Boson Method in Superconductivity: Application to the Study of Vortex Lines

L. Leplae, F. Mancini, and H. Umezawa
Department of Physics, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201
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The boson method in superconductivity, developed in previous articles, is extended and applied to the problem of vortices in neutral and type-II superconductors. In the approximation considered, the distributions of current and magnetic field of a single vortex are given in the whole domain, up to the center of the flux line. Expressions for the vortex self-energy and the interaction energy between two vortices are also derived. In the limiting case in which $\kappa \gg 1$ (where κ is the Ginzburg-Landau parameter) and the structure of the core is approximated by a δ function, our results agree with those of Abrikosov's theory, based on the Ginzburg-Landau equations.

I. STRUCTURE OF BOSON METHOD

We have devoted several papers[1,2] to the role played by the bosons in the recovery of gauge invariance and current conservation in the theory of superconductivity. In the course of this study it appeared that many phenomena could be presented from a new angle. We called this approach the boson method. In the previous papers we have essentially limited our study to the case of a position-independent order parameter, although the space-dependent case has been briefly touched upon in Ref. 2.

The purpose of this paper is to extend our previous considerations to space-dependent superconductors and to show that this approach presents certain advantages. In order to check our general argument we applied it to an extreme case of space dependence: the vortex lines in type-II superconductors. We would like to emphasize that although some of our results are similar to the Landau-Ginzburg-Gor'kov theory,[3] our method presents some essential differences which will be shown explicitly in the following pages.

It has been known for many years that, in order to have a theory of superconductivity where the currents are conserved and which is gauge invariant, one has to take into account terms of higher order than in the BCS theory (the reduced Hamiltonian is not gauge invariant and the current obtained through a simple Bogoliubov transformation is not conserved).[4] It is also well known that already at second order[5] there appear bosons in the theory and that these bosons play a crucial role in the symmetry recovery.

If one wants to extend this program to the case of a space-dependent order parameter, one is faced immediately with great difficulties, since already the first-order[5] problem (where no bosons appear) is very hard to solve.[3]

The boson method suggests a line of approach which allows us to bypass some of these difficulties. A keystone in this approach is the introduction of some invariant transformations, under which the field equations stay invariant while the ground-state expectation values of various observables may change. Through these transformations we can move from the space-independent solution to space-dependent ones. We have already derived the operator forms of most of the important observables (such as the current, density, and Hamiltonian) expressed in terms of bosons and quasifermions. These expressions were obtained by solving the electron equations for the space-independent superconductor, and they manifest the gauge invariance and current conservation. Space-dependent phenomena can be derived from these results simply by operating the invariant transformations on the observables. In this way the space dependence can be treated without solving the Gor'kov equations, while the gauge invariance and current conservation are guaranteed by the boson effect. It will be shown that the order parameter thus obtained satisfies the Gor'kov equations.

To be more specific let us first consider the case of a neutral superconductor. In Ref. 1 we have solved the electron equations derived from the BCS Hamiltonian and found the quasifermion operators $\alpha_{ks}(s=\uparrow, \downarrow)$ and quasiboson operators B_l.

The BCS Hamiltonian can be rewritten in terms of these new operators in the form

$$H=\sum_{k,s} E_k\, \alpha_{k,s}^\dagger\, \alpha_{k,s}+\sum_l \omega_l B_l^\dagger B_l\ , \tag{1.1}$$

where $\omega_l \simeq 3^{-1/2} v_F l = v_0 l$ for $l\xi \ll 1$ (ξ is the coherence length). The ground state $|0\rangle$ is assumed to be position independent. The current and density operators become

$$\begin{aligned}\rho(\vec{x}, t) &= \rho^{(1)}(\vec{x}, t)+\rho^{(2)}(\vec{x}, t)\ ,\\ \vec{j}(\vec{x}, t) &= \vec{j}^{(1)}(\vec{x}, t)+\vec{j}^{(2)}(\vec{x}, t)\ ,\end{aligned} \tag{1.2}$$

where $\rho^{(1)}$, $\rho^{(2)}$, $\vec{j}^{(1)}$, and $\vec{j}^{(2)}$ are, respectively, the quasifermion density, quasiboson density, quasifermion current, and quasiboson current. $\rho^{(2)}$ and $\vec{j}^{(2)}$ can be expressed in terms of the boson field $B(\vec{x},t)$ and its canonical conjugate $\pi(\vec{x},t)$ in the following manner:

$$\rho^{(2)}(\vec{x},t) = -\eta(\vec{\nabla})\pi(\vec{x},t), \ (\pi = \dot{B}) \tag{1.3}$$

$$\vec{j}^{(2)}(\vec{x},t) = v_0^2\,\eta(\vec{\nabla})\vec{\nabla}B(\vec{x},t) \ . \tag{1.4}$$

The coefficient $\eta(\vec{\nabla})$ is given by $\eta(\vec{\nabla}) = -2\sqrt{2}\,\Delta\,[R(\vec{\nabla})]^{1/2}$, where Δ is the energy gap and $R(\vec{\nabla})$ is defined in Ref. 2.[6] As we have shown, $n(\vec{\nabla})$ is properly a temperature-dependent function: $\eta(\vec{\nabla}) \equiv \eta(\vec{\nabla}, T)$; in the following, the variable T will be omitted.

The conservation law for the bosons is guaranteed by the field equation

$$\left[\frac{\partial^2}{\partial t^2} - v_0^2\,\vec{\nabla}^2\right] B(\vec{x},t) = 0 \ . \tag{1.5}$$

It has also been shown in Refs. 1 and 2 that the electron fields can be expressed in terms of the quasiparticle fields as

$$\psi_{\uparrow,\downarrow} = \exp\{[i/\eta(\vec{\nabla})]B(\vec{x},t)\}F[\phi(\vec{x},t), \partial B(\vec{x},t)], \tag{1.6}$$

where $\phi(\vec{x},t)$ is the quasifermion field and F is an unknown function.

An interesting property of the boson operator is that, under the transformation induced by the generator

$$N_f^{(2)} = \int d^3y\, f(\vec{y})\rho^{(2)}(\vec{y}), \tag{1.7}$$

it transforms in the following way:

$$B(\vec{x},t) \to B_f(\vec{x},t) = e^{-iN_f^{(2)}} B(\vec{x},t) e^{iN_f^{(2)}}$$
$$= B(\vec{x},t) + \int d^3y\, f(\vec{y})\eta(\vec{\nabla}_y) d(\vec{x}-\vec{y}), \tag{1.8}$$

where

$$i\,d(\vec{x}-\vec{y}) = [B(\vec{x},t), \pi(\vec{y},t)]. \tag{1.9}$$

Another important result is that the transformation induced by $N_f^{(2)}$ leaves the equations of motion and the q-number part of the Hamiltonian invariant if $f(\vec{x})$ obeys the Laplace equation

$$\vec{\nabla}^2 f(\vec{x}) = 0 \ . \tag{1.10}$$

For instance the equations of motion

$$\left[\epsilon(\vec{\nabla}) + \frac{1}{i}\frac{\partial}{\partial t}\right]\psi_\uparrow = -\lambda\,\psi_\downarrow^\dagger(\psi_\downarrow\psi_\uparrow) \ ,$$
$$\left[\epsilon(\vec{\nabla}) + \frac{1}{i}\frac{\partial}{\partial t}\right]\psi_\downarrow = -\lambda\psi_\uparrow^\dagger(\psi_\uparrow\psi_\downarrow), \tag{1.11}$$

where $\epsilon(\vec{\nabla}) = -\frac{1}{2}(\vec{\nabla}^2 + k_F^2)$, become after the transformation

$$\left[\epsilon(\vec{\nabla}) + \frac{1}{i}\frac{\partial}{\partial t}\right]\psi_\uparrow^{(f)} = -\lambda\,\psi_\downarrow^{(f)\dagger}\psi_\downarrow^{(f)}\psi_\uparrow^{(f)} \ ,$$
$$\left[\epsilon(\vec{\nabla}) + \frac{1}{i}\frac{\partial}{\partial t}\right]\psi_\downarrow^{(f)} = -\lambda\,\psi_\uparrow^{(f)\dagger}\psi_\uparrow^{(f)}\psi_\downarrow^{(f)} \ , \tag{1.12}$$

where

$$\psi_{\uparrow,\downarrow}^{(f)} = e^{-iN_f^{(2)}}\psi_{\uparrow,\downarrow}\,e^{iN_f^{(2)}}$$
$$= \exp\{i[1/\eta(\vec{\nabla})]\,B_f\}\,F[\phi, \partial B_f] \ . \tag{1.13}$$

Equations (1.12) are easily derived observing that, owing to (1.3), (1.5), and (1.7),

$$\dot{N}_f^{(2)} = \left[\frac{\partial}{\partial t},\ N_f^{(2)}\right] = \frac{\partial}{\partial t}\int d^3y\, f(\vec{y})\rho^{(2)}(\vec{y},t)$$
$$= -\int d^3y\, f(\vec{y})\eta(\vec{\nabla})\frac{\partial^2}{\partial t^2}B(\vec{y},t)$$
$$= -v_0^2\int d^3y\, f(\vec{y})\eta(\vec{\nabla})\vec{\nabla}^2 B(\vec{y},t) \tag{1.14}$$

and after partial integration[7]

$$\dot{N}_f^{(2)} = \left[\frac{\partial}{\partial t}, N_f^{(2)}\right] = -v_0^2\int d^3y\,\vec{\nabla}^2 f(\vec{y})\eta(\vec{\nabla})B(\vec{y},t) = 0 \ , \tag{1.15}$$

since, by hypothesis, $f(\vec{y})$ obeys the Laplace equation.

The proof that the transformation induced by $N_f^{(2)}$ leaves invariant the q-number part of the Hamiltonian, modifying only the c-number part, requires more detailed arguments and is presented in Appendix A.

It is also interesting to notice the invariance of the boson field equation (1.5) under the transformation (1.8).

In the following, the transformation induced by $N_f^{(2)}$ is called "the boson transformation." As was shown in Ref. 2, this coincides with the gauge transformation when f is very smooth.[8] It is important to distinguish the boson transformation from the gauge transformation because the electron equation is invariant under the boson transformation even when f is not smooth. When f is a constant, the boson transformation is simply a phase transformation of ψ. The invariance of the electron equations under the phase transformation of ψ is well known and is obvious from (1.11). It is, however, remarkable that there exists a position-dependent transformation which keeps the electron equations invariant. Such an invariance is not explicitly seen from the electron equation (1.11), because the boson transformation induces a complicated nonlocal transformation of ψ as is shown by (1.13). (The invariance becomes explicit when we focus our attention on the boson.)

Using these results, our approach in a space-dependent problem is the following. We first look for an $f(\vec{x})$ satisfying the Laplace equation as well as the boundary conditions of the problem and then induce the transformation generated by the corresponding $N_f^{(2)}$. For instance the new boson current obtained after the transformation is, using (1.4),

$$\vec{j}^{(2)}(\vec{x},t) = v_0^2\eta(\vec{\nabla})\vec{\nabla}B(\vec{x},t) + \eta^2\vec{\nabla}_x v_0^2\int d^3y\, c(\vec{x}-\vec{y})f(\vec{y}), \tag{1.16}$$

where η means $\eta(0)$ and

$$c(\vec{x}-\vec{y})=[\eta(\vec{\nabla}_x)\eta(\vec{\nabla}_y)/\eta^2]d(\vec{x}-\vec{y}). \qquad (1.17)$$

Everywhere in Ref. (2) we approximated $\eta(l)$ by $\eta(0)$ so that $c(\vec{x})$ itself was the boson commutator. In Ref. (2) we took

$$c(\vec{x}-\vec{y})=\sum_{|l|<l_0} e^{il\cdot(\vec{x}-\vec{y})}\ , \qquad (1.18)$$

where $l_0=2\Delta/v_0$. This is of course an oversimplified choice. In the present paper we do not specify the precise structure of this function. Use is made only of the fact that this function has a range of the order of ξ and that it is normalized:

$$\int d^3x\, c(\vec{x}-\vec{y})=\int d^3x\, d(\vec{x}-\vec{y})=1. \qquad (1.19)$$

The ground-state current is obtained immediately as the average value of $\vec{j}^{(2)}(\vec{x},t)$ with respect to the (space-independent) ground state $|0\rangle$:

$$\langle 0|\vec{j}^{(2)}(\vec{x},t)|0\rangle=\eta^2v_0^2\,\vec{\nabla}_x\int d^3y\, c(\vec{x}-\vec{y})f(\vec{y}). \qquad (1.20)$$

The position-dependent order parameter is obtained simultaneously:

$$\Delta(\vec{x})=\langle 0|\exp\{[2i/\eta(\vec{\nabla})]B_f(\vec{x},t)\}F^2(\phi,\partial B_f)|0\rangle\ . \qquad (1.21)$$

The absolute value $|\Delta(\vec{x})|$ depends on $\vec{x}$ because B_f contains f [cf. (1.8)] which is a space-dependent c number.

The Gor'kov equations for the Green's functions can be derived from Eq. (1.12) in the usual way. In other words, $\Delta(\vec{x})$ given by (1.21) satisfies the Gor'kov equations. One should note that the ground state to be used is still the original space-independent ground state $|0\rangle$, the space dependence having been introduced in the operators $\psi^{(f)}$ [see Eq. (1.13)].

We thus obtain *simultaneously* the Gor'kov equations and their solutions. On the other hand the correct ground-state current is given by expression (1.20).

The fact that the transformation induced by $N_f^{(2)}$ leaves the q-number part of the Hamiltonian invariant implies that the system described after transformation is the same system as before but influenced by new boundary conditions. The invariance of the electron equations does not mean that the energy of the system is unchanged. The transformation induces a c-number part in the Hamiltonian, and thus, for instance, modifies the ground-state energy. For instance let us consider the contribution of the persistent current to the ground-state energy. The boson part of the Hamiltonian (1.1) can be put in the form[2]

$$H(B)=\tfrac{1}{2}\int d^3x\,[\pi^2(\vec{x})+v_0^2\vec{\nabla}B(\vec{x})\cdot\vec{\nabla}B(\vec{x})]. \qquad (1.22)$$

After the boson transformation the expectation value of the transformed Hamiltonian H_f in the ground state is

$$\langle H_f(B)\rangle=(1/2\eta^2v_0^2)\int d^3x\,\vec{j}(\vec{x})\cdot\vec{j}(\vec{x}). \qquad (1.23)$$

One of the principal merits of the Boson method is that it relates through a simple transformation (simple only when expressed in terms of boson fields) different states of a superconductor, each of these situations being characterized by a solution of the Laplace equation (1.10) obeying certain boundary conditions.

The method described above is not modified when the Coulomb interaction among electrons is taken into account.[2] We include in the Hamiltonian the Coulomb interaction

$$\tfrac{1}{2}e^2\int d^3x\int d^3y\,\rho(\vec{x},t)\rho(\vec{y},t)/|\vec{x}-\vec{y}| \qquad (1.24)$$

and diagonalize the new Hamiltonian. The expression of the Hamiltonian in terms of quasifermion and quasiboson operators is again given by (1.1), where ω_p is now a plasma frequency. The following relations are still true:

$$\rho^{c(2)}(\vec{x},t)=-\eta(\vec{\nabla})\pi^c(\vec{x},t),$$

$$\vec{j}^{c(2)}(\vec{x},t)=v_0^2\eta(\vec{\nabla})\vec{\nabla}B^c(\vec{x},t),$$

$$[B^c(\vec{x},t),\pi^c(\vec{y},t)]=i\,d(\vec{x}-\vec{y}),$$

where the superscript c designates the operators under the influence of the Coulomb interaction. The boson field equation (1.5) is modified by the Coulomb effect to become

$$\left(\frac{\partial^2}{\partial t^2}-v_0^2\vec{\nabla}^2\right)B^c(\vec{x},t)-\frac{\mu^2v_0^2}{4\pi}\int d^3y\,\frac{1}{|\vec{x}-\vec{y}|}\,\vec{\nabla}^2B(\vec{y},t)=0,$$
$$v_0^2\vec{\nabla}^2\to v_0^2\vec{\nabla}^2 \qquad (1.25)$$

where $\mu=(4\pi)^{1/2}e\eta$. This shows that the plasma energy ω_p is equal to $(v_0^2p^2+\mu^2v_0^2)$. As was shown in Ref. 2, $\mu^2v_0^2$ is about equal to $4\pi e^2n_s/m$ where n_s is the superelectron density. The relation between π and $\dot{B}$ is also modified by the Coulomb effect

$$\dot{B}^c(\vec{x},t)=\pi^c(\vec{x},t)+\frac{\mu^2}{4\pi}\int d^3y\frac{\pi^c(\vec{y},t)}{|\vec{x}-\vec{y}|}\ . \qquad (1.26)$$

Making use of the relations (1.25) and (1.26), we obtain

$$\frac{\partial}{\partial t}\pi^c(\vec{x},t)=v_0^2\nabla^2B^c(\vec{x},t). \qquad (1.27)$$

The current conservation law is now obvious from this relation. It is also obvious that the boson equation (1.25) is invariant under the boson transformation as it should be. It is not difficult to show that the Gor'kov equations and their solutions can again be generated simultaneously by the new generator

$$N_f^{c(2)}=\int d^3y\,\rho^{c(2)}(\vec{y},t)f(\vec{y}),$$

where $f(\vec{y})$ obeys the Laplace equation.

The inclusion of a vector potential presents more problems. Part of the vector potential can be included directly by gauge-invariance considerations. $\vec{\nabla}B(\vec{x},t)$ is replaced everywhere by

$$\vec{\nabla}B(\vec{x},t) - e\int d^3y\, \vec{A}(\vec{y})\,\eta(\vec{\nabla}_y)\,d(\vec{x}-\vec{y}). \qquad (1.28)$$

The gauge invariance of this expression is obvious if one remembers the relation (1.13). By doing the substitution (1.28) one includes, so to speak, the vector potential attached to the bosons (or condensed electrons). Since the description of vortex lines presented in the rest of the present article is intended as an illustration of the method described in this section, we simplified the calculations by including only the part of the vector potential appearing in (1.28). The core current may consequently be incomplete. A more detailed calculation will be presented later. From these considerations it is easy to see that in the case of a charged superconductor the boson transformation, generated by $N_f^{c(2)}$, induces the following ground-state current (or persistent current):

$$\vec{J}(\vec{x}) = e\eta^2 v_0^2 [\nabla_x \int c(\vec{x}-\vec{y}) f(\vec{y})\, d^3y - e\int c(\vec{x}-\vec{y})\vec{A}(\vec{y})\, d^3y]. \qquad (1.29)$$

Performing the boson transformation in the boson Hamiltonian in the charged case, we find that the expectation value of the transformed Hamiltonian H_f in the ground state is

$$\langle H_f(B)\rangle = (1/2e^2\eta^2 v_0^2)\int d^3x\, \vec{J}(\vec{x})\cdot\vec{J}(\vec{x}). \qquad (1.30)$$

It should be noted that the boson transformation induces also the following operator term in H_f:

$$(1/2\eta)\int d^3x\, \vec{\nabla}\vec{B}(\vec{x})\cdot\vec{J}(\vec{x})$$

which, however, is zero owing to the conservation law $\vec{\nabla}\cdot\vec{J}=0$. Thus, the ground state remains an eigenstate of the Hamiltonian under the boson transformation.

Another interesting application of the boson method is the Josephson effect. This effect can be obtained through the boson transformation generated by $N_f^{(2)}$, where f obeys the Laplace equation and is subject to appropriate boundary conditions. Thus the Josephson effect becomes a problem of the Laplace equation. A study of the Josephson effect along these lines will be presented in a separate article.

II. EQUATIONS FOR VECTOR POTENTIAL $\vec{A}$

If we introduce (1.29) in the Maxwell equation[9]

$$4\pi\vec{J} = \vec{\nabla}\times\vec{\nabla}\times\vec{A}, \qquad (2.1)$$

we get the following equation for $\vec{A}$:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{x}) + (1/\lambda_L^2)\int c(\vec{x}-\vec{y})\vec{A}(\vec{y})\, d^3y = 4\pi e\vec{j}(\vec{x}), \qquad (2.2)$$

where λ_L is the London penetration depth

$$1/\lambda_L^2 = 4\pi n_s e^2/m \simeq 4\pi e^2\eta^2 v_0^2 .$$

The general solution of this equation is given by $\vec{A}=\vec{A}_1+\vec{A}_2$, where $\vec{A}_1$ is the general solution of the associated homogeneous equation and $\vec{A}_2$ is a particular solution:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}_1(\vec{x}) + (1/\lambda_L^2)\int c(\vec{x}-\vec{y})\vec{A}_1(\vec{y})\, d^3y = 0, \qquad (2.3)$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}_2(\vec{x}) + (1/\lambda_L^2)\int c(\vec{x}-\vec{y})\vec{A}_2(\vec{y})\, d^3y = 4\pi e\vec{j}(\vec{x}). \qquad (2.4)$$

Equations (2.3) and (2.4) show that $\vec{A}_1$ is induced by the external field, while $\vec{A}_2$ is the self-consistent field induced by the persistent current. Equation (2.3) is clearly the generalized London equation.

Using the expansion

$$\int c(\vec{x}-\vec{y})\vec{A}_2(\vec{y})\, d^3y = (1+G\vec{\nabla}^2)\vec{A}_2(\vec{x}) + O(\xi^4/\lambda^4),$$

where G is a constant of the order ξ^2, we have

$$G = \tfrac{2}{3}\pi\int_0^\infty r_y^4 c(r_y)\, dr_y ,$$

and observing that the substitution of this expansion in (2.4), together with the conservation of the neutral current $\vec{j}$, implies that

$$\vec{\nabla}\cdot\vec{A}_2 = 0,$$

the Eq. (2.4) can be approximately[10] put in the following form:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}_2(\vec{x}) + (1/\lambda_G^2)\vec{A}_2(\vec{x}) = 4\pi e(\lambda_L^2/\lambda_G^2)\vec{j}(\vec{x}) . \qquad (2.5)$$

Here it should be noted that $\lambda_G = (\lambda_L^2 - G)^{1/2}$ is not to be confused with the effective penetration depth.

Then solving the Eq. (2.5) and using (2.1) we obtain

$$\vec{J}(\vec{x}) = \vec{J}_1(\vec{x}) + \vec{J}_2(\vec{x}) , \qquad (2.6)$$

$$\vec{J}_1(\vec{x}) = -(1/4\pi\lambda_L^2)\int c(\vec{x}-\vec{y})\vec{A}_1(\vec{y})\, d^3y, \qquad (2.7)$$

$$\vec{J}_2(\vec{x}) = \int D(\vec{x}-\vec{y})\vec{j}(\vec{y})\, d^3y , \qquad (2.8)$$

where

$$D(\vec{x}-\vec{y}) = \frac{e\lambda_L^2}{\lambda_G^2}\left[\delta(\vec{x}-\vec{y}) - \frac{1}{4\pi\lambda_G^2}\,\frac{1}{|\vec{x}-\vec{y}|} \times \exp\left(-\frac{|\vec{x}-\vec{y}|}{\lambda_G}\right)\right] . \qquad (2.9)$$

The equations for the magnetic fields $\vec{H}_1 = \vec{\nabla}\times\vec{A}_1$ and $\vec{H}_2 = \vec{\nabla}\times\vec{A}_2$ are

$$\int c(\vec{x}-\vec{y})\,\vec{H}_1(\vec{y})\, d^3y + \lambda_L^2\, \vec{\nabla}\times\vec{\nabla}\times\vec{H}_1(\vec{x}) = 0 , \qquad (2.10)$$

$$\vec{H}_2(\vec{x}) + \lambda_G^2\, \vec{\nabla}\times\vec{\nabla}\times\vec{H}_2(\vec{x}) = 4\pi e\lambda_L^2\, \vec{\nabla}\times\vec{j}(\vec{x}). \qquad (2.11)$$

Since $\vec{H}_1$ and $\vec{J}_1$ do not penetrate into the metal, we mainly consider $\vec{H}_2$ and $\vec{J}_2$. In Sec. III, $\vec{J}_2$ will be denoted simply by $\vec{J}$.

Summarizing, our computation procedure involves the following steps: Look for solutions of the Laplace equation (1.10) under appropriate boundary conditions to compute $f(\vec{x})$, then calculate the neutral current $\vec{j}(\vec{x})$ by means of (1.20); the charged current $\vec{J}(\vec{x})$ is then given by (2.6)–(2.8);

the ground-state energy of the neutral and charged systems can be computed by making use of (1.23) and (1.30), respectively.

III. FLUX LINES

In this remaining part of the paper we apply the boson method to the problem of flux lines in type-II superconductors. We consider a particular solution of the Laplace equation (1.10):

$$f(\vec{x}) = \tfrac{1}{2}\nu\,\phi(\vec{x}). \tag{3.1}$$

This is the simplest solution with a cylindrically symmetric nature: $\phi(\vec{x})$ is the polar angle of the vector $\vec{x}$ expressed in cylindrical coordinates and ν has to be an integer in order that the order parameter $\langle \psi_\uparrow \psi_\downarrow \rangle$ be single valued.[2] By using this solution we calculate in this section the distributions of current and magnetic field of a single vortex. The results cover the entire system, including the core. The function $c(\vec{x}-\vec{y})$ given in (1.17) plays a crucial role in the case: We show that it describes the structure of the vortex core. In Sec. IV, we consider the problem of the interaction between two vortices. When the distance d between two vortex lines is less than the coherence length, we find that two terms contribute to the interaction force: One is repulsive (this is the only term present for $d \gg \xi$), and the other is attractive. The latter contribution increases when the distance decreases.

In the special case where the core structure is approximated by a δ function and in the limit $\kappa = \lambda/\xi \gg 1$, we find exactly the same results as the Abrikosov theory,[11] based on the Ginzburg-Landau equations.

In the present framework of the boson method the results presented in this paper are valid for any temperature below T_c and in the approximation which neglects terms of the order of $1/\kappa^4$.

Finally, we would like to mention that there are two new approaches to the theory of type-II superconductors which try to give a description of the structure of the vortex line throughout the entire H, T plane. Bardeen *et al.*[12] have recently developed a new method for solving the Bogoliubov equations in the WKBJ approximation, and they are principally concerned with such quantities as pair potential and magnetic field in the vicinity of the core. Eilenberger[13] has constructed a method of solving the Gor'kov equations and in a recent paper,[14] in collaboration with Büttner, gives the description of a vortex line far from the center.

Neither of these methods is easy to use because the equations that must be solved contain the order parameter and the vector potential which must be determined self-consistently. In our approach, as we have pointed out in Sec. I, this complication is by-passed because the space-dependent order parameter and the solutions of the Gor'kov equations are obtained simultaneously through the boson transformation. Our method at the present time may still be incomplete because we take into account only part of the vector potential, using gauge-invariance considerations. Extension of the method is under study.

A. Neutral Case

Since $f(\vec{y})$ does not depend on the third coordinate y_3, we can define a new function

$$\bar{c}(\vec{y}) = \int dy_3\, c(\vec{y}) \tag{3.2}$$

and write for the neutral current

$$\vec{j}(\vec{x}) = \eta^2 v_0^2 \vec{\nabla} \int d^2y\, \bar{c}(\vec{x}-\vec{y}) f(\vec{y}). \tag{3.3}$$

Now $f(\vec{x})$, given by (3.1), is a multiply connected function; in order to calculate the current we have to define the differentiation in (3.3). First we write

$$C_f(\vec{x}) = \int d^2y\, \bar{c}(\vec{x}-\vec{y}) f(\vec{y})\,;$$

then for an infinitesimal translation $\vec{x} \to \vec{x} + \vec{\delta}$ we have

$$C_f(\vec{x}+\vec{\delta}) = \int d^2y\, \bar{c}(\vec{x}+\vec{\delta}-\vec{y}) f(\vec{y}) = \int d^2y\, \bar{c}(\vec{x}-\vec{y}) f(\vec{y}+\vec{\delta}).$$

In the last step we made an essential definition that $f(\vec{y}) = \frac{1}{2}\nu\,\phi(\vec{y})$ varies continuously with $\vec{y}$: This is because the boson transformation $B \to B_f$, with $f(\vec{x})$ given by (3.1), requires that the domain of $\phi(\vec{y})$ not be restricted to $0 < \phi < 2\pi$ but extend from $-\infty$ to $+\infty$.

Thus, defining

$$\vec{\nabla}_i \int d^2y\, \bar{c}(\vec{x}-\vec{y}) f(\vec{y}) = \lim_{\delta \to 0} \frac{C_f(\vec{x}+\delta\vec{e}_i) - C_f(\vec{x})}{\delta}$$

$$(\text{for } \vec{\delta} = \delta\vec{e}_i, \quad i = 1,\ 2),$$

we find that

$$\vec{j}(\vec{x}) = \eta^2 v_0^2 \int d^2y\, \bar{c}(\vec{x}-\vec{y}) \vec{\nabla} f(\vec{y}). \tag{3.4}$$

After a change of variables it is simple to calculate the angular part of the integral in (3.4), so that the persistent current for a neutral superconductor takes the value

$$\vec{j}(\vec{x}) = \frac{2\pi\eta^2 v_0^2 \nu}{2} \frac{\vec{e}(\phi_x)}{r_x} \int_0^{r_x} r_y\, \bar{c}(r_y)\, dr_y\,, \tag{3.5}$$

where $r_x = |\vec{x}|$ and $\vec{e}(\phi_x)$ is the unit vector in the azimuthal direction, i.e., $\vec{e}(\phi_x) \equiv (-x_2/r_x,\ x_1/r_x)$. Since $c(\vec{x})$ and $\bar{c}(\vec{x})$ are functions only of $|\vec{x}|$, we frequently denote them as $c(r_x)$ and $\bar{c}(r_x)$. If we recall that the c function is normalized by (1.19), i.e.,

$$2\pi \int_0^\infty r\bar{c}(r)\, dr = 1,$$

we can rewrite (3.5)

$$\vec{j}(\vec{x}) = \frac{\eta^2 v_0^2 \nu}{2} \frac{\vec{e}(\phi_x)}{r_x} \left(1 - 2\pi \int_{r_x}^\infty r_y\, \bar{c}(r_y)\, dr_y\right). \tag{3.6}$$

This describes the current distribution in the neutral system. From (3.5) and (3.6) we easily obtain

$$\lim_{r_x\to 0}\vec{j}(\vec{x})=0,\quad \vec{\nabla}\cdot\vec{j}(\vec{x})=0,$$

$$\vec{\nabla}^2\vec{j}(\vec{x})=\pi\eta^2 v_0^2\nu\vec{e}(\phi_x)\bar{c}'(r_x), \tag{3.7}$$

$$\vec{\nabla}\times\vec{j}(\vec{x})=\pi\eta^2 v_0^2\nu\,\vec{e}_3\bar{c}(r_x)\ ,\ \oint\vec{j}(\vec{x})\cdot d\vec{l}=\pi\eta^2 v_0^2\nu,$$

where

$$\bar{c}'(r_x)=\frac{d\bar{c}}{dr_x},$$

$\vec{e}_3$ is the unit vector along the third axis, and the path of the last integral is a circle of radius $R\to\infty$. From these results we can derive the following conclusions: The circulation of velocity is quantized in units of $2\pi/2m$:

$$K=\oint\vec{v}_s\cdot d\vec{l}=(2\pi/2m)\,\nu.$$

The persistent current goes to zero at the center, behaves far from the axis like the well-known result

$$\vec{j}(\vec{x})=\frac{n_s K}{2\pi}\frac{\vec{e}(\phi_x)}{r_x}\quad\text{for } r_x\gg\xi\ , \tag{3.8}$$

and, in the intermediate region, is strongly modified owing to the c function, as is shown by (3.6).

From (3.8) we can deduce how the superelectron density varies:

$$n_s(\vec{x})=2\pi n_s\int_0^{r_x} r_y\bar{c}(r_y)\,dr_y\ ,$$

in other words,

$$\frac{n_s(\vec{x})}{n_s}=\begin{cases}c_1 r_x^2 & \text{for } r_x\ll\xi\\ 1-c_2/r_x^2 & \text{for } r_x\gg\xi,\end{cases}$$

where c_1 and c_2 are two constants:

$$c_1=\pi\bar{c}(0),\quad c_2=\tfrac{1}{2}\pi\int_0^\infty r^3 c(r)\,dr.$$

To illustrate these results let us use the naive expression (1.18) for $c(\vec{x})$. In this case we obtain

$$\bar{c}(\vec{x})=\frac{1}{2\pi\xi r_x}\,J_1\left(\frac{r_x}{\xi}\right)\ ,$$

where J_1 is the Bessel function of first order. Therefore

$$n_s(\vec{x})=n_s\left[1-J_0\left(\frac{r_x}{\xi}\right)\right].$$

B. Charged Case

Now that the problem is settled for the neutral case, we consider a charged superconductor. Substituting the expression (3.5) in (2.8) we get, after a straightforward calculation (see Appendix B), the following vortex current:

$$\vec{J}(\vec{x})=\frac{\nu}{4e\lambda_G^3}\vec{e}(\phi_x)\left[K_1\left(\frac{r_x}{\lambda_G}\right)\int_0^{r_x} r_y\bar{c}(r_y)I_0\left(\frac{r_y}{\lambda_G}\right)dr_y - I_1\left(\frac{r_x}{\lambda_G}\right)\int_{r_x}^\infty r_y\bar{c}(r_y)K_0\left(\frac{r_y}{\lambda_G}\right)dr_y\right]\ , \tag{3.9}$$

where K_i, $I_i(i=0,\ 1)$ are the usual modified Bessel functions. From (2.11) we can express the magnetic field $\vec{h}\equiv\vec{H}_2$ in terms of the neutral and realistic currents:

$$\vec{h}(\vec{x})=4\pi\vec{\nabla}\times[e\lambda_L^2\,\vec{j}(\vec{x})-\lambda_G^2\,\vec{J}(\vec{x})]\ .$$

Then, using the expressions (3.5) and (3.9), we find (see Appendix C)

$$\vec{h}(\vec{x})=\frac{\pi\nu}{e\lambda_G^2}\vec{e}_3\left[K_0\left(\frac{r_x}{\lambda_G}\right)\int_0^{r_x} r_y\bar{c}(r_y)I_0\left(\frac{r_y}{\lambda_G}\right)dr_y + I_0\left(\frac{r_x}{\lambda_G}\right)\int_{r_x}^\infty r_y\bar{c}(r_y)\ K_0\left(\frac{r_y}{\lambda_G}\right)dr_y\right]. \tag{3.10}$$

Now observing that $\vec{h}(\vec{x})\to 0$ as $\exp[-r_x/\lambda_G]\,r_x^{-1/2}$ for $r_x\gg\lambda_G$, we can immediately calculate the total magnetic flux; we integrate (2.11) over a circular surface of radius $R\gg\lambda$:

$$\int d\vec{S}_n\cdot\vec{h}(\vec{x})+\lambda_G^2\int d\vec{S}_n\cdot\vec{\nabla}\times\vec{\nabla}\times\vec{h}(\vec{x}) = 4\pi e\lambda_L^2\int d\vec{S}_n\cdot\vec{\nabla}\times\vec{j}(\vec{x}).$$

The second integral does not give any contribution and we are left with

$$\Phi\equiv\int d\vec{S}_n\cdot\vec{h}(\vec{x})=4\pi e\lambda_L^2\int d\vec{S}_n\cdot\vec{\nabla}\times\vec{j}(\vec{x})=(\pi/e)\,\nu\ , \tag{3.11}$$

where use was made of (3.7). An equivalent way to get this result is to compute the flux of the magnetic field directly by using expression (3.10): The calculation is very simple as is shown in Appendix C and the result agrees with (3.11), as it should. Applying similar considerations to Eq. (2.10) and making use of (1.19) we find that the Meissner magnetic field $\vec{H}_1$ does not contribute to the magnetic flux. Thus Φ accounts for the total magnetic flux, which turns out, from (3.11), to be quantized in units π/e. It is interesting that the total flux is induced by the line integral of the *neutral* current.

Let us analyse the results obtained, recalling that the function $\bar{c}(r_x)$ is practically confined in the domain $0\le r_x\le\xi$, and using the asymptotic expressions for the Bessel functions for small values of the argument:

$$I_0(x)=1+O(x^2),\quad K_0(x)=\ln(1/x),$$

$$I_1(x)=\tfrac{1}{2}x+O(x^3),\quad K_1(x)=1/x\ .$$

At first we would like to emphasize that in the approximation considered (see Sec. I) the expressions given above represent the distribution of the current and of the magnetic field at any point inside the superconductor and are valid for any value of the parameter $\kappa(1/\sqrt{2}<\kappa<\infty)$ when the c function satisfies

$$\int c(\vec{x}-\vec{y})c(\vec{y}-\vec{z})\,d^3y=c(\vec{x}-\vec{z}).$$

Otherwise the error is of the order of $(1/\kappa)^4$. Thus, the result may not require $\lambda\gg\xi$: e.g., $1/\kappa^4\simeq 10^{-2}$ for $\lambda\simeq 3\xi$. We observe in (3.9) and (3.10) that two terms contribute to the current and magnetic field: The first one represents the behavior for large values of r_x(i.e., $r_x \gg \xi$), the second one is appreciable only for $r_x \leq \xi$ and thus represents the modification due to the effect of the core.

(a) $r_x \to 0$:

$$\lim_{r_x \to 0} \vec{J}(\vec{x}) = 0,$$

$$h(0) = \frac{\Phi}{\lambda_G^2} \int_0^\infty r_y \bar{c}(r_y) K_0\left(\frac{r_y}{\lambda_G}\right) dr_y. \qquad (3.12)$$

At the center, the current goes naturally to zero, and the magnetic field reaches its maximum value.

(b) $0 < r_x \leq \xi$: We find that inside the core the current $\vec{J}$ does not have the same behavior as the neutral one $\vec{j}$, but is modified by the second term in (3.9). The magnetic field decreases slowly.

(c) $r_x \gg \xi$:

$$\vec{J}(\vec{x}) = \frac{\Phi}{4\pi\lambda_G^3} \vec{e}(\phi_x) K_1\left(\frac{r_x}{\lambda_G}\right) \int_0^{r_x} r_y \bar{c}(r_y) I_0\left(\frac{r_y}{\lambda_G}\right) dr_y, \qquad (3.13a)$$

$$\vec{h}(\vec{x}) = \frac{\Phi}{\lambda_G^2} \vec{e}_3 K_0\left(\frac{r_x}{\lambda_G}\right) \int_0^{r_x} r_y \bar{c}(r_y) I_0\left(\frac{r_y}{\lambda_G}\right) dr_y. \qquad (3.13b)$$

The effect of the core is practically negligible.

(*d*) $r_x \gg \lambda_G$: The current and field fall off rapidly, owing to the superconducting screening effect manifested by $K_1(r_x/\lambda_G)$ and $K_0(r_x/\lambda_G)$. From the expressions (3.13) we can conclude that our theory agrees with the results of the usual theory[11,15] when we consider the limit $\kappa \gg 1$ (in which case $\lambda_G \approx \lambda_L$) and distances far from the center:

$$\vec{h}(\vec{x}) = \frac{\Phi}{2\pi\lambda_L^2} K_0\left(\frac{r_x}{\lambda_L}\right) \vec{e}_3 \text{ for } r_x \gg \xi, \quad \lambda \gg \xi.$$

$$\vec{J}(\vec{x}) = \frac{\Phi}{8\pi^2\lambda_L^3} K_1\left(\frac{r_x}{\lambda_L}\right) \vec{e}(\phi_x),$$

Let us close this section by noting that when we use for $c(\vec{x}-\vec{y})$ the simple expression (1.18), the following results can be obtained:

$$\vec{J}(\vec{x}) = \frac{\Phi}{8\pi^2\lambda_L^2} \vec{e}(\phi_x) \int_0^{1/\xi} \frac{t^2\, dt}{t^2 + 1/\lambda_L^2} J_1(r_x t),$$

$$\vec{h}(\vec{x}) = \frac{\Phi}{2\pi\lambda_L^2} \vec{e}_3 \int_0^{1/\xi} \frac{t\, dt}{t^2 + 1/\lambda_L^2} J_0(r_x t), \qquad (3.14)$$

where J_0 and J_1 are Bessel functions. It is interesting to compare the expression for the magnetic field at the center that can be derived from (3.14)

$$h(0) = (\Phi/4\pi\lambda_L^2) \ln(1 + \lambda_L^2/\xi^2)$$

and the results obtained by Neumann and Tewordt[16] by solving numerically the G-L equations. These authors calculate the magnetic field in units of the critical field H_c for the values $\kappa = 1/\sqrt{2}$, 1, 2, 5. The agreement is very good for $\kappa = 5$, but not so good for smaller values of κ.

IV. INTERACTION ENERGY BETWEEN FLUX LINES

A. Neutral Case

The energy due to the persistent current can be computed by putting (3.5) into (1.23). We find that the self-energy per unit length of a flux line has the value

$$E = \pi^3\eta^2 v_0^2 \nu^2 \int_0^\infty r_y \bar{c}(r_y)\, dr_y \int_{r_y}^\infty r_z \bar{c}(r_z) \ln\left(\frac{R}{r_z}\right) dr_z,$$

where R is the radius of the vortex.

Now let us consider a system composed of two flux lines and study the static situation when the centers of the lines are in position $\vec{a}$ and $\vec{b}$, respectively. According to (1.23) the total energy of the system is

$$E = (1/2\eta^2 v_0^2) \int d^3x\, [\vec{j}^{(a)}(\vec{x}) + \vec{j}^{(b)}(\vec{x})] \cdot [\vec{j}^{(a)}(\vec{x}) + \vec{j}^{(b)}(\vec{x})]$$
$$= E^{(a)} + E^{(b)} + (1/\eta^2 v_0^2) \int d^3x\, \vec{j}^{(a)}(\vec{x}) \cdot \vec{j}^{(b)}(\vec{x}),$$

where $E^{(a)}$ and $E^{(b)}$ are the self-energies, and the last term represents the interaction energy between the lines. Let us study this term. By using the expression (3.5) we find that the interaction energy per unit length is given by

$$E^{(ab)} = \pi^2\eta^2 v_0^2 \nu^2 \int d^2x \frac{\vec{e}(\phi_{xa}) \cdot \vec{e}(\phi_{xb})}{r_{xa} r_{xb}} \times \int_0^{r_{xa}} r_y \bar{c}(r_y)\, dr_y \int_0^{r_{xb}} r_z \bar{c}(r_z)\, dr_z,$$

where the following notation is used:

$$r_{xm} = |\vec{x} - \vec{m}|, \quad \vec{e}(\phi_{xm}) = \left(-\frac{x_2 - m_2}{r_{xm}}, \frac{x_1 - m_1}{r_{xm}}\right) \quad (\vec{m} = \vec{a}, \vec{b}).$$

The calculation of this expression gives[17]

$$E^{(ab)} = -\frac{\pi\eta^2 v_0^2 \nu^2}{2} \left[\ln d + 2\pi \int_d^\infty r_y \bar{c}(r_y) \ln\left(\frac{r_y}{d}\right) dr_y\right]$$
$$+ \pi^2\eta^2 v_0^2 \nu^2 \int_0^\infty r_y \bar{c}(r_y)\, dr_y \int_0^\infty r_z \bar{c}(r_z)\, dr_z \times \int d^2x \frac{\vec{e}(\phi_x) \cdot \vec{e}(\phi_{xd})}{r_x r_{xd}} \theta(r_y - r_x)\, \theta(r_z - r_{xd}),$$

where $\vec{d} = \vec{b} - \vec{a}$ and d is the distance between centers of two vortices, i.e., $d = |\vec{d}|$. The last term in this expression comes from the core-core interaction; it is easily seen that this term is zero when $d \gg \xi$, while for small distances it seems to be predominant. When $d \gg \xi$ we have the usual result

$$E^{(ab)} = -\tfrac{1}{2}\pi\eta^2 v_0^2 \nu^2 \ln d.$$

Therefore for large values of the separation distance we find the usual repulsive force between the two lines

$$\vec{F}(d) = -\vec{\nabla}_d E^{(ab)} = \tfrac{1}{2}\pi\eta^2 v_0^2 \nu^2 \, \vec{d}/d^2 ,$$

while for small distances the results are strongly modified by the effect of the core-core interaction.

B. Charged Case

In the charged case there is a contribution to the total energy that comes from the magnetic field associated with the lines. Therefore the energy of the system is given by the expression

$$\langle H\rangle = \int d^3x \left[\frac{h^2(\vec{x})}{8\pi} + \frac{\vec{J}(\vec{x})\cdot\vec{J}(\vec{x})}{2e^2\eta^2 v_0^2}\right] ,$$

where use was made of (1.30). By using the equation for the magnetic field (2.11) the last expression can be written as

$$\langle H\rangle = \tfrac{1}{2}\lambda_G^2 \int d^3x\, \vec{\nabla}\cdot[\vec{h}(\vec{x})\times\vec{J}(\vec{x})] + \tfrac{1}{2}e\lambda_L^2 \int d^3x\, \vec{h}(\vec{x})\cdot\vec{\nabla}\times\vec{j}(\vec{x}) + 2\pi(\lambda_L^2 - \lambda_G^2)\int d^3x\, \vec{J}(\vec{x})\cdot\vec{J}(\vec{x}). \qquad (4.1)$$

Now because $\vec{h}(\vec{x})$ and $\vec{J}(\vec{x})$ are regular functions (i.e., they do not possess any singularity at the center), it is easy to see that the first term in (4.1) does not contribute. The third term represents a correction to the energy of the order of ξ^2/λ^2: For the sake of simplicity, in the following, we shall not consider this term, though this quantity can be calculated by means of expression (3.9) for the charged current. Now the energy of the system is

$$\langle H\rangle = \tfrac{1}{2}e\lambda_L^2 \int d^3x\, \vec{h}(\vec{x})\cdot\vec{\nabla}\times\vec{j}(\vec{x}).$$

From the results (3.7) and (3.10) we obtain for the line self-energy (per unit length)

$$E = \frac{\Phi^2}{2\lambda_G^2}\int_0^\infty r_x \bar{c}(r_x) K_0\left(\frac{r_x}{\lambda_G}\right) dr_x \times \int_0^{r_x} r_y\, \bar{c}(r_y) I_0\left(\frac{r_y}{\lambda_G}\right) dr_y . \qquad (4.2)$$

The usual theory gives for this quantity the expression

$$E = (\Phi/4\pi\lambda_L)^2 \ln(\lambda_L/\xi). \qquad (4.3)$$

This result represents a rough approximation of (4.2) in the limit $\lambda \gg \xi$.[18] In a similar way we can calculate the interaction energy per unit length between two lines, whose centers are separated by a distance d:

$$E(d) = \frac{\Phi^2}{4\pi\lambda_G^2}\int d^2x\, \bar{c}(r_{xd})\left[K_0\left(\frac{r_x}{\lambda_G}\right)\int_0^{r_x} r_y \bar{c}(r_y) I_0\left(\frac{r_y}{\lambda_G}\right) dr_y + I_0\left(\frac{r_x}{\lambda_G}\right)\int_{r_x}^\infty r_y \bar{c}(r_y) K_0\left(\frac{r_y}{\lambda_G}\right) dr_y\right] . \qquad (4.4)$$

In this expression the second term is due to the core-core interaction and is practically zero when $d \gg \xi$. It can be proved, by using the asymptotic expressions for the Bessel functions, that in the limit $\lambda \gg \xi$ and for distances $d \gg \xi$, expression (4.4) gives the well-known result[11,15]

$$E(d) = \frac{\Phi^2}{8\pi^2\lambda_L^2} K_0\left(\frac{d}{\lambda_L}\right) \quad \text{for } \lambda \gg \xi, \quad d \gg \xi.$$

By using in (4.2) and (4.4) the naive expression (1.18) for $c(\vec{x})$ we obtain

$$E = \left(\frac{\Phi}{1\pi\lambda_L}\right)^2 \frac{1}{2}\ln\left(1 + \frac{\lambda_L^2}{\xi^2}\right) ,$$

$$E(d) = \frac{\Phi^2}{8\pi^2\lambda_L^2}\int_0^{1/\xi} \frac{t\,dt}{t^2 + 1/\lambda_L^2} J_0(dt).$$

We notice in these expressions that the self-energy of a vortex is related to the value of the magnetic field at its center

$$E = (\Phi/8\pi)h(0)$$

and that the interaction energy between two vortices can be expressed in terms of the magnetic field of a line as

$$E(d) = (\Phi/4\pi)h(d).$$

The potential energy (4.4) leads to the following interaction force in the direction of $\vec{d}$:

$$F(d) = -\frac{\Phi^2}{4\pi\lambda_G^3}\int d^2x\, \bar{c}(r_x)\frac{\partial r_{xd}}{\partial d}\left[K_1\left(\frac{r_{xd}}{\lambda_G}\right)\int_0^{r_{xd}} r_y \bar{c}(r_y) I_0\left(\frac{r_y}{\lambda_G}\right) dr_y - I_1\left(\frac{r_{xd}}{\lambda_G}\right)\int_{r_{xd}}^\infty r_y \bar{c}(r_y) K_0\left(\frac{r_y}{\lambda_G}\right) dr_y\right] .$$

Because the Bessel functions are positive and the integral of the function $\bar{c}(r_x)$ is positive definite, we find that the force is composed of two terms, one repulsive and one attractive. It should be noted that the attractive part is present only when $d \le \xi$.

V. CONCLUSIONS

Our method of computation can be summarized as follows: Solve the Laplace equation for the phase, calculate the neutral persistent current, and, from the latter result, calculate the charged persistent current; the ground-state energy and the magnetic field can be obtained from these currents. All these results are expressed in terms of the c function. We have not studied the c function in detail; it is one of our future programs to compute it by the boson formalism itself. As an application we have studied the vortices and computed the currents, magnetic fields, vortex self-energies, and the mutual interactions among the vortices. The approximation used is to ignore the terms of order $(1/\kappa)^4(\kappa=\lambda/\xi)$ so that the results can be applied to a wide range of κ.

Our expressions for the currents and magnetic fields are valid even in the vortex core. These quantities have been computed by many authors; in the region far from the center our results coincide with theirs. The mutual force between two vortices was obtained for arbitrary distances between the vortices. For large distances it agrees with the result found in the literature, whereas at short distances there appears an extra attractive force. Our result for the self-energy of a single vortex agrees with the known result only when we approximate the c function by the δ function. The precise form of the self-energy is important for the computation of the critical magnetic field.

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APPENDIX A

In the case where $f(\vec{x})$ is not confined to a finite domain, so that the partial integration is not permitted, the proof of Eq. (1.15) can be modified as follows. Let us consider the commutator

$$[\dot{N}_f^{(2)},\ \pi(\vec{x},t)]=-iv_0^2\int d^3y\,f(\vec{y})\eta(\vec{\nabla}_y)\vec{\nabla}_y^2 d(\vec{y}-\vec{x}),$$

where use was made of Eqs. (1.14) and (1.19). We can now perform the partial integration because $d(\vec{y}-\vec{x})$ is a function of finite range. We then find that

$$[\dot{N}_f^{(2)},\ \pi(\vec{x},t)]=-iv_0^2\int d^3y\,\vec{\nabla}^2 f(\vec{y})\eta(\vec{\nabla}_y)d(\vec{y}-\vec{x})=0,$$

owing to the Laplace equation for $f(\vec{y})$. Furthermore, it is obvious that $\dot{N}_f^{(2)}$ commutes with $B(\vec{x})$ and $\phi(\vec{x})$ (the quasielectron field). Thus $\dot{N}_f^{(2)}$ must be a c number. We then have

$$\dot{N}_f^{(2)}=\langle 0\,|\,\dot{N}_f^{(2)}\,|\,0\rangle;$$

on the other hand

$$\langle 0\,|\,\dot{N}_f^{(2)}\,|\,0\rangle=0$$

because

$$\langle 0\,|\,B\,|\,0\rangle=0.$$

We have thus proved that $\dot{N}_f^{(2)}$ is zero.

This proof, however, is too formal, because we know that $N_f^{(2)}$ is not well defined as an operator acting on the Hilbert space of the quasifermions and quasibosons when $f(\vec{x})$ extends to an infinite domain. As was pointed out in Ref. 19, we consider in such a case a finite tridimensional domain $D(L)$, defined as the set of points $\vec{x}$ such that $|\vec{x}|\le L$, and introduce a function $g(\vec{x})$ such that

$$g(\vec{x})=\begin{cases}1 & \text{for } \vec{x}\in D(L)\\ 0 & \text{for } |\vec{x}|\gg L\,.\end{cases}$$

We then construct a new operator $N_{fg}^{(2)}$ defined as

$$N_{fg}^{(2)}=\int d^3x\,g(\vec{x})f(\vec{x})\rho^{(2)}(\vec{x})=-\int d^3x\,g(\vec{x})f(\vec{x})\,\eta(\vec{\nabla})\pi(\vec{x}),$$

where $f(\vec{x})$ satisfies the Laplace equation (1.10). The boson transformation of any operator $A(\vec{x})$ is then defined by

$$A_f(x)=\lim_{L\to\infty}\exp(-iN_{fg}^{(2)})A(\vec{x})\exp(iN_{fg}^{(2)})\ , \qquad \text{(A1)}$$

which is, in many cases, well defined, while

$$\lim_{L\to\infty}\exp(iN_{fg}^{(2)})$$

is not. For instance for $\vec{x}\in D(L)$

$$\begin{aligned}B_f(\vec{x})&=\lim_{L\to\infty}\exp(-iN_{fg}^{(2)})\,B(\vec{x})\exp(iN_{fg}^{(2)})\\&=B(\vec{x})+i\lim_{L\to\infty}\int d^3y\,g(\vec{y})f(\vec{y})\,\eta(\vec{\nabla}_y)\,d(\vec{x}-\vec{y})\\&=B(\vec{x})+i\int d^3y\,f(\vec{y})\,\eta(\vec{\nabla}_y)\,d(\vec{x}-\vec{y})\end{aligned}$$

because $d(\vec{x}-\vec{y})$ has a short range. Therefore for $\vec{x}\in D(L)$ the definition (A1) of the boson transformation coincides with (1.8) given in the text.

Following an argument similar to the one applied to $\dot{N}_f^{(2)}$, we can show that $\dot{N}_{fg}^{(2)}$ commutes with $\pi(\vec{x})$, $B(\vec{x})$, and $\phi(\vec{x})$ when $\vec{x}\in D(L)$. We then conclude that $\dot{N}_{fg}^{(2)}$ acts as a null operator in the algebra of operators in $D(L)$. This conclusion with the limit $L\to\infty$ is the precise meaning of the previous statement giving $\dot{N}_f^{(2)}=0$.

Since $D(L)$ represents the domain of the system under consideration, the integration of any observable $A(\vec{x})$,

$$\int d^3x\,A(\vec{x}),$$

should be limited to the domain $D(L)$ with L being practically infinite. In this way such an operator belongs to the algebra of operators in $D(L)$. We then see that the boson transformation of these observables can be performed simply by the replacement (1.8). For example, the boson transformation induces the following change in the Hamiltonian (1.1) [see (1.22)]:

$$v_0^2 M_f + c \text{ number},$$

with

$$M_f=\int d^3x\,\vec{\nabla}B(\vec{x})\cdot\vec{\nabla}f(\vec{x}),$$

$$M_f=\int d^3x\,\vec{\nabla}B\cdot\vec{\nabla}\int d^3y\,f(\vec{y})\eta(\vec{\nabla}_y)d(\vec{x}-\vec{y}).$$

It is easy to show that M_f commutes with $\pi(\vec{x})$, $B(\vec{x})$, and $\phi(\vec{x})$ when $\vec{x}\in D(L)$ and therefore that M_f is a null operator. In this sense the boson transformation leaves invariant the q part of the Hamiltonian and modifies it only by a c number. It must be noted that this assertion is not in conflict with the result previously found that $\mathring{N}_{fg}^{(2)}$ is a null operator in the algebra $D(L)$. As we have already mentioned, if $h(\vec{x})$ is the Hamiltonian density, our Hamiltonian is given by

$$H=\int d^3x\, h(\vec{x}),$$

where the integration domain is confined to $D(L)$ so that our Hamiltonian belongs to the algebra $D(L)$. On the other hand $N_{fg}^{(2)}$ is defined on an algebra bigger then $D(L)$, because g extends out of $D(L)$. This means that

$$\dot{N}_{fg}^{(2)}\neq[H,N_{fg}^{(2)}].$$

To give an intuitive description of the argument exposed in this appendix we note that as a first step in our approach we solve the equations of motion (1.11) in an homogeneous system that extends to infinity. Until this point all the quantities are defined in the whole domain. When, through the boson transformation, we induce the space dependence, we must limit ourselves to the dimensions, extremely big but finite, of the system under consideration. The boson transformation and the invariance of the equations under this transformation are properly defined in this restricted domain.

APPENDIX B

Calculation of the charged current. According to (2.8) the charged current $\vec{J}$ is given by

$$\vec{J}(\vec{x})=\frac{\lambda_L^2}{\lambda_G^2}\left[e\vec{j}(x)-\frac{e}{4\pi\lambda_G^2}\int d^3y\,\exp\left(-\frac{|\vec{x}-\vec{y}|}{\lambda_G}\right)\times\frac{\vec{j}(\vec{y})}{|\vec{x}-\vec{y}|}\right].$$

Since the neutral current $\vec{j}(\vec{x})$ does not depend on the third coordinate, we can immediately calculate the integral over y_3:

$$\int dy_3\exp\left(-\frac{|\vec{x}-\vec{y}|}{\lambda_G}\right)\frac{1}{|\vec{x}-\vec{y}|}=2K_0\left(\frac{|\vec{x}-\vec{y}|}{\lambda_G}\right),$$

where K_0 is the modified Bessel function of zero order; then using the expression (3.6) we obtain[20]

$$\vec{J}(\vec{x})=\frac{\lambda_L^2}{\lambda_G^2}\left[e\vec{j}(\vec{x})-\frac{e\eta^2v_0^2\nu}{4\pi\lambda_G^2}\int d^2y\,K_0\left(\frac{r_{xy}}{\lambda_G}\right)\frac{\vec{e}(\phi_y)}{r_y}\times\left(1-2\pi\int_{r_y}^{\infty}r_z\bar{c}(r_z)\,dr_z\right)\right].$$

The calculation of the first integral gives us

$$\int d^2y\,K_0\left(\frac{r_{xy}}{\lambda_G}\right)\frac{\vec{e}(\phi_y)}{v_y}=2\pi\lambda_G^2\frac{\vec{e}(\phi_x)}{r_x}\left[1-\frac{r_x}{\lambda_G}K_1\left(\frac{r_x}{\lambda_G}\right)\right],$$

where K_1 is the modified Bessel function of first order and use was made of the formula (see 6.5618 of Ref. 21)

$$\int_0^1 x^{\nu+1}K_\nu(ax)\,dx=2^{-\nu-2}\Gamma(\nu+1)-a^{-1}K_{\nu+1}(a)\qquad(\mathrm{Re}\,\nu>-1).$$

When we define

$$\vec{P}(\vec{x})=\frac{1}{\lambda_G}\int d^2y\,K_0\left(\frac{r_{xy}}{\lambda_G}\right)\frac{\vec{e}(\phi_y)}{r_y}\int_{r_y}^{\infty}r_z\bar{c}(r_z)\,dr_z$$

the charged current can be written as

$$\vec{J}(\vec{x})=\frac{\lambda_L^2}{\lambda_G^3}\left\{\frac{e\eta^2v_0^2\nu}{2\lambda_G}\vec{e}(\phi_x)\left[K_1\left(\frac{r_x}{\lambda_G}\right)-\frac{2\pi\lambda_G}{r_x}\int_{r_x}^{\infty}r_y\bar{c}(r_y)\,dr_y\right]+\vec{P}(\vec{x})\right\}.\qquad(\mathrm{B1})$$

To calculate the quantity $\vec{P}(\vec{x})$ let us take, for simplicity, $\vec{x}$ on the first axis and consider the second component of $\vec{e}(\phi_y)$ [the first component of $\vec{e}(\phi_y)$ gives zero as a result]:

$$P(\vec{x})=\frac{1}{\lambda_G}\int_0^{\infty}r_z\bar{c}(r_z)\,dr_z\int_0^{r_z}dr_y\times\int_0^{2\pi}d\phi_y K_0\left(\frac{r_{xy}}{\lambda_G}\right)\cos\phi_y.$$

In order to calculate the angular part let us use the integral representation of K_0 (see 8.4327 of Ref. 21)

$$K_\nu(xz)=\frac{z^\nu}{2}\int_0^{\infty}\exp\left[-\frac{x}{2}\left(t+\frac{z^2}{t}\right)t^{-\nu-1}dt\right]\qquad(|\arg z|<\tfrac{1}{4}\pi)$$

so that

$$\int_0^{2\pi}d\phi_y K_0\left(\frac{r_{xy}}{\lambda_G}\right)\cos\phi_y=\tfrac{1}{2}\int_0^{2\pi}d\phi_y\cos\phi_y\times\int_0^{\infty}\exp\left[-\frac{1}{2\lambda_G}\left(t+\frac{r_{xy}^2}{t}\right)\right]\frac{dt}{t}$$
$$=\pi\int_0^{\infty}\frac{dt}{t}\exp\left[-\frac{t}{2\lambda_G}-\left(\frac{r_y^2+r_x^2}{2\lambda_G t}\right)\right]I_1\left(\frac{r_xr_y}{\lambda_G t}\right)$$
$$=2\pi\left[I_1\left(\frac{r_y}{\lambda_G}\right)K_1\left(\frac{r_x}{\lambda_G}\right)\theta(r_x-r_y)+I_1\left(\frac{r_x}{\lambda_G}\right)K_1\left(\frac{r_y}{\lambda_G}\right)\theta(r_y-r_x)\right],$$

where in the last step we used the formula (see 6.6531 of Ref. 21)

$$\int_0^{\infty}\exp\left(-\tfrac{1}{2}x-\frac{1}{2x}(a^2+b^2)\right)I_n\left(\frac{ab}{x}\right)\frac{dx}{x}=\begin{cases}2I_n(a)K_n(b) & 0<a<b\\ 2I_n(b)K_n(a) & 0<b<a\end{cases}\qquad(\mathrm{Re}\,n>-1).$$

Therefore

$$\vec{P}(x)=\frac{2\pi}{\lambda_G}\vec{e}(\phi_x)\int_0^\infty r_z\bar{c}(r_z)\,dr_z\int_0^{r_z}dr_y\left[I_1\left(\frac{r_y}{\lambda_G}\right)K_1\left(\frac{r_x}{\lambda_G}\right)\theta(r_x-r_y)+I\left(\frac{r_x}{\lambda_G}\right)K_1\left(\frac{r_y}{\lambda_G}\right)\theta(r_y-r_x)\right].$$

By recalling that

$$I_1(z)=\frac{d}{dz}I_0(z),\quad K_1(z)=-\frac{d}{dz}K_0(z),$$

the integration in r_y is easily performed to give the result

$$\vec{P}(x)=\vec{e}(\phi_x)\left\{2\pi\left[K_1\left(\frac{r_x}{\lambda_G}\right)\int_0^{r_x}r_z\bar{c}(r_z)I_0\left(\frac{r_z}{\lambda_G}\right)dr_z-I_1\left(\frac{r_x}{\lambda_G}\right)\int_{r_x}^\infty r_z\bar{c}(r_z)K_0\left(\frac{r_z}{\lambda_G}\right)dr_z\right]-K_1\left(\frac{r_x}{\lambda_G}\right)+\frac{2\pi\lambda_G}{r_x}\int_{r_x}^\infty r_z\bar{c}(r_z)\,dr_z\right\},$$

where use was made of the well-known Wronskian property

$$I_n(z)K_{n+1}(z)+I_{n+1}(z)K_n(z)=1/z.$$

Substituting the expression obtained for $\vec{P}(x)$ in (B1) we find the result (3.9).

APPENDIX C

Calculation of the magnetic field. As we have seen in Sec. III B, we need to compute the curl of the charged current to obtain the magnetic field:

$$\vec{\nabla}\times\vec{J}(\vec{x})=\frac{\Phi}{4\pi\lambda_G}\vec{\nabla}\times\vec{e}(\phi_x)\left[K_1\left(\frac{r_x}{\lambda_G}\right)\int_0^{r_x}r_y\bar{c}(r_y)I_0\left(\frac{r_y}{\lambda_G}\right)dr_y-I_1\left(\frac{r_x}{\lambda_G}\right)\int_{r_x}^\infty r_y\bar{c}(r_y)K_0\left(\frac{r_y}{\lambda_G}\right)dr_y\right].$$

Now using the formula

$$\vec{\nabla}\times[\vec{e}(\phi_x)g(r_x)]=\vec{e}_3\left[\frac{g(r_x)}{r_x}+\frac{dg(r_x)}{dr_x}\right],$$

where $g(r_x)$ is any regular function of the modulus $|\vec{x}|=r_x$, and using the recurrence formulas for the Bessel functions

$$\frac{d}{dz}K_1(z)=-K_0(z)-\frac{1}{z}K_1(z),\qquad\frac{d}{dz}I_1(z)=I_0(z)-\frac{1}{z}I_1(z),$$

we obtain

$$\vec{\nabla}\times\vec{J}(\vec{x})=\frac{\Phi}{4\pi\lambda_G^3}\vec{e}_3\left\{r_x\bar{c}(r_x)\times\left[K_1\left(\frac{r_x}{\lambda_G}\right)I_0\left(\frac{r_x}{\lambda_G}\right)+I_1\left(\frac{r_x}{\lambda_G}\right)K_0\left(\frac{r_x}{\lambda_G}\right)\right]-\frac{1}{\lambda_G}K_0\left(\frac{r_x}{\lambda_G}\right)\int_0^{r_x}r_y\bar{c}(r_y)I_0\left(\frac{r_y}{\lambda_G}\right)dr_y-\frac{1}{\lambda_G}I_0\left(\frac{r_x}{\lambda_G}\right)\int_{r_x}^\infty r_y\bar{c}(r_y)K_0\left(\frac{r_y}{\lambda_G}\right)dr_y\right\}.$$

This, together with the fourth relation in (3.7) and the Wronskian property for the Bessel functions, gives the value of the magnetic field (3.10).

Calculation of the magnetic flux. In this last part of the appendix we want to show how to calculate the magnetic flux directly from the expression (3.10) for the magnetic field:

$$\Phi=\int\vec{h}(\vec{x})\cdot dS_n=\frac{2\pi^2\nu}{e\lambda_G^2}\int_0^\infty r_x\,dr_x\left[K_0\left(\frac{r_x}{\lambda_G}\right)\int_0^{r_x}r_y\bar{c}(r_y)I_0\left(\frac{r_y}{\lambda_G}\right)dr_y+I_0\left(\frac{r_x}{\lambda_G}\right)\int_{r_x}^\infty r_y\bar{c}(r_y)K_0\left(\frac{r_y}{\lambda_G}\right)dr_y\right].$$

By using the formulas

$$\int_1^\infty xK_0(ax)\,dx=\frac{1}{a}K_1(a),\quad\int_0^1 xI_0(ax)\,dx=\frac{1}{a}I_1(a)$$

we can perform the integration over r_x:

$$\Phi=\frac{2\pi^2\nu}{e\lambda_G}\int_0^\infty r_y^2\bar{c}(r_y)\left[I_0\left(\frac{r_y}{\lambda_G}\right)K_1\left(\frac{r_y}{\lambda_G}\right)+K_0\left(\frac{r_y}{\lambda_G}\right)I_1\left(\frac{r_y}{\lambda_G}\right)dr_y\right]=\frac{\pi\nu}{e},$$

where use was made of the Wronskian property for the Bessel functions and of the normalization of the c function.

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[1]L. Leplae and H. Umezawa, Nuovo Cimento 44, 410 (1966).

[2]L. Leplae and H. Umezawa, Math. Phys. 10, 2038 (1969).

[3]For a review of the Ginzburg-Landau-Gor'kov theory see N. R. Werthamer, in *Superconductivity*, edited by R. D. Parks (M. Dekker, New York, 1969), Chap. 6.

[4]It has been shown in Ref. 1 that in order to diagonalize the Hamiltonian it is necessary to express the electron operators as an infinite series of normal products of quasiparticle operators. (This transformation has been referred to as the "dynamical map" in our previous papers.) The Bogoliubov transformation, which corresponds to the first-order terms of this series, diagonalizes only the so-called reduced Hamiltonian.

[5]"First order", "second order" refer to first- and

second-order terms in the dynamical map.

[6]In Ref. 2 we presented R in the Fourier form, i. e., $R(\vec{l})$. $R(\vec{\nabla})$ is obtained from $R(\vec{l})$ by replacing $(\vec{l})$ by $(1/i)\vec{\nabla}$. There is a misprint in Ref. 2: Eq. (16a) should be replaced by

$$R(\vec{l}) = \frac{1}{4}\int\frac{d^3K}{(2\pi)^3}\,\frac{E_{K+l/2}+E_{K-l/2}}{E_{K+l/2}\;E_{K-l/2}} \times \frac{1}{(E_{K+l/2}+E_{K-l/2})^2-\omega_l^2} \quad .$$

[7]To extend our argument to the case where $f(\vec{y})$ is not confined in a finite domain the proof can be modified as is shown in Appendix A.

[8]When f is not smooth, the gauge transformation differs from the boson transformation by the effect of $\rho^{(1)}$.

[9]We use as system of units $\hbar = c = 1$.

[10]In the case where the function $c(\vec{x}-\vec{y})$ satisfies the equation $\int c(\vec{x}-\vec{y})c(\vec{y}-\vec{z})\,d^3y = c(\vec{x}-\vec{z})$, the Eq. (2.5) is exact with $\lambda_G = \lambda_L$.

[11]A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. 32, 1442 (1957)[Soviet Phys. JETP 5, 1174 (1957)]. For a review of the theory of type-II superconductors see A. L. Fetter and P. C. Hohenberg, in *Superconductivity*, edited by R. D. Parks (M. Dekker, New York. 1969), Chap. 14.

[12]J. Bardeen, R. Kümmel, A. E. Jacobs, and L. Tewordt, Phys. Rev. 187, 556 (1969).

[13]G. Eilenberger, Z. Physik 214, 195 (1968).

[14]G. Eilenberger and H. Büttner, Z. Physik 224, 335 (1969).

[15]P. G. De Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966).

[16]L. Neumann and L. Tewordt, Z. Physik 189, 55 (1966).

[17]Here we disregard an irrelevant constant which is independent of d.

[18]In the limit $\lambda \gg \xi$ we can derive the result (4.3) from (4.2) if we approximate $\bar{c}(r)$ by the two-dimensional δ function $\bar{c}(r_x) = \delta(\xi - r_x)/2\pi r_x = \delta^{(2)}(\vec{x})$.

[19]L. Leplae, R. N. Sen, and H. Umezawa, Nuovo Cimento 49B, 1(1967).

[20]We shall use the notation defined in Secs. III and IV.

[21]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965).