\exp\left(\frac{\{-[C_1+r-x_0-v_0(t-t_0)]^2+D_1^2+2iD_1[C_1+r-x_0-v_0(t-t_0)]\}}{2(\Delta x)^2}\right.$$
$$\left.\times[1-2D_2+2iC_2-i(\hbar/m)(t-t_0)](\Delta k)^2+iC_0+D_0+ik_0(r-x_0)-i\omega_0(t-t_0)\right), \tag{6}$$

where

$$\omega_0=\omega(k_0),\qquad F=\frac{2^{1/2}\Delta k}{[1+i(\Delta k)^2(\hbar/m)(t-t_0)]^{1/2}}, \tag{7}$$

$$b_0(k)/k\approx(1/k_0)\exp[iA_0+iA_1(k-k_0)+iA_2(k-k_0)^2+B_0+B_1(k-k_0)+B_2(k-k_0)^2],$$

$$q_1(k)/k\approx(1/k_0)\exp[iC_0+iC_1(k-k_0)+iC_2(k-k_0)^2+D_0+D_1(k-k_0)+D_2(k-k_0)^2],$$

$$\Delta x(t)=\frac{\{1+[(\hbar/m)(t-t_0)(\Delta k)^2]^2\}^{1/2}}{\Delta k}, \tag{8}$$

V. MOTION OF CENTER OF MASS AND SIDE JUMP Δy

The average y coordinate for the wave packet before scattering ($t \ll 0$) is zero, since $\psi \approx \psi_0$. After wave-packet scattering ($t \gg 0$), it is found from Eqs. (4)–(8). We keep only the terms of first order in spin-orbit perturbation and lowest order in Δk. The limits $\Delta x \Delta k = 1$ and $\Delta x \Delta k \gg 1$ give the same result

$$\langle y \rangle = \langle y \rangle_{v-\mathrm{so}} + \langle y \rangle_{0-\mathrm{so}}, \tag{9}$$

$$\langle y \rangle_{v-\mathrm{so}} \approx \{8/3k_0^2[\Delta x(t=0)]^2\}\{v_0 t \,\mathrm{Im}[b_0^*(k_0)q_1(k_0)] - \tfrac{1}{2}(D_1 - B_1)\,\mathrm{Re}[b_0^*(k_0)q_1(k_0)]\}, \tag{10}$$

$$\langle y \rangle_{0-\mathrm{so}} \approx -\sqrt{2}\,\mathrm{Re}q_1(k_0)/k_0^3[\Delta x(t=0)]^2. \tag{11}$$

To zero order in spin-orbit perturbation, the total scattering probability is, with similar approximations,

$$P \approx \iiint |\Delta\psi_v|^2 d^3x \approx 4\,|b_0(k_0)|^2/k_0^2[\Delta x(t=0)]^2. \tag{12}$$

The average y coordinate of those electrons which have actually been scattered is therefore

$$\langle y \rangle_{\mathrm{sc}} = \langle y \rangle / P. \tag{13}$$

As expected, Eqs. (2)–(3) and (9)–(13) show that $\langle y \rangle_{\mathrm{sc}}$ varies in a uniform and linear manner with increasing time after scattering ($t \gg 0$). The time-dependent term [first term of Eq. (10)] corresponds to an electron having acquired average velocity and momentum towards left or right, on scattering. It is caused by "asymmetric scattering" [Fig. 2(a)], i.e., a left-right symmetry of the differential cross section,[1–4] and is consistent with Eq. (19) of Smit.[1]

On the other hand, the time-independent part of $\langle y \rangle_{\mathrm{sc}}$ after collision is equivalent to a discontinuous finite lateral jump performed by the electron [Fig. 2(b)] on scattering. From Eqs. (9)–(13)

$$\Delta y = \lim_{t \to +0} \langle y \rangle_{\mathrm{sc}} = \frac{-\tfrac{1}{3}(D_1 - B_1)\,\mathrm{Re}[b_0^*(k_0)q_1(k_0)] - \mathrm{Re}[q_1(k_0)/2^{3/2}k_0]}{|b_0(k_0)|^2}, \tag{14}$$

where b_0 and q_1 are given by Eqs. (2) and (3).

VI. BORN EXPANSION

If the potential strength V_0 is assumed weak, we can expand b_0 and q_1 [Eqs. (2) and (3)] in powers of $V_0 = -(\hbar^2/2mR^2)(\alpha^2 - \beta^2)$.

To first order:

$$b_0 = i(\beta - \sin\beta\cos\beta)(\alpha^2 - \beta^2)/2\beta^2. \tag{15}$$

To second order:

$$q_1 = \left(\frac{\hbar}{2mcR}\right)^2 \frac{3}{\beta}\left(-\cos\beta + \frac{\sin\beta}{\beta}\right)^2 (\alpha^2 - \beta^2)\left(i - \frac{\alpha^2 - \beta^2}{\beta^3}[\beta^2 + i\beta + (i\cos\beta - \sin\beta)(-\beta^2\cos\beta + \beta\sin\beta + \sin\beta)]\right). \tag{16}$$

Then Eq. (14) gives, neglecting terms of order higher than zero,

$$\Delta y = -\left(\frac{\hbar}{2mcR}\right)^2 \frac{[-\cos\beta + (\sin\beta)/\beta]^2}{1 - [(\sin\beta)/\beta]\cos\beta}\left[2(D_1 - B_1) - \left(\frac{2^{1/2}3}{k_0}\right)\frac{1 + \sin\beta\cos\beta - \sin^2\beta(\beta^{-1} + \beta^{-2})}{1 - [(\sin\beta)/\beta]\cos\beta}\right], \tag{17}$$

where D_1 and B_1 are given by Eqs. (7) and (8),

$$B_1 = k_0 \frac{d(\ln|b_0(k_0)/k_0|)}{dk_0},$$

$$D_1 = k_0 \frac{d(\ln|q_1(k_0)/k_0|)}{dk_0}.$$

Hence, to lowest order, Δy is independent of potential strength V_0.

If we add to the assumption $\alpha^2 - \beta^2 \ll 1$ (Born) the assumption $\alpha \ll 1$, $\beta \ll 1$ (short-range limit), the second term of Eq. (9) or Eq. (17) vanishes and we obtain finally

$$\Delta y = \tfrac{1}{6}k_0 \lambda\!\!\!^{-}_c{}^2 \approx 3 \times 10^{-16}\ \mathrm{m}, \tag{18}$$

where $\lambda\!\!\!^{-}_c = \hbar/mc$ is the "rationalized" Compton wavelength, and where $k_0 \approx 10^{10}\ \mathrm{m}^{-1}$ at the Fermi level. Note that this Δy value is independent of both depth V_0 and radius R of the scattering potential well. Note also that Δy arises already from the first-order Born wave. This, of course, was not true for Smit asymmetric scattering. In physical terms, the phase of the first-order wave has a left-right asymmetry, even for incident plane wave. This phase asymmetry becomes important when forming and scattering wave packets, and results in the existence of the side jump Δy.

This small free-electron value will now be shown to be enhanced by $\approx 10^4$ through band effects.

VII. EFFECTIVE SPIN-ORBIT INTERACTION FOR BAND ELECTRONS

We consider band electrons in a dilute alloy. Added to the spin-orbit Hamiltonian created by the impurity scattering potential $V(r)$ [Eq. (1)], we now have[1] the periodic spin-orbit interaction associated with the lattice potential $U(\mathbf{x})$,

$$H_{so}=-(1/2m^2c^2)\mathbf{S}\cdot(\nabla U(\mathbf{x})\times\mathbf{p}). \tag{19}$$

The interaction of Eq. (1) can be actually neglected as compared to Eq. (19) whenever the impurity perturbation $V(r)$ is not too strong.

Both Eq. (19) and the Hamiltonian $-e\mathbf{E}\mathbf{x}$ of the applied electric field have d-d or p-d interband matrix elements, and it is best[10,12] to eliminate this interband mixing by diagonalization. The new, redefined, bands $E(\mathbf{k})$ are called "physical bands."[10]

Even after interband diagonalization, the position operator $\mathbf{x}$ may be written $\mathbf{x}=\mathbf{R}+\mathbf{q}$. The first part $\mathbf{R}$ is the usual Wannier coordinate. The second part $\mathbf{q}$ represents[16,10,12] a periodic dielectric polarization inside each lattice cell,

$$\mathbf{R}=(\hbar/i)\frac{\partial}{\partial\mathbf{k}}, \tag{20}$$

$$\langle\mathbf{q}\rangle_k\approx-\tfrac{1}{2}\mathbf{k}\times\mathrm{D},\qquad \mathbf{D}\parallel\mathbf{S}. \tag{21}$$

The interaction of the electric dipole $e\langle\mathbf{q}\rangle_k$ of a Bloch wave with the field $\mathbf{E}$ causes a k-dependent energy shift, and the Fermi surface has no center of inversion anymore (asymmetric physical bands[1,12]),

$$\epsilon(\mathbf{k})\approx(\hbar^2/2m^*)k^2,\qquad E(\mathbf{k})=\epsilon(\mathbf{k})-e\mathbf{E}\langle\mathbf{q}\rangle_k, \tag{22}$$

and resembles the Fermi surface of a ferroelectric material.

As shown by Smit,[1] by Adams and Blount,[10] and in more detail by Fivaz,[12] the combined effect of the dipole [Eq. (21)] and of the impurity potential $V(r)$ is to yield an effective spin-orbit Hamiltonian,

$$H_{so}^{\mathrm{eff}}=-(\tau_q/\hbar m^*)R^{-1}(\partial V/\partial R)(\mathbf{R}\times\mathbf{p})_zS_z, \tag{23a}$$

$$\tau_q=(m^*/3\hbar^3)\sum_{m\neq n}(\chi\bar{\xi}\rho^2/\Delta E_{mn})\,|\langle m\,|\,\mathbf{q}\times\mathbf{p}\,|\,n\rangle|^2, \tag{23b}$$

similar in form to the free-electron interaction of Eq. (1) but with a coupling parameter enhanced in the ratio

$$(2m^2c^2/m^*\hbar)\tau_q\approx3.4\times10^4, \tag{24}$$

where the energy gap between $3d$ bands m and n is assumed $\Delta E_{mn}\approx0.5$ eV, the overlap integral $\chi_m\approx0.3$, the nearest-neighbor distance $\rho\approx2.5\times10^{-10}$ m, and the atomic spin-orbit parameter for iron

$$\bar{\xi}=-(\hbar^2/2m^2c^2)\langle r^{-1}(\partial U/\partial r)\rangle\approx0.1\text{ eV}.$$

Of course, the effective-mass formalism used to derive Eq. (23a) strictly applies only to slowly varying scattering potentials, an unrealistic assumption in metals, and the results have only qualitative significance. Interband elements of $V(r)$ are neglected.

Combining Eqs. (18), (23b), and (24), we obtain

$$\Delta y=(\tfrac{1}{6}k_0\lambda\!\!\!^{-}_c{}^2)(2m^2c^2/m^*\hbar)\tau_q=\tfrac{1}{3}v_0\tau_q\approx2(\tfrac{1}{9}\chi\rho^2k_0)\frac{\bar{\xi}}{\Delta E_{mn}}\approx0.8\times10^{-11}\text{ m} \tag{25}$$

for band electrons, in the limit of weak and short-ranged scattering potential.

Note that, although the effective interaction [Eq. (23a)] arises indirectly from the existence of the electric-dipole energy shift of Eq. (22), nevertheless this interaction involves explicitly only the usual part $\mathbf{R}=(\hbar/i)\partial/\partial\mathbf{k}$ of the coordinate. Thus the side jump of Eq. (25) represents a time variation of $\mathbf{R}$, not of the dielectric polarization $\mathbf{q}$. This agrees with the fact that, as we have shown above, the side jump exists even in the case of free-electron scattering, where $\mathbf{q}=0$.

VIII. TRANSPORT THEORY AT ARBITRARY $\omega_c\tau$

We assume a model with one band of holes and one band of electrons. The induction $\mathbf{B}$ is $\parallel z$. The "classical" current densities $\mathbf{j}_e$, $\mathbf{j}_h$ of electrons and holes arise from the time variation of the coordinate $\mathbf{R}=(\hbar/i)\partial/\partial\mathbf{k}$, but not including the side jumps. The nonclassical current densities, associated with the side jumps happening at every collision, are

$$\mathbf{J}_e=s_en_ee_e\Delta\mathbf{x}_e,\qquad \Delta\mathbf{x}_e=(\Delta y_e/v_ee_en_e)\mathbf{j}_e\times\boldsymbol{\alpha}$$
$$\mathbf{J}_h=s_hn_he_h\Delta\mathbf{x}_h,\qquad \Delta\mathbf{x}_h=(\Delta y_h/v_he_hn_h)\mathbf{j}_h\times\boldsymbol{\alpha} \tag{26}$$

where $\Delta\mathbf{x}_e$ and $\Delta\mathbf{x}_h$ represent vector side jumps averaged over nonequilibrium Fermi distributions, n_e and n_h are electron and hole densities per unit volume, v_e and v_h are Fermi velocities, and $\boldsymbol{\alpha}$ is a unit vector parallel to $\mathbf{M}$. The relaxation frequencies s_e, s_h associated with the dilute scattering impurities are simple scalars if we neglect asymmetric scattering.[4] Equations (26) insure that $\Delta\mathbf{x}_e$, $\Delta\mathbf{x}_h$ remain, respectively, perpendicular to $\mathbf{j}_e$, $\mathbf{j}_h$ even when these, at $B\neq0$, are not parallel to $\mathbf{E}$ anymore. The total current densities are

$$\mathbf{I}_e=\mathbf{j}_e+\mathbf{J}_e,\qquad \mathbf{I}_h=\mathbf{j}_h+\mathbf{J}_h. \tag{27}$$

In the stationary state, we write for the total momenta $\mathbf{p}_e$, $\mathbf{p}_h$ of electrons and holes per unit volume,

$$0=d\mathbf{p}_e/dt=n_ee_e\mathbf{E}+\mathbf{I}_e\times\mathbf{B}-(m_e/e_e)s_e\mathbf{j}_e$$
$$0=d\mathbf{p}_h/dt=n_he_hE+\mathbf{I}_h\times\mathbf{B}-(m_h/e_h)s_h\mathbf{j}_h, \tag{28}$$
$$e_e<0,\qquad 0=e_e+e_h.$$

The structure of the right-hand side of Eqs. (28) is based on the idea that the accelerating effect of the fields $\mathbf{E}$ and $\mathbf{B}$ and the relaxation effect of the scattering potentials are independent and simply additive, at least as far as transverse spin-dependent phenomena are concerned. For example, even in the presence of scatter-

ing, there is no anomalous "driving term" of the form $n_e e_e s_e(\mathbf{E}\times\mathbf{S})$. This is easy to prove in the case of the wave function ψ of the first Born approximation. We write

$$\psi=\sum_k a_k(t)\varphi(\mathbf{k},t),$$

where

$$\varphi(\mathbf{k},t)=\exp i[\mathbf{k}\cdot\mathbf{x}-(\hbar k^2 t/2m)+(\mathbf{k}\cdot\mathbf{E}t^2 e/2m)]$$

is[17] the wave function to first order in $\mathbf{E}$ in the absence of scattering. Then we find, for $\mathbf{k}\neq\mathbf{k}_0$, in presence of scattering,

$$|a_k(t)|^2\approx\hbar^{-1}|\langle k|V(r)+H_{\rm so}|k_0\rangle|^2 \times f((\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{E},\ t,\ \omega(k)-\omega(k_0)),$$

where $\mathbf{k}_0$ is the incident wave vector, and where $\mathbf{S}$ appears only in $H_{\rm so}$. Thus the scattering probability is symmetric with respect to the $(\mathbf{E},\mathbf{S})$ plane, at least for the leading term of the Born expansion.

Similarly, the "inside" current $\mathbf{J}$ contributes to the Lorentz force in the same way as the "outside" current $\mathbf{j}$ does, in Eqs. (28). We have found that a failure to take into account this $\mathbf{J}\times\mathbf{B}$ term may actually result in a negative value for the resistivity $\rho=E_x/(I_x^e+I_x^h)$ at $B\neq0$, in contradiction with the second law of thermodynamics.

Combining Eqs. (26)–(28), we obtain

$$0=n_e e_e\mathbf{E}+\mathbf{I}_e\times\mathbf{B}-(m_e/e_e)[t^e]\mathbf{I}_e,$$
$$0=n_h e_h\mathbf{E}+\mathbf{I}_h\times\mathbf{B}-(m_h/e_h)[t^h]\mathbf{I}_h, \tag{29}$$

where $[t^e]$ and $[t^h]$ are linear operators with antisymmetric off-diagonal elements,

$$t_{xx}^e=t_{yy}^e=\frac{s_e}{1+(\alpha_z\Delta y_e/\Lambda_e)^2},$$
$$t_{xy}^e=-t_{yx}^e=-\frac{s_e\alpha_z(\Delta y_e/\Lambda_e)}{1+(\alpha_z\Delta y_e/\Lambda_e)^2}, \tag{30}$$

and where $\Lambda_e=v_e/s_e$ and $\Lambda_h=v_h/s_h$ are mean free paths.

Equations (29) are isomorphous to Eqs. (4) of our work[4] on asymmetric scattering, and thus have a similar formal solution,

$$\tan\Phi_H=E_y/E_x$$
$$=-\left[u_e\left(\frac{T_{xy}^e-B_z}{T_{xx}^e}\right)+u_h\left(\frac{T_{xy}^h-B_z}{T_{xx}^h}\right)\right]\Big/(u_e+u_h), \tag{31}$$

$$\rho=E_x/(I_x^e+I_x^h)=[(u_e+u_h)(1+\tan^2\Phi_H)]^{-1}>0, \tag{32}$$

$$\rho_H=E_y/(I_x^e+I_x^h)=\tan\Phi_H/(u_e+u_h)(1+\tan^2\Phi_H), \tag{33}$$

where the boundary condition is $I_y^e+I_y^h=0$, where the tensors

$$[T^e]=(m_e/e_e)[t^e],\qquad [T^h]=(m_h/e_h)[t^h]$$

have the dimension of an inverse mobility (induction), and where

$$u_e=\frac{e_e n_e/T_{xx}^e}{1+[(T_{xy}^e-B_z)/T_{xx}^e]^2}, \tag{34}$$

$$u_h=\frac{e_h n_h/T_{xx}^h}{1+[(T_{xy}^h-B_z)/T_{xx}^h]^2} \tag{35}$$

are field-dependent conductivities.

In the low-field limit $|B/T_{xx}^e|\ll1$, $|B/T_{xx}^h|\ll1$, Eqs. (31)–(33) become, if $|T_{xy}^e/T_{xx}^e|\ll1$, $|T_{xy}^h/T_{xx}^h|\ll1$,

$$\tan\Phi_H\approx\frac{(\sigma_e B_z/S_e)+(\sigma_h B_z/S_h)}{\sigma_e+\sigma_h}+\frac{(\sigma_e\Delta y_e/\Lambda_e)+(\sigma_h\Delta y_h/\Lambda_h)}{\sigma_e+\sigma_h}\alpha_z, \tag{36}$$

$$\rho\approx(\sigma_e+\sigma_h)^{-1},\qquad \sigma_e=n_e e_e/T_{xx}^e>0,\qquad \sigma_h=n_h e_h/T_{xx}^h>0, \tag{37}$$

$$\rho_H\approx\frac{(\sigma_e^2/n_e e_e)+(\sigma_h^2/n_h e_h)}{(\sigma_e+\sigma_h)^2}B_z+\frac{(\sigma_e\Delta y_e/\Lambda_e)+(\sigma_h\Delta y_h/\Lambda_h)}{(\sigma_e+\sigma_h)^2}\alpha_z. \tag{38}$$

Note that Eq. (38) is of the usual form $\rho_H=R_0B_z+R_sM_z$. If we introduce the "extraordinary Hall conductivity" $\gamma_{\rm Hs}=R_sM_z/\rho^2$, then, at $|B/T_{xx}^e|\ll1$, $|B/T_{xx}^h|\ll1$,

$$\gamma_{\rm Hs}\approx\alpha_z[(n_e e_e^2\Delta y_e/\hbar k_e)+(n_h e_h^2\Delta y_h/\hbar k_h)]. \tag{39}$$

Thus $\gamma_{\rm Hs}$ is a constant independent of impurity concentration, and therefore $R_s\propto\rho^2$ for varying impurity concentration, in agreement with room-temperature data[7] discussed in the Introduction. Moreover, Δy, and hence $\gamma_{\rm Hs}$, is predicted to be roughly independent of the radius, strength, and sign of the central impurity potential, in agreement with these data which show the relation between R_s and ρ to be the same for many different types of impurity in iron and even for phonon scattering in pure iron.

As shown by Eqs. (38) and (39), the relative contribution of each band to ρ_H or $\gamma_{\rm Hs}$ is actually independent of its relative mobility. This paradoxical situation is possible only because of the nonclassical side-jump mechanism, and explains how the low-mobility $3d$ electrons can possibly dominate the extraordinary Hall effect of ferromagnets.

A one-band estimate of $\gamma_{\rm Hs}$ gives $\gamma_{\rm Hs}\approx ne^2\Delta y/\hbar k\approx 1.0\times10^4\ \Omega^{-1}\ {\rm m}^{-1}$, assuming $n=5\times10^{28}\ e/{\rm m}^3$, $\Delta y=0.8\times10^{-11}$ m, $k=10^{10}\ {\rm m}^{-1}$. This is within an order of magnitude of the experimental value[7] $\gamma_{\rm Hs}=10\times10^4$ for iron dilute alloys. This is a satisfactory agreement in view of the rather uncertain numerical value of the multiplication factor of Eq. (24). Nevertheless, this result suggests that $\Delta y\approx10^{-10}$ m would be experimen-

tally more correct than the predicted value $\approx 10^{-11}$ m. Note also the very simple and revealing form for one band, at $|B/T_{xx}| \ll 1$,

$$\tan\Phi_H = \alpha_z \Delta y/\Lambda. \tag{40}$$

A more accurate form of Eq. (37) would add to the right-hand side small (positive or negative) linear magnetoresistance terms such that $\Delta\rho/\rho \approx (B_z/T_{xx}^e)(\Delta y_e/\Lambda_e)\alpha_z$, and also the more usual quadratic positive terms such that $\Delta\rho/\rho \approx (B_z/T_{xx}^e)^2$.

In the high-field limit

$$|B_z/T_{xx}^e| \gg 1, \qquad |B_z/T_{xx}^h| \gg 1,$$
$$\Lambda_e = \text{const}, \quad \Lambda_h = \text{const},$$

Eqs. (31)–(33) become, if the metal is uncompensated ($n_e \neq n_h$),

$$\tan\Phi_H = B_z(n_e e_e + n_h e_h)/(n_e e_e T_{xx}^e + n_h e_h T_{xx}^h), \tag{41}$$

$$\rho = (n_e e_e T_{xx}^e + n_h e_h T_{xx}^h)/(n_e e_e + n_h e_h)^2 > 0, \tag{42}$$

$$\rho_H = B_z/(n_e e_e + n_h e_h). \tag{43}$$

To first order, these formulas are independent of Δy_e and Δy_h. Thus the side jumps are of no influence on the high-field properties of an uncompensated magnetic metal. On the other hand, if the metal is compensated ($n_e = n_h$), Eqs. (31)–(33) become[4] at high fields

$$\tan\Phi_H = -\frac{|T_{xx}^e|(\Delta y_e/\Lambda_e) + |T_{xx}^h|(\Delta y_h/\Lambda_h)}{|T_{xx}^e| + |T_{xx}^h|} a_z, \tag{44}$$

$$\rho = B_z^2/n_e|e_e|(|T_{xx}^e| + |T_{xx}^h|) > 0, \tag{45}$$

$$\rho_H = -\frac{|T_{xx}^e|(\Delta y_e/\Lambda_e) + |T_{xx}^h|(\Delta y_h/\Lambda_h)}{n_e|e_e|(|T_{xx}^e| + |T_{xx}^h|)^2} \alpha_z B_z^2. \tag{46}$$

Since T_{xx}^e, T_{xx}^h, $\Delta y_e/\Lambda_e$, and $\Delta y_h/\Lambda_h$ are proportional to the impurity concentration, and since $\alpha_z \propto M_z$, Eqs. (44)–(46) imply

$$\tan\Phi_H \propto M_z^1 B_z^0 c^1, \tag{47}$$

$$\rho \propto M_z^0 B_z^2/c, \tag{48}$$

$$\rho_H \propto M_z^1 B_z^2 c^0. \tag{49}$$

As in the case of asymmetric scattering,[4] we obtain a finite asymptotic Hall angle at high fields, here of order $-\Delta y_e/\Lambda_e$, and a Hall resistivity proportional to $M_z B_z^2$. However, while the high-field asymptotic behavior is not too difficult to reach experimentally[4] in iron for the case of asymmetric scattering, it would actually require $|(B/T_{xx}^e)(\Delta y_e/\Lambda_e)| \gg 1$ in the present case. This would mean fields of the order $B \approx 10^4$ T.

Note [Eqs. (41)–(49)] that the relation $\rho_H \propto \rho^2$ or $R_s \propto \rho^2$, valid at low fields, is predicted to break down at high fields.

Note also that Kohler's rule, valid[4] for asymmetric scattering, breaks down for the nonclassical side-jump model. This results from the fact that $|(\omega_c)_e \tau_e| = |B/S_e|$ and $\Delta y_e/\Lambda_e$ are two independent parameters, appearing in our formulas. Thus the high-field limit has to be defined carefully, stating whether Λ_e is to be kept constant or not.

IX. DISCUSSION OF OTHER POSSIBLE NONCLASSICAL MECHANISMS

The theory by Karplus and Luttinger[18] was the first one to predict $R_s \propto \rho^2$ and a correct order of magnitude for R_s. It is based on the transverse current arising from the time variation of the periodic electric dipole [see our Eq. (21)]. However, as pointed out by Smit,[19] the time variation caused by the accelerating action of the applied electric field on $\mathbf{k}$ must be cancelled in the stationary state by the decelerating action of impurities and phonons.[20] Thus this polarization current cannot be the correct physical mechanism for the dc Hall effect.

More recently, Doniach[11] and also Fivaz[12] have proposed that a Fermi distribution for the asymmetric bands $E(\mathbf{k})$ arising from the dipole energy shifts [our Eq. (22)] cannot be considered to be a state of thermodynamic equilibrium. Due to the asymmetry of the density of states, any relaxation process associated with impurities or phonons would rather push the electron system towards some different distribution. Correspondingly, an anomalous "dipole driving term" would exist in the Boltzmann equation. However, the Born calculations made by these authors to illustrate that point seem actually to lead to the opposite conclusion. For example, Eqs. (12) and (13) of Doniach [or Eqs. (55) and (56) of Fivaz] show that the electron system will indeed relax to the equilibrium Fermi distribution associated with $E(\mathbf{k})$ [rather than to the one associated with $\epsilon(\mathbf{k})$]. And, of course, the total current arising from the $E(\mathbf{k})$ equilibrium Fermi distribution and from the $\partial E/\partial \mathbf{k}$ velocity operator vanishes[19,20] even when the band is asymmetric. The "dipole driving term" disappears when the correct energy E is used (instead of ϵ) as an independent variable. Actually, the arguments of Doniach and Fivaz, if they were correct, would lead to the existence of persistent currents in any impure metal without inversion and time-reversal symmetry (impure and magnetic ferroelectric), at $\mathbf{E}=0$.

This does not mean that an anomalous driving term[10] cannot exist in the Boltzmann equation, or in our drift equations (28), where it would be of the form $n_e e_e s_e(\mathbf{E}\times\mathbf{S})$ or $n_h e_h s_h(\mathbf{E}\times\mathbf{S})$. However, as we have shown earlier, it would appear only in the higher orders of Born expansion.

We have also found that the addition of such $n_e e_e s_e(\mathbf{E}\times\mathbf{S})$ or $n_h e_h s_h(\mathbf{E}\times\mathbf{S})$ anomalous driving term to Eq. (28) may lead to $\rho < 0$ at $B \neq 0$, in contradiction

with the second law of thermodynamics, unless the Lorentz-force term is correspondingly modified by the addition of a $s_e(\mathbf{I}_e\times\mathbf{B})\times\mathbf{S}$ or $s_h(\mathbf{I}_h\times\mathbf{B})\times\mathbf{S}$ term. This is to be expected, since $\mathbf{E}$ and $\mathbf{B}$ transform into each other under a Lorentz transformation.

After correction, it is easy to show that the formulas then obtained for ρ, ρ_H, and $\tan\Phi_H$ are identical to our Eqs. (31)–(33), so that there is no direct way of experimentally differentiating between anomalous driving term and side-jump mechanism, except by the Born order. However, the physical natures of the two mechanisms are quite different.

X. CONCLUSIONS AND FINAL REMARKS

The side jump Δy is an important nonclassical mechanism for the Hall effect of ferromagnets at room temperature. The side jump per collision is of order 10^{-10}–10^{-11} m and, to lowest order, is independent of the range and strength of the scattering potential. Of course, classical asymmetric scattering is also important in metals and dilute alloys at low temperatures. Side-jump theory is developed for a two-band model, at arbitrary values of $\omega_c\tau$.

While the constant Δy derived from the lowest order of Born expansion [Eq. (18)] may be sufficient in the case of phonon scattering, one would expect higher-order terms to be important [Eq. (14)] for impurity scattering. This should especially happen for Cr, V impurities in iron, which form $3d$ virtually bound states at the Fermi level.[21] However, the data of Fig. 1 hardly show any difference between the various impurities, and are consistent with the lowest-order theory (constant Δy). This puzzling fact suggests that the effect of higher-order Born terms of Δy turns out to be nearly cancelled by the effect of the "anomalous driving term" mentioned above.

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