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Theory of Optical Parametric Amplification from a Focused Gaussian Beam*

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The theory of parametric amplification of focused Gaussian beams in a uniaxial medium in the approximation of no pump depletion is studied using the mode theory developed in previous papers. We consider a Gaussian pump beam interacting with a Gaussian signal beam both propagating perpendicular to the optic axis of the medium. We show how the gain varies with the focusing parameter of the pump, the relative focusing of the pump and signal, phase matching, and a degeneracy parameter defined as $\mu=(k_3-2k_1)/k_3$, where k_3 is the wave number of the pump beam and k_1 that of the signal beam. The conditions under which the resultant idler field may be approximated by a Gaussian beam are then studied. It is shown that when the interaction takes place in the near field of both beams, the idler field may be assumed to be Gaussian with spot size given approximately by $1/w_{02}^2=1/w_{03}^2+1/w_{01}^2$, and when the pump and signal are focused for maximum gain the idler field is approximately Gaussian with spot size given approximately by $1/w_{03}^2=1/w_{02}^2+1/w_{02}^2$.

I. INTRODUCTION

In this paper, the theory of parametric amplification is developed using the theory of optical-mode interaction in nonlinear media described in a previous paper, [1] which will hereafter be referred to as I. The problem of parametric amplification is considerably complicated by the presence of three optical fields and the three sets of beam parameters, so in order to keep the theory as simple as possible we will retain all the simplifications used previously. The interaction is considered to take place with the three beams propagating perpendicular to the optic axis of a uniaxial crystal, which for definiteness is taken to be lithium niobate. The coordinate system is set up so that the fields propagate in the z direction and each of the fields is focused in the plane $z=0$. The nonlinear medium is assumed to be an infinite slab bounded by the planes $z=-z_1$ and $z=z_1$ (the entry and exit face, respectively) embedded in a linear medium of equal refractive index so that there are no reflections at the boundaries. Absorption has been neglected but can be included by a trivial extension of the theory as indicated in I.

The following theory, although set out for the case when the pump is propagated as an extraordinary beam and the signal and idler as ordinary beams, is relevant with minor modifications to other three-field interactions.

We consider in this paper the amplification of a Gaussian signal beam by a Gaussian pump in the small-gain and no-pump-depletion approximation. This analysis will apply directly to the singly resonant oscillator in the case when the signal field is enclosed in a resonator. The actual field in the resonator, and thus the gain in this case, can be calculated by a simple self-consistent field calculation as for second-harmonic generation (SHG). [2,3]

Boyd and Ashkin, [4] and Boyd and Kleinman[5] have studied the restricted problem of the case when both the signal and idler fields are enclosed in resonators, so that the spot size of all three of the interacting fields is defined. In general, in an amplifier the idler field is created by the interaction of the pump and signal beams and it is not possible to define a unique spot size for the idler field. In the first part of this paper we develop a general theory which is independent of the form of the idler field and show how the amplification of the signal varies with the various parameters.

In the second part, the theory of the interaction of three Gaussian beams is developed under the same approximations and a comparison is made between the three-mode theory and the general theory developed in Sec. I. We then show under what conditions the general theory can be approximated by the three-mode theory. This approximation is essential in order to extend the theory into the regions of pump deletion and high gain. The traveling wave field at each frequency ω_l may be written in terms of the modes[1]

$$\mathcal{E}_{nm}^{\omega_l}(x,y,z)=\frac{\sigma(1+i\xi_l)^{(n+m)/2}}{w_{0l}(2^{n+m+1}n!m!\pi)^{1/2}(1-i\xi_l)^{(n+m+2)/2}}\times H_n\left(\frac{\sigma x\sqrt{2}}{w_l}\right)H_m\left(\frac{y\sqrt{2}}{w_l}\right)\exp\left(-\frac{\sigma^2x^2+y^2}{w_{0l}^2(1-i\xi_l)}\right), \quad (1.1)$$

where

$$\xi_l=2z/k_lw_{0l}^2\,,\quad w_l=w_{0l}(1+\xi_l^2)^{1/2},$$

and W_{0l} denotes the spot size at the focus plane $z=0$ of the field at frequency ω_l with wave vector k_1 in the z direction, and $H_n(x)$ is the Hermite polynomial of degree n. For the ordinary modes propagating in the uniaxial medium $\sigma=1$ and $\sigma=(\epsilon_z/\epsilon_x)^{1/2}$ for the extraordinary modes propagating perpendicular to the optic axis.

We also define at this point $z_{0l}=\frac{1}{2}k_l w_{0l}^2$ which is one-half the confocal parameter of the mode. The electric field due to the nmth mode is then given by

$$\vec{E}=E_{nm}^{\omega_l}(x,y,z)e^{i\omega_l t}\hat{\vec{\alpha}}=\mathcal{E}_{nm}^{\omega_l}(x,y,z)e^{-i(k_l z-\omega_l t)}\hat{\vec{\alpha}}\ , \quad (1.2)$$

where $\hat{\vec{\alpha}}$ is the unit vector in the y direction for the ordinary modes and the unit vector in the x direction for the extraordinary modes. These modes of the open resonator obey the orthonormality relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{E}_{nm}^{\omega_l}(x,y,z)\mathcal{E}_{n'm'}^{\omega_l}(x,y,z)dx\,dy=\delta_{nn'}\delta_{mm'} \quad (1.3)$$

and the completeness relation[6]

$$\sum_{m,n}\mathcal{E}_{nm}^{\omega_l}(x,y,z)\mathcal{E}_{nm}^{\omega_l}(x',y',z')$$

$$=\frac{\sigma k_l}{2\pi i(z'-z)}\exp\left(\frac{ik_l[\sigma^2(x-x')^2+(y-y')^2]}{2(z'-z)}\right), \quad (1.4)$$

which reduces to the usual form of a completeness relation in the case when $z=z'$,

$$\sum_{m,n}\mathcal{E}_{nm}^{\omega_l}(x,y,z)\mathcal{E}_{nm}^{\omega_l}(x',y',z)=\delta(x-x')\delta(y-y')\ . \quad (1.5)$$

Using these relations, the signal and idler fields which propagate as ordinary beams may be written, in general, as

$$E^{\omega_{1,2}}(x,y,z)=\sum_{m,n}A_{mn}^{\omega_{1,2}}\mathcal{E}_{mn}^{\omega_{1,2}}(x,y,z)e^{-ik_{1,2}z}, \quad (1.6a)$$

and the pump which propagates as an extraordinary beam may be written

$$E^{\omega_3}(x,y,z)=\sum_{m,n}B_{mn}^{\omega_3}\mathcal{E}_{mn}^{\omega_3}(x,y,z)e^{-ik_3 z}\ , \quad (1.6b)$$

where in the linear theory of propagation the A_{mn} and B_{mn} are the mode amplitudes, which are constants determined by the boundary conditions on some initial plane.

From Maxwell's equations, the equation governing the propagation of the electric field at frequency ω_l in the nonlinear medium is

$$\nabla\times(\nabla\times\vec{E}^{\omega_l})-\frac{\omega_l^2}{c^2}\overleftrightarrow{\epsilon}\cdot\vec{E}^{\omega_l}=\frac{4\pi\omega_l^2}{c^2}\vec{P}_{NL}^{\omega_l}\ . \quad (1.7)$$

Substituting Eqs. (1.6) into the appropriate Eq. (1.7), allowing the mode amplitudes A and B to be slowly varying functions of z, the coupled mode equations for parametric amplification in lithium niobate may be derived[1]:

$$\frac{dA_{mn}^{\omega_1}}{dz}=C_{mnrskl}^{\omega_1}A_{rs}^{\omega_2 *}B_{kl}^{\omega_3}\ , \quad (1.8a)$$

$$\frac{dA_{mn}^{\omega_2}}{dz}=C_{mnrskl}^{\omega_2}A_{rs}^{\omega_1 *}B_{kl}^{\omega_3}\ , \quad (1.8b)$$

$$\frac{dB_{mn}^{\omega_3}}{dz}=C_{mnrskl}^{\omega_3}A_{rs}^{\omega_1}A_{kl}^{\omega_2}\ . \quad (1.8c)$$

The summation convention is understood in these equations, and the coupling coefficients C^{ω} are given by

$$C_{mnrskl}^{\omega_1}=\frac{-2\pi i\omega_1^2 d_{15}}{k_1 c^2}e^{-i\Delta kz}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{E}_{mn}^{\omega_1 *}\mathcal{E}_{rs}^{\omega_2 *}\mathcal{E}_{kl}^{\omega_3}dx\,dy\ , \quad (1.9)$$

where $\Delta k=k_3-k_1-k_2$, C^{ω_2}, and C^{ω_3} are similarly defined.

It will be convenient to calculate the signal gain using the Manley-Rowe relation between the signal and idler fields which is obtained as follows.

We note from Eq. (1.9) and the similar expression for C^{ω_2} that

$$\eta_1\omega_2 C_{mnrskl}^{\omega_1}=\eta_2\omega_1 C_{rsmnkl}^{\omega_2}\ , \quad (1.10)$$

where $k_1=\eta_1\omega_1/c$, $k_2=\eta_2\omega_2/c$, η_1, and η_2 are the appropriate refractive indices of the medium at frequencies ω_1 and ω_2, respectively. Multiplying Eqs. (1.8a) and (1.8b) by $A_{mn}^{\omega_1 *}$ and $A_{mn}^{\omega_2 *}$, respectively, and then summing over m and n we have

$$\eta_1\omega_2 A_{mn}^{\omega_1 *}\frac{dA_{mn}^{\omega_1}}{dz}=\eta_2\omega_1 A_{mn}^{\omega_2 *}\frac{dA_{mn}^{\omega_2}}{dz}\ . \quad (1.11)$$

Adding the complex conjugate of this equation to it we have the Manley-Rowe relation

$$\frac{\eta_1}{\omega_1}\frac{d}{dz}\left(\sum_{m,n}|A_{mn}^{\omega_1}|^2\right)=\frac{\eta_2}{\omega_2}\frac{d}{dz}\left(\sum_{mn}|A_{mn}^{\omega_2}|^2\right)\ . \quad (1.12)$$

We assume that at the entry face of the nonlinear medium the idler field is identically zero, hence

$$A_{mn}^{\omega_2}(-z_1)\equiv 0 \ \text{ for all } m,n\ . \quad (1.13)$$

Thus, integrating Eq. (1.12) we have

$$\frac{\eta_1}{\omega_1}\sum_{m,n}\left(|A_{mn}^{\omega_1}(z_1)|^2-|A_{mn}^{\omega_1}(-z_1)|^2\right)$$

$$=\frac{\eta_2}{\omega_2}\sum_{m,n}|A_{mn}^{\omega_2}(z_1)|^2\ . \quad (1.14)$$

The signal gain G is given by

$$G=\omega_1\eta_2\sum_{m,n}|A_{mn}^{\omega_2}(z_1)|^2/\omega_2\mu_1\sum_{m,n}|A_{mn}^{\omega_1}(z_1)|^2\ . \quad (1.15)$$

II. SIGNAL GAIN, IDLER FIELD UNSPECIFIED

We consider first the gain of a Gaussian signal beam interacting with a Gaussian pump when the pump is undepleted and the signal gain in small. Thus the pump field has only one nonzero amplitude B_{00} and this is constant.

The lowest-order (Gaussian) signal mode amplitude may be written as

$$A_{00}^{\omega_1}(z)=A_{00}^{\omega_1}+\Delta A^{\omega_1}(z)\ , \quad (2.1)$$

where

$$|\Delta A^{\omega_1}(z)|\ll|A_{00}^{\omega_1}|$$

and all the other signal modes will be of order ΔA^{ω_1}. In this case Eqs. (1.8) may be approximated in the zeroth order in ΔA^{ω_1} by one set of equations,

$$\frac{dA_{mn}^{\omega_2}}{dz} = C_{mn0000}^{\omega_2} A_{00}^{\omega_1 *} B_{00}^{\omega_3} , \qquad (2.2)$$

which can be integrated immediately to give

$$A_{mn}^{\omega_2}(z_1) = A^{\omega_1 *} B^{\omega_2} \int_{-z_1}^{z_1} C_{mn}^{\omega_2}(z)\, dz . \qquad (2.3)$$

In this equation the supscripts 00 have been dropped from the mode amplitudes and coupling coefficient and the z dependence of the coupling coefficient has been emphasized. The total idler-power is given by

$$P = \frac{c\eta_2}{4\pi} \sum_{m,n} |A_{mn}^{\omega_2}(z_1)|^2$$

$$= \frac{c\eta_2}{4\pi} |A^{\omega_1}|^2 |B^{\omega_3}|^2 \int_{-z_1}^{z_1}\int_{-z_1}^{z_1} \sum_{m,n} C_{mn}^{\omega_2}(z') C_{mn}^{\omega_2} dz\, dz' . \qquad (2.4)$$

Substituting for the coupling coefficients the sum over them becomes

$$\sum_{m,n} C_{mn}^{\omega_2}(z') C_{mn}^{\omega_2 *}(z) = \frac{4\pi^2 \omega_1^4 d_{15}^2}{k_2^2 c^4} e^{-i\Delta k(z'-z)}$$

$$\times \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} \mathcal{E}_{00}^{\omega_1}(z) \mathcal{E}_{00}^{\omega_1 *}(z') \mathcal{E}_{00}^{\omega_3}(z')$$

$$\times\ \mathcal{E}_{00}^{\omega_3 *} \sum_{m,n} \mathcal{E}_{mn}^{\omega_2}(z) \mathcal{E}_{mn}^{\omega_2 *}(z')\, dx\, dy\, dx'\, dy' .$$

Substituting from Eq. (1.4) for the sum over the idler modes, this equation then becomes

$$\sum_{m,n} C_{mn}^{\omega_2}(z') C_{mn}^{\omega_2 *}(z) = \frac{2\pi\omega_1^4 d_{15}^2 e^{-i\Delta k(z'-z)}}{ik_2 c^4 (z'-z)}$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}_{00}^{\omega_1}(z) \mathcal{E}_{00}^{\omega_1 *}(z') \mathcal{E}_{00}^{\omega_3}(z') \mathcal{E}_{00}^{\omega_3 *}(z')$$

$$\times \exp\left(\frac{ik[(x-x')^2 + (y-y')^2]}{2(z'-z)}\right) dx\, dx'\, dy\, dy' .$$

The quadruple integral splits into two double integrals of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(au^2 + buu' + cu'^2)} du\, du' = 2\pi/(4ac - b^2)^{1/2} ,$$

on substituting for the explicit form of the mode functions. After some algebra, we find that the sum is given by

$$\sum_{m,n} C_{mn}^{\omega_2}(z') C_{mn}^{\omega_2 *}(z) = \frac{8\pi\omega_2^4 d_{15}^2 \sigma e^{-i\Delta k(z'-z)}}{k_2^2 c^4 w_{01}^2 w_{03}^2}$$

$$\times \{[a_1 + a_3 zz' + ia_2(z'+z)]^{1/2}[b_1 + b_3 zz' + b_5(z'^2 + z^2)$$

$$+ ib_2(z'+z) + ib_5(z'z)zz']\}^{-1/2} , \qquad (2.5)$$

where[7]

$$a_1 = 1/w_{01}^2 + 1/w_{03}^2 , \qquad (2.6a)$$

$$a_2 = \frac{1}{k_2}\left[\frac{k_3}{k_1 w_{01}^4} + \frac{2}{w_{01}^2 w_{03}^2}\left(\frac{k_1}{k_3} - 1 + \frac{k_3}{k_1}\right) + \frac{k_1}{k_3 w_{03}^4}\right] , \qquad (2.6b)$$

$$a_3 = \frac{4}{w_{01}^2 w_{03}^2}\left(\frac{1}{k_1^2 w_{01}^2} + \frac{1}{k_3^2 w_{03}^2}\right) , \qquad (2.6c)$$

$$b_1 = 1/w_{01}^2 + \sigma^2/w_{03}^2 ,$$

$$b_2 = \frac{1}{k_2}\left[\frac{k_3}{k_1 w_{01}^4} + \frac{2}{w_{01}^2 w_{03}^2}\left(\frac{k_1}{k_3} - 1 + \frac{\sigma^2 k_3}{k_1}\right) \qquad (2.6d)\right.$$

$$\left. + \frac{1}{w_{03}^4}\left(\frac{k_1}{k_3} + \sigma^2(\sigma^2 - 1)\right)\right] , \qquad (2.6e)$$

$$b_3 = \frac{4}{w_{01}^2 w_{03}^2}\left(1 + \frac{(\sigma^2 - 1)k_3}{k_2}\right)$$

$$\times\left[\frac{1}{k_3^2 w_{03}^2}\left(1 + \frac{(\sigma^2 - 1)k_3}{k_1}\right) + \frac{1}{k_1 w_{01}^2}\right] , \qquad (2.6f)$$

$$b_4 = \frac{(\sigma^2 - 1)}{k_3 k_1^2 w_{01}^2 w_{03}^2} , \qquad (2.6g)$$

$$b_5 = \frac{(\sigma^2 - 1)}{k_1 k_2 w_{01}^2 w_{03}^2}\left(\frac{1}{w_{01}^2} + \frac{\sigma^2}{w_{03}^2}\right) . \qquad (2.6h)$$

It can be seen that a_1, a_2, and a_3 tend to b_1, b_2, and b_3, respectively, as σ tends to one. When this sum is substituted into Eq. (2.4) for the total idler power at the exit face of the crystal, we have

$$P = \frac{2\mu_2\omega_2^2 d_{15}^2 \sigma |A^{\omega_1}|^2 |B^{\omega_3}|^2}{k_2^2 c^3 w_{01}^2 w_{03}^2} \int_{-z_1}^{z_1}\int_{-z_1}^{z_1} \frac{e^{-i\Delta k(z'-z)} dz\, dz'}{[a_1 + a_3 zz' + ia_2(z'-z)]^{1/2}[b_1 + b_3 zz' + b_5(z'^2 + z^2) + ib_2(z'-z) + ib_4 zz'(z'-z)]^{1/2}} . \qquad (2.7)$$

A. Near Field

When the nonlinear medium lies entirely within the near field of both the signal and pump beams, we may approximate the integrand of Eq. (2.7) exactly as for the case of SHG.[3] At the optimum phase-matched position, the signal varies according to the expression

$$G = \frac{8\pi\omega_1\omega_2 d_{15}^2 \sigma l^2 |B^{\omega_3}|^2}{\eta_1\eta_2 c^2 (w_{01}^2 + w_{03}^2)^{1/2} (\sigma^2 w_{01}^2 + w_{03}^2)^{1/2}} . \qquad (2.8)$$

This expression has no absolute maximum as a function of the spot sizes of the two beams. It implies that these must be as small as possible within the limits of the theory ($w_0 \gg \lambda$, λ the wavelength of the field). The output varies as the length of the crystal squared, as in the plane-wave case.

B. General Case

In order to simplify Eq. (2.7) and integrate it, we define new dimensionless variables. We change the variable of integration to $\tau = z/l$, where l is the length of the nonlinear crystal, then the phase-mismatch parameter becomes $\xi = \Delta k l$. The focusing parameter of the pump beam is defined as $u = l/z_{03} = 2l/k_3 w_{03}^2$. This is the standard focusing parameter, and we also define a parameter $v_1 = z_{01}/z_{03}$, which describes the relative focusing of the pump and signal. Last, we define a degeneracy parameter μ such that[7]

$$\mu = (k_3 - 2k_1)/k_3 \tag{2.9}$$

In terms of these parameters the expression for the signal gain is

$$G = \frac{\pi k_3^2 d_{15}^2 (1-\mu^2)(1+\mu)\sigma l u |B^{\omega_3}|^2}{\eta_1^2 \eta_2^2} \times \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{e^{-i\xi(\mu'-\mu)}\, d\mu\, d\mu'}{[\alpha_1 + \alpha_3 \tau\tau' + i\alpha_2(\tau'-\tau)]^{1/2}[\beta_1 + \beta_3\tau\tau' + \beta_5(\tau'^2+\tau^2) + i\beta_2(\tau'-\tau) + i\beta_4\tau\tau'(\tau'-\tau)]^{1/2}} , \tag{2.10}$$

where

$$\begin{aligned}
\alpha_1 &= (2v_1 + 1 - \mu) , \\
\alpha_2 &= u[(1+v_1^2)(1-\mu) + v_1(3+\mu^2)]/v_1(1+\mu) , \\
\alpha_3 &= u^2[2 + (1-\mu)v_1]/v_1 , \\
\beta_1 &= (2\sigma^2 v_1 + 1 - \mu) , \\
\beta_2 &= u[v_1^2\sigma^2(2\sigma^2 - 1 - \mu) \\
&\quad + v_1(4\sigma^2 - 1 + \mu^2) + 1 - \mu]/v_1(1+\mu) , \\
\beta_3 &= u^2[(2\sigma^2 - 1 + \mu)(v_1(2\sigma^2 - 1 - \mu) + 2)]/v_1(1+\mu) , \\
\beta_4 &= u^3(\sigma^2 - 1)(2\sigma^2 - 1 + \mu)/v_1(1+\mu) , \\
\beta_5 &= -u^2(\sigma^2 - 1)/v_1(1+\mu) .
\end{aligned} \tag{2.11}$$

The integral has been evaluated numerically as a function of $u = l/z_{03}$, μ, $v_1 = z_{01}/z_{03}$, and $\xi = \Delta k l$, and a sample of the results is presented in Figs. 1–8. In all the figures the *square root of the gain* has been plotted as the abscissa, so that the small side features in Figs. 1–5 show up and the figures are in line with those for SHG published previously.[3]

The gain computed from these figures will be in units of $\pi d_{15}^2 k_3^3 |B^{\omega_3}|^2 l/\eta_1^2\eta_2^2$, which taking the approximate values $d_{15} = 1.85 \times 10^{-8}$, $\eta_{1,2} = 2.23$, $k_3 = 2.46 \times 10^5$, $l = 1$, and pump power = 50 mW, has the value

$$G = (1.1 \times 10^{-5}) \times (\text{figure value})^2 .$$

Figures 1–3 show the variation of the gain with phase matching for the almost degenerate amplifier[8] for various values of the focusing parameter. These curves have the familiar appearance of the similar curves for SHG. Figures 4 and 5 show the variation of the gain with phase matching for the nondegenerate case when $\mu = 0.5$ at two different values of focusing. These two curves have slight differences from the almost degenerate case but the general features are the same. Figures 6–8 show the variation of the signal gain with the focusing parameter of the pump $u = l/z_{03}$ for various values of the relative focusing of the signal beam to the pump beam $v_1 = z_{01}/z_{03}$.

Figure 6 shows this for the almost degenerate

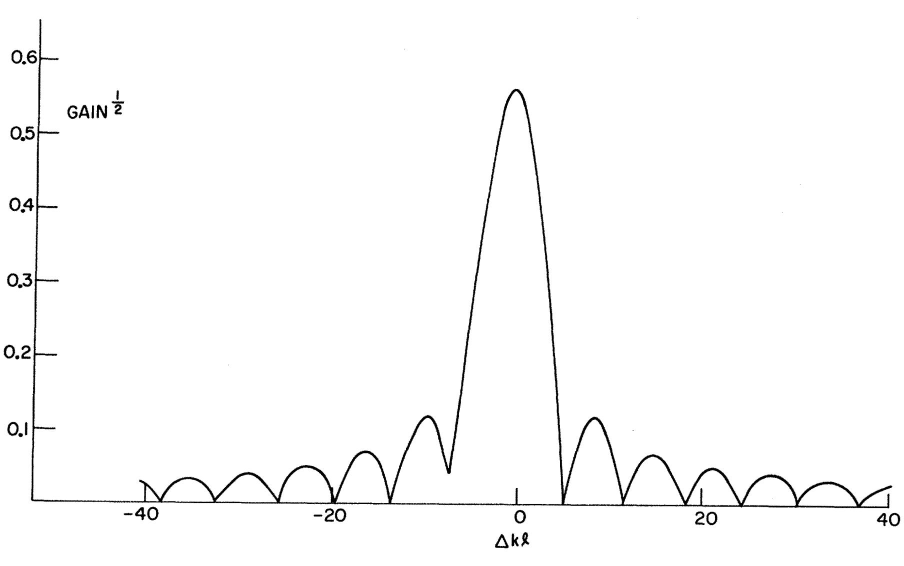

FIG. 1. Variation of the total gain with phase matching for relative crystal length $u = 0.5$, degeneracy parameter $\mu = 0$, and relative focusing of signal to pump $v_1 = 1.0$.

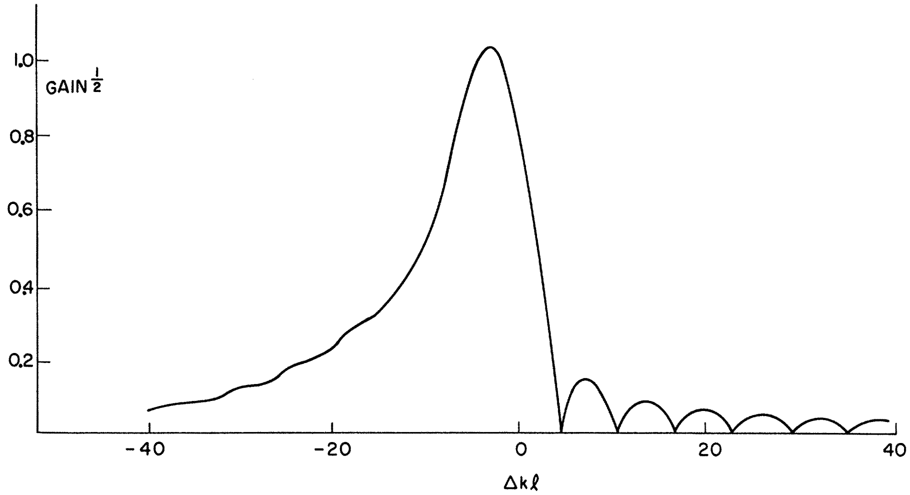

FIG. 2. Variation of the total gain with phase matching for relative crystal length $u = 5.55$, degeneracy parameter $\mu = 0$, and relative focusing of signal to pump $v_1 = 1.0$.

amplifier $\mu = 0$. The maximum gain occurs when both signal and idler are focused to the same degree at the value $u = 5.56$. It may easily be seen that for any given value of the pump focusing there exists an optimum focusing for the signal beam given by the envelope of the curves in this figure.

As the degeneracy parameter moves away from zero, Figs. 7 and 8, the maximum possible gain decreases but the curves retain the same form and the maximum gain occurs approximately at the same values of the two focusing parameters. When $\mu = 0.5$, which corresponds to $k_1 = \frac{1}{4}k_3$ and $k_2 = \frac{3}{4}k_3$, optimum focusing occurs when $u = 5.7$, and when $\mu = -0.5$ corresponding to $k_1 = \frac{3}{4}k_3$, $k_2 = \frac{1}{4}k_3$, optimum focusing occurs when $u = 4.8$ for each case, the optimum occurs when signal and pump are focused equally.

III. GAUSSIAN MODE IDLER

In order to see under what conditions the previous theory may be approximated by a theory in which it is assumed that the idler field is also a Gaussian mode, we now derive the signal gain for this case and compare with the expression found in II B.

This approximation is important since it is a necessary step towards extending the theory into the regions of large gain and pump depletion. We must then have some guide to which modes are contributing significantly to the interaction. This section will also apply to the case of a singly resonant oscillator when the idler field is enclosed in a resonator and the signal field is not. The doubly resonant oscillator should strictly be treated via a standing wave theory, but the following theory gives some guide to this case.

Since we assume small gain and no pump depletion again, we need only consider Eq. (1.8b) for the lowest-order idler-mode amplitude

$$\frac{dA_{00}^{\omega_2}}{dz} = C_{000000}^{\omega_2} A_{00}^{\omega_1} B_{00}^{\omega_3} \quad . \tag{3.1}$$

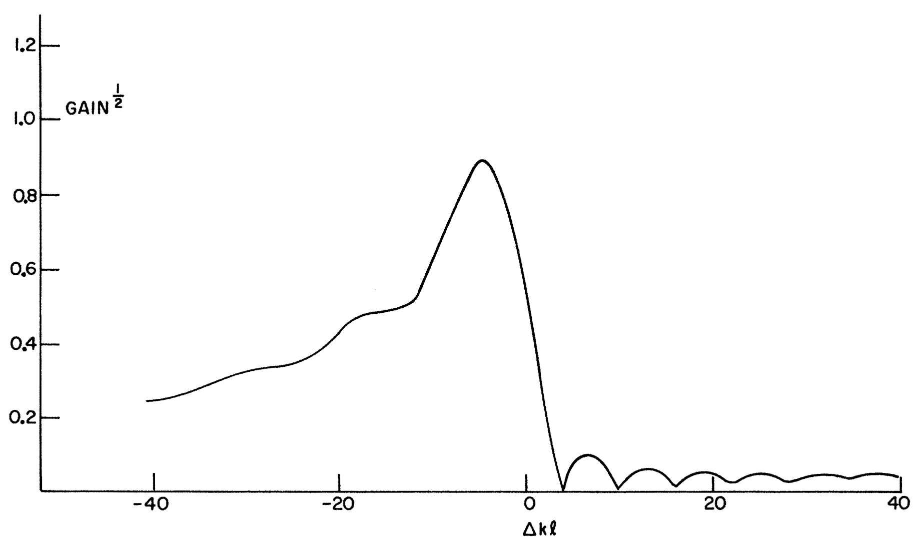

FIG. 3. Variation of the total gain with phase matching for relative crystal length $u = 20.0$, degeneracy parameter $\mu = 0.0$, and relative focusing of signal to pump $v_1 = 1.0$.

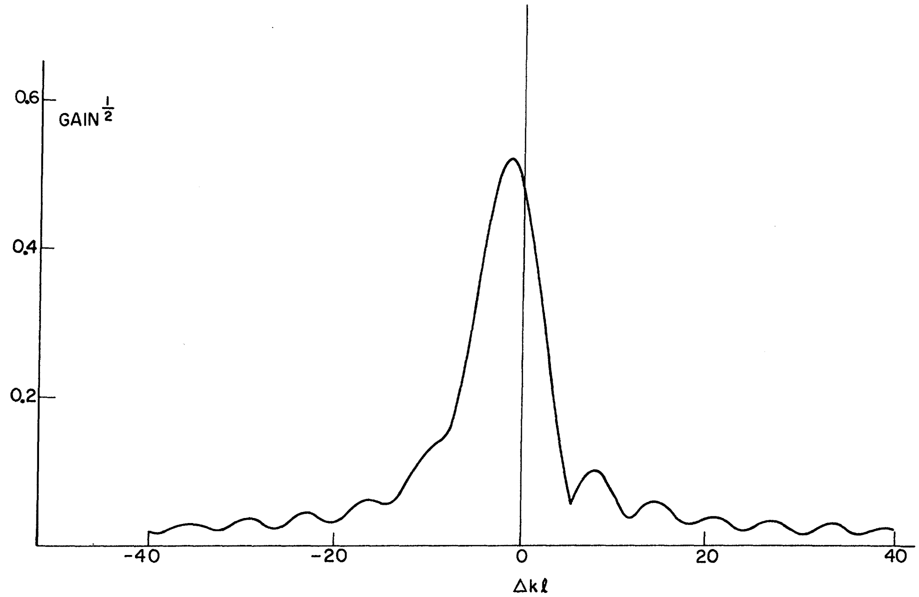

FIG. 4. Variation of the total gain with phase matching for relative crystal length $u=0.5$, degeneracy parameter $\mu=0.5$, and relative focusing of signal to pump $v=1.0$.

Dropping the subscripts 00 and integrating, the output into the lowest-order idler mode is given by

$$P_0=\frac{c\eta_2}{4\pi}\,|A^{\omega_1}|^2|\,B^{\omega_3}|^2$$
$$\times\int_{-z_1}^{z_1}\int_{-z_1}^{z_1} C^{\omega_2}(z')\,C^{\omega_2 *}(z)\,dz\,dz'\,. \qquad (3.2)$$

The coupling coefficient $C^{\omega_2}(z)$ is defined by the Eq. (1.9)

$$C^{\omega_2}(z)=\frac{-2\pi i\omega_2^2 d_{15}\,e^{-i\Delta k z}}{k_2c^2}$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{-\infty}\mathcal{E}_{00}^{\omega_1 *}\,\mathcal{E}_{00}^{\omega_2 *}\,\mathcal{E}_{00}^{\omega_3 *}\,dx\,dy\,. \qquad (3.3)$$

Substituting for the mode functions and carrying out the integration, we have

$$C^{\omega_2}(z)=-\frac{4(2\pi)^{1/2}i\omega_2^2d_{15}\sigma^{1/2}}{k_2c^2w_{03}w_{02}w_{01}}$$
$$\times\frac{e^{-i\Delta kz}}{(a_7+ia_8z)^{1/2}(b_7+b_9z^2+ib_8z)^{1/2}}\,, \qquad (3.4)$$

where

$$a_7=\frac{1}{\omega_{01}^2}+\frac{1}{w_{02}^2}+\frac{1}{w_{03}^2}\,, \qquad (3.5\text{a})$$

$$a_8=2\left(\frac{k_3}{k_1k_2w_{01}^2w_{02}^2}+\frac{k_1}{k_3k_2w_{03}^2w_{02}^2}\right.$$
$$\left.+\frac{k_2}{k_1k_3w_{03}^2w_{01}^2}\right)\,, \qquad (3.5\text{b})$$

$$b_7=\frac{1}{w_{01}^2}+\frac{1}{w_{02}^2}+\frac{\sigma^2}{w_{03}^2}\,, \qquad (3.5\text{c})$$

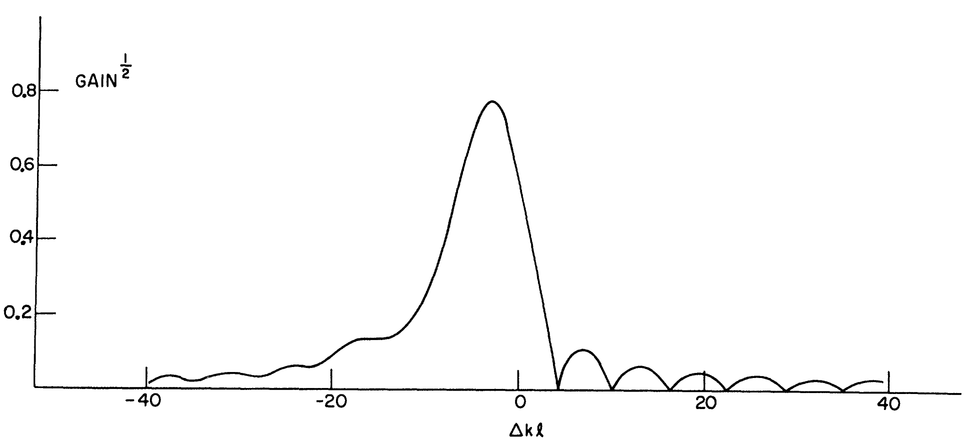

FIG. 5. Variation of the total gain with phase matching for relative crystal length $u=5.55$, degeneracy parameter $\mu=0.5$, and relative focusing of signal to pump $v=1.0$.

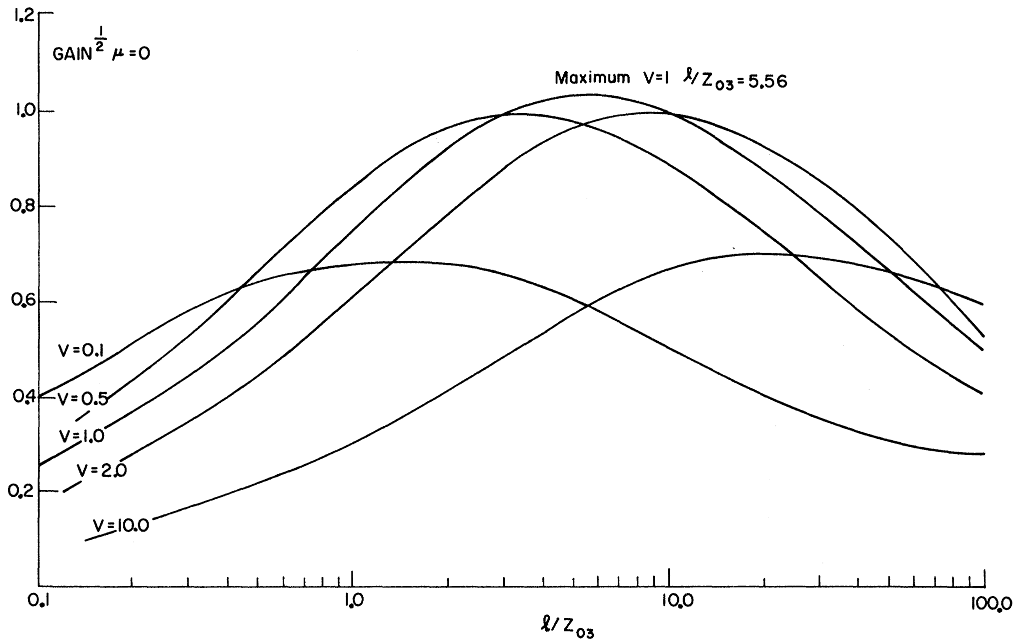

FIG. 6. Variation of the total gain with focusing of the pump beam for various values of the relative focusing of the signal to pump, degeneracy parameter $\mu = 0$.

$$b_8 = 2\left(\frac{k_3}{k_1k_2w_{01}^2w_{02}^2} + \frac{k_1+(\sigma^2-1)k_3}{k_2k_3w_{03}^2w_{02}^2} + \frac{k_2+(\sigma^2-1)k_3}{k_1k_3w_{03}^2w_{01}^2}\right), \qquad (3.5d)$$

$$b_9 = \frac{-4(\sigma^2-1)}{k_1k_2w_{01}^2w_{02}^2w_{03}^2}, \qquad (3.5e)$$

and using Eq. (1.15), the expression for the gain of the signal beam due to the Gaussian mode of an idler beam with spot size w_{02} is given by

$$G_0 = \frac{32\pi\omega_2\omega_1 d_{15}^2\sigma|B^{\omega_3}|^2}{\eta_1\eta_2c^2w_{01}^2w_{02}^2w_{03}^2} \times \int_{-z_1}^{z_2}\int_{-z_1}^{z_2} \frac{e^{-i\Delta k(z'-z)}\,dz\,dz'}{[a_7^2+a_8^2zz'+ia_7a_8(z'-z)]^{1/2}[b_7^2+b_8^2zz'+b_7b_9(z'^2+z^2)+ib_7b_8(z'-z)-ib_8b_9zz'(z'-z)]^{1/2}}, \qquad (3.6)$$

where the term in b_9^2 has been neglected since it is of the second order in σ^2-1 and thus negligible for normal values of focusing $l/z_{03} < 10$.

We now compare Eq. (3.6) with Eq. (2.7) to see under what condition Eq. (2.7) can be approximated by Eq. (3.6), i.e., under what conditions the idler field can be represented by a Gaussian beam.

A. Near Field

When the nonlinear medium lies entirely within the near field of both the signal and idler beams, Eq. (3.6) reduces to

$$G_0 = \frac{32\pi\omega_2\omega_1 d_{15}^2\sigma l^2|B^{\omega_3}|^2}{\eta_1\eta_2c^2w_{02}^2w_{02}^2w_{03}^2(1/w_{01}^2+1/w_{02}^2+1/w_{03}^2)(1/w_{01}^2+1/w_{02}^2+\sigma^2/w_{03}^2)}. \qquad (3.7)$$

Comparing this with Eq. (2.8) for the total gain we have

$$\frac{G_0}{G} = \frac{4(1/w_{01}^2+1/w_{03}^2)^{1/2}(1/w_{01}^2+\sigma^2/w_{03}^2)^{1/2}}{w_{02}^2(1/w_{01}^2+1/w_{02}^2+1/w_{03}^2)(1/w_{01}^2+1/w_{02}^2+\sigma^2/w_{03}^2)}. \qquad (3.8)$$

This expression has the maximum with respect to w_{02}, the idler spot size, at

$$\frac{1}{w_{02}^2} = \left(\frac{1}{w_{01}^2}+\frac{1}{w_{03}^2}\right)^{1/2}\left(\frac{1}{w_{01}^2}+\frac{\sigma^2}{w_{03}^2}\right)^{1/2}. \qquad (3.9)$$

For this value of the idler spot size, the ratio of the two gains to the second order in (σ^2-1) is

$$\frac{G_0}{G} = 1 - \frac{(\sigma^2-1)^2}{64(w_{03}^2/w_{01}^2+1)^2}, \qquad (3.10)$$

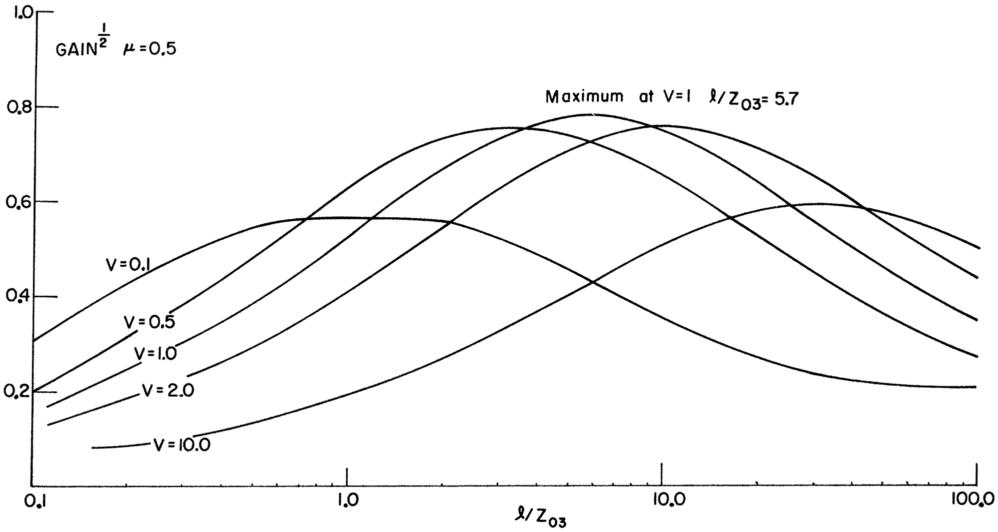

FIG. 7. Variation of the total gain with focusing of the pump beam for various values of the relative focusing of the signal to pump, degeneracy parameter $\mu = 0.5$.

which is very close to 1. Thus when the nonlinear crystal lies within the near field of both signal and pump, the idler field may be represented by a Gaussian mode with spot size given by Eq. (3.9) to a very good approximation.

B. General Case

Equations (3.6) and (2.7) are of exactly similar form, as was the case in SHG[6]; for the integrals to be equal we require

$$a_2/a_1 = a_8/a_7 , \quad a_3/a_1 = a_8^2/a_7^2 . \tag{3.11}$$

Substituting for these quantities we find that the equalities can only be satisfied if

$$k_1 w_{01}^2 + k_3 w_{03}^2 = 0 . \tag{3.12}$$

This certainly cannot be satisfied for normal parametric amplification. However, it should be noted that it can be for backward wave amplification when k_1 is negative. Thus, for normal parametric amplification there does not exist a relation for the equivalent idler spot size which is independent of the degree of focusing of the pump as there does for SHG.[6] Writing Eq. (3.6) in terms of the parameters defined for the total output and the parameter

$$v_2 = z_{02}/z_{03} = k_2 w_{02}^2 / k_3 w_{03}^2 ,$$

we have

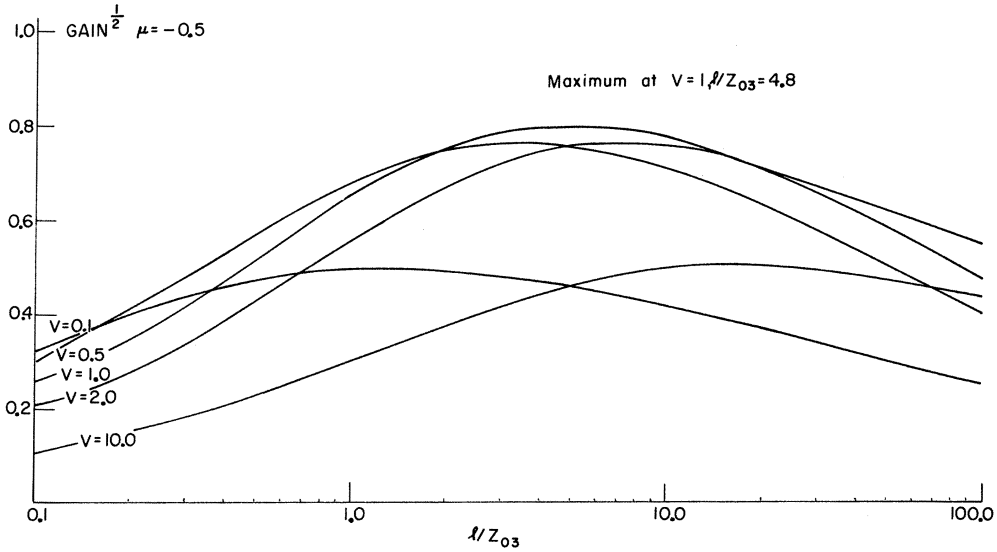

FIG. 8. Variation of the total gain with focusing of the pump beam for various values of the relative focusing of the signal to pump, degeneracy parameter $\mu = -0.5$.

$$G_0 = \frac{\pi d_{15}^2 k_3^3 |B^{\omega_3}|^2 4(1-\mu^2)^2 v_1 v_2 u l \sigma}{\eta_2^2 \eta_1^2}$$
$$\times \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{e^{-i(\tau'-\tau)\xi}}{[\alpha_7^2+\alpha_8^2\tau\tau'+i\alpha_7\alpha_8(\tau'-\tau)]^{1/2}[\beta_7^2+\beta_8^2\tau\tau'+\beta_7\beta_9(\tau'^2+\tau^2)+i\beta_7\beta_8(\tau'-\tau)-i\beta_8\beta_9\tau\tau'(\tau'-\tau)]^{1/2}}, \tag{3.13}$$

where

$$\begin{aligned}
\alpha_7 &= 2v_1v_2 + (1+\mu)v_1 + (1-\mu)v_2, \\
\alpha_8 &= u[2 + (1-\mu)v_1 + (1+\mu)v_2], \\
\beta_7 &= 2\sigma^2 v_1 v_2 + (1+\mu)v_1 + (1-\mu)v_2, \\
\beta_8 &= u[2 + (1-\mu+2\sigma)v_1 + (1+\mu+2\sigma)v_2], \\
\beta_9 &= -2u^2\sigma.
\end{aligned} \tag{3.14}$$

This expression was integrated numerically and the results compared with the previous calculation for the total gain. Figure 9 shows the gain at optimum phase matching when the pump and signal have the same confocal parameter ($v_1 = 1$) as a function of v_2, the idler confocal parameter relative to the pump for various values of the pump focusing $u = l/z_{03}$.

The maxima of these curves are very close to the total output for this case, as shown in the figure. For example, at $u = 5.5$, very near the optimum value, the total gain is given by

$$G = (1.0370)^2 \pi d_{15}^2 k_3^3 |B^{\omega_3}|^2/\eta_2^2\eta_1^2. \tag{3.15}$$

Then if the idler field is assumed to be made up of a set of modes of confocal parameter equal to that of the pump ($v_2 = 1$), the signal gain due to the lowest-order Gaussian mode of the idler is

$$G_0 = (1.0328)^2 \pi d_{15}^2 k_3^3 |B^{\omega_3}|^2/\eta_2^2\eta_1^2, \tag{3.16}$$

which is 99.2% of the total gain. Thus the idler field may be assumed to be a Gaussian of this confocal parameter to a very good approximation.

In Fig. 10, we show how the idler confocal parameter, for which optimum gain occurs, varies as a function of the pump focussing when the signal and pump are focused equally ($v_1 = 1$). This shows the confocal parameter which should be assigned to the idler for a given degree of focusing of the pump. The curve is correctly asymptotic to the near field condition given approximately by

$$1/w_{02}^2 = 1/w_{01}^2 + 1/w_{03}^2,$$

and note that for the optimum value of focusing ($u = 5.56$) the value of the parameter v_2 is unity, which is just the condition

$$1/w_{03}^2 = 1/w_{01}^2 + 1/w_{02}^2,$$

which was the case considered by Boyd and Ashkin,[4]

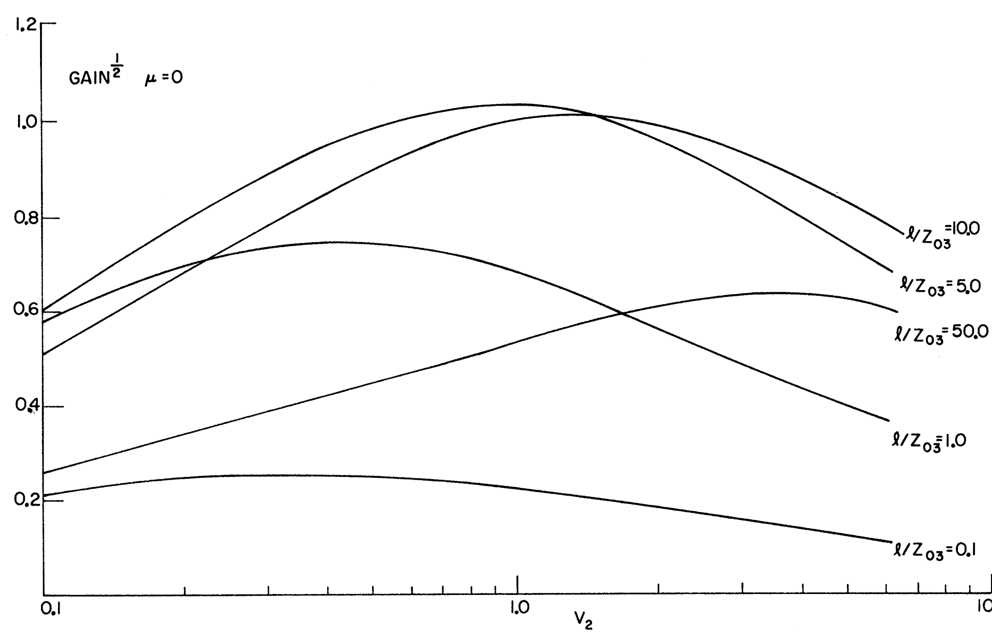

FIG. 9. Variation of the gain due to the Gaussian idler mode with the confocal parameter assigned to the idler modes relative to that of the pump for various values of pump focusing.

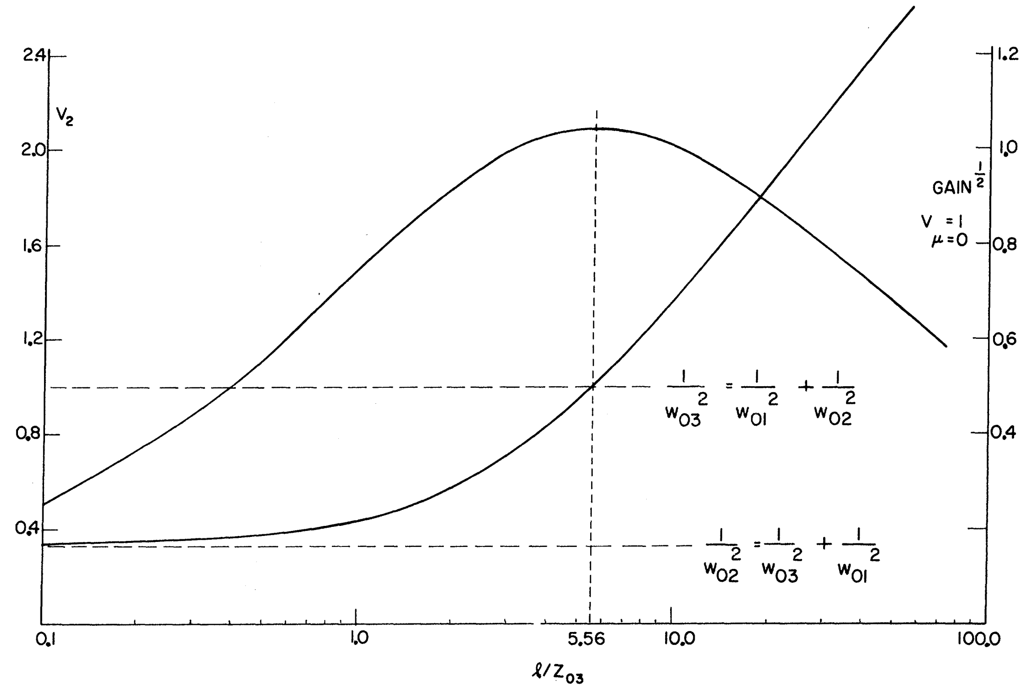

FIG. 10. Variation of the optimum idler confocal parameter with pump focusing.

and Boyd and Kleinman.[5]

This condition is only approximately true when the amplifier is nondegenerate, for example when $\mu = 0.5$ at the optimum $u = 5.7$, $v_2 = 1$, the optimum with respect to v_2 occurs at $v_2 = 1.02$.

Lastly, it should be noted that if $v_1 = v_2$, the expression (3.13) for the gain due to a Gaussian idler field is an even function of μ, the degeneracy parameter. Thus, for two Gaussian fields at frequencies ω_1 and ω_2, the gain if the field at ω_1 is the signal field is the same as that if the field at ω_2 is the signal field, for a given signal and pump input power.

That the maxima in Figs. 7 and 8 do not occur at precisely the same value of l/z_{03} is a reflection of the fact that the optimum condition $v_1 = v_2 = 1$ is only approximate, depending on μ (the degeneracy parameter) and the crystal anisotropy.

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[7]In deriving these equations the approximation $k_3 = k_1 + k_2$ has been used. Δk is only significant in the exponential phase-mismatch term $e^{-i\Delta k(z'-z)}$.

[8]The case $\mu = 0$, i.e., $2k_1 = 2k_2 = k_3$ is referred to as an almost degenerate amplifier, since the way the theory in this paper is set up the signal and idler fields are always distinct fields, whereas, of course, for a true degenerate amplifier the signal and idler fields are one and the same.