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Instability and Echo-Pulse Amplification in a Ferrimagnetic Spin System in an Inhomogeneous Field*

G. F. Herrmann, R. M. Hill, and D. E. Kaplan
Lockheed Palo Alto Research Laboratory, Palo Alto, California 94304
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Nonlinear interactions are analyzed for the spin system of a ferromagnetic medium in an inhomogeneous magnetic field. Attention is focused on a certain class of excitations referred to as quasistationary spin waves which are in some respects analogous to plane spin waves in a uniform field. These waves have the property that at some time during their evolution they correspond to a uniform precession of the magnetization. Such waves are formed by sharp spatially uniform incident pulses, and conversely, are responsible for the radiation of sharp so-called "echo" pulses. Large quasistationary waves are unstable in a manner reminiscent of the Suhl instability of the uniform precession in a homogeneous field. This instability is accompanied by the exponential growth of small perturbing quasistationary waves and can be observed experimentally in the form of amplified echo pulses. The theoretical model accounts for the principal experimental features of the amplified echo, namely, the initial exponential increase with pulse separation τ and the rapid decrease at large τ, as well as the requirement for an inhomogeneous field.

I. INTRODUCTION

The spin system of a ferromagnet is noted for its nonlinear interactions and its excitations are often prone to instability. In 1956, Suhl[1] showed that the uniform precessional motion of the magnetization was unstable against certain spin-wave disturbances and that whenever the precession exceeded a given value, conditions were created under which pairs of spin waves of opposite wave vector would grow exponentially, drawing their energy from the uniform precession. This instability may be regarded as the prototype of a class of processes characterized by systematic energy transfer from a large excitation of the spin system into small perturbing disturbances. A directly observable example of such a process is the phenomenon of amplified echoes,[2] whose origin can be attributed[3] to the nonlinear interactions among disturbances impressed on the system by a series of very short pulses.

The ferromagnetic echo is generically related to Hahn's spin echo[4] and to a broad class of similar echo phenomena which occur as a consequence of nonlinearity in many complex resonant media. Typically, a series of resonant pulses incident on a medium is followed by a number of reradiated echo pulses at time intervals equal to those between incident pulses, or to sum or difference combinations of these intervals. The general properties of these echoes have been discussed by many authors[5] and it has been shown that the replication of time intervals does not depend on any particular physical interaction but is a consequence of phase-coherence requirements analogous to those which are responsible for the preferential generation of sum and difference frequencies in the nonlinear mixing of continuous signals. In the following, we shall restrict ourselves exclusively to the simplest echo sequence (Fig. 1) which consists of two incident pulses of which the first is very weak, followed by an echo pulse emitted from the sample after an additional time interval τ equal to the interval between the incident pulses. In a ferromagnetic crystal, under proper conditions, the echo pulse may be many orders of magnitude larger than the first pulse and we then talk of an "amplified echo." As a rule, the amplified echo initially increases exponentially with the pulse interval τ, a behavior indicative of an unstable excitation. Beyond a certain value of τ, amplification drops sharply, suggesting that the instability disappears at large τ.

The phenomenon of echo amplification occurs only in an inhomogeneous magnetic field. This can be concluded from purely theoretical considerations that apply to echoes in general[6] and is also confirmed experimentally. Echo amplification was first observed in irregularly shaped samples with highly inhomogeneous demagnetizing fields. Conclusive evidence was obtained more recently in experiments employing spherical samples placed

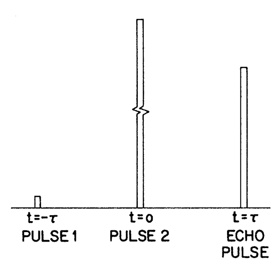

FIG. 1. Simple amplified echo sequence.

in a controlled inhomogeneous field. With increasing inhomogeneity, echo amplification increased from zero in a homogeneous field to values as high as 10^3. The theoretical study of this class of phenomena therefore requires an analysis of the nonlinear interactions in a ferromagnet in an inhomogeneous field.

Media in which echoes are observed can generally be described as comprising large numbers of resonances spread over some frequency range. In the present case this would suggest a theoretical approach based on a normal mode expansion in terms of the stationary disturbances of the spin system. This procedure would relate directly to a formal theory developed in an earlier paper[6] where it was shown, that in a system of nonlinearly coupled oscillation modes, instability and echo amplification may indeed take place. However, the application of this method of analysis to an actual inhomogeneous ferromagnetic medium is extremely difficult. The stationary disturbances, in contrast to the homogeneous case, have a complex spatial dependence which does not easily lend itself to mathematical manipulation. Moreover, in the most interesting configurations, orthogonality and completeness of the mode system is not assured.

An alternative approach, which we shall follow here, is to focus on the simplest types of disturbances directly associated with echo phenomena, namely, disturbances which are produced by very sharp spatially uniform electromagnetic pulses. These disturbances are not stationary and cannot be treated as normal modes of the system. They start initially as uniform precessions of the magnetization with constant amplitude and phase over the sample, but evolve with the passage of time into more complex configurations. Under experimental conditions, exchange plays only a secondary role in this motion and the amplitudes and local frequencies vary only slightly with time. It is therefore appropriate to refer to these solutions as "quasistationary spin waves" (QS waves). We shall see that at any given time, the local properties of these solutions are in fact approximately those of ordinary spin waves, and that the nonlinear interaction among them is analogous to the interaction among truly stationary spin waves in a homogeneous medium. In particular, the 4-magnon process associated with the Suhl instability[1] has exact counterparts in the inhomogeneous medium, and is directly responsible for echo amplification.

From an experimental point of view, the most important property of these QS waves is that at some specific time they are represented by a uniform precession, i.e., a precession at equal phase and equal amplitude over the sample. At that particular time, and at that time alone, there is strong coupling to the electromagnetic field. This coupling is exhibited either in the absorption of a sharp pulse from the field, or the emission of an "echo." According to this picture, the echo is the product of nonlinear coupling among QS waves, and echo amplification is the manifestation of an instability which causes the exponential growth of these waves.

It does not seem inappropriate to admit from the outset that no general analysis of nonlinear interactions in an inhomogeneous ferromagnetic medium is being attempted. On the contrary, we shall deal mostly with idealized physical models which are mathematically tractable. Our first and foremost objective is to formulate a problem whose solution is mathematically straightforward and physically clear and which predicts the principal features of the echo mechanism. Once this is accomplished, we are in a better position to discuss the more general problem and evaluate the various complicating factors which exist in a realistic physical context.

We proceed first with the development of a qualitative model which provides a physical explanation of the process. This is followed in Sec. III by an equally idealized mathematical model. Sections IV and V are again qualitative in nature, and attempt to relate the results to realistic physical configurations. A brief discussion of the canonical formulation of the problem is given in the Appendix.

II. QUALITATIVE MODEL

When a simple model is used to describe a complicated system, it would seem most appropriate to define in advance a strict regime of applicability and list all underlying assumptions. We shall, however, defer this until later in order not to clutter the presentation at this stage, and also since it can be done more profitably *ex post facto* once some mathematical results are available. We confine ourselves initially to a one-dimensional (or planar) model by assuming $\vec{\nabla} H \cong$ const and restricting consideration to disturbances which vary only along the direction of $\vec{\nabla} H$. The magnetic field is therefore constant along the plane "wave fronts," and one avoids the difficulties arising from the anisotropy of the medium relative to spin-wave propagation (see Sec. V). We also ignore boundary effects.

Let us at first neglect exchange. We can then define a local precession frequency, $\omega(\vec{r})$, which depends on the field at the position $\vec{r}$. Let $a(\vec{r})$ denote a canonical coordinate[7] representing the precessing component of the magnetization. In the linear approximation $a(\vec{r})$ varies with time as $e^{i\omega(\vec{r})t}$. Consider now a disturbance initiated as a

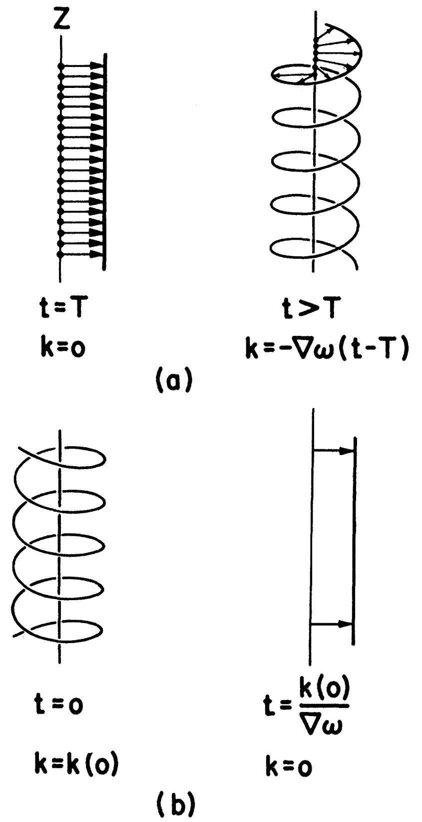

FIG. 2. (a) Evolution from uniform precession into a QS wave. Vectors represent the precessing components of the magnetization and are a function of position along the vertical line. Locus of the vector end points evolves into a right-handed helix. (b) Unwinding of a left-handed spin wave into a uniform precession (only the locus of vector end points shown).

uniform precession at $t=T$ by a spatially uniform pulse that is sharp enough to encompass in its Fourier spectrum the entire frequency range in the sample. Such a disturbance can be written in the form

$$a_T(\vec{r}, t) = Ae^{i\omega(\vec{r})(t-T)}, \tag{1}$$

where A is a constant.

For $t>T$, a_T develops into a wave and one may define an instantaneous wave vector $\vec{k} = -\vec{\nabla}\phi(\vec{r})$, where ϕ is the local phase angle of the precession. With $\phi = \omega(\vec{r})(t-T)$ one obtains

$$\vec{k} = \vec{k}_T(t) = -\vec{\nabla}\omega(t-T). \tag{2}$$

The evolution from uniform precession to a spin wave of wave vector $\vec{k}_T$ is shown pictorially in Fig. 2 for the case where $\vec{\nabla}H$ is parallel to $\vec{H}$. The precessing component of magnetization when plotted as a function of the appropriate position coordinate describes a helix. For $t>T$, the vector $\vec{k}$ points in a direction opposite to $\vec{\nabla}\omega$ and the helix is right handed.[8] With increasing time the helix becomes increasingly tightly wound. Conversely, any spin wave for which $\vec{k}$ is *along* $\vec{\nabla}\omega$ represents a left-handed helix, which at an appropriate time unwinds into the uniform precession. Thus, if at $t=0$, $\vec{k}=\vec{k}(0)$, uniform precession occurs at $t=k(0)/\nabla\omega$ [Fig. 2(b)].

The time $t\sim T$ corresponds to the incidence of a pulse or, alternatively, the radiation of a sharp "echo" pulse. At other times, coupling of the magnetization to the electromagnetic field is very small, because of the disparity between the respective wavelengths.

Disturbances of the form of Eq. (1) will be called QS waves. At any instant, they possess the spatial characteristics of spin waves of constant amplitude and wave number. Our analysis is based on the idea that interactions among such waves retain important characteristics of spin-wave interactions in a homogeneous medium.

The simplest type of echo experiment consists of a sequence of two incident pulses, a very weak one, at $t=-\tau$, followed by a strong one at $t=0$. At $t=\tau$ an echo pulse is emitted from the medium. When the echo is very much more intense than pulse 1, it is referred to as an amplified echo (Fig. 1). In general, pulse 1 may possess structure in the form of modulation and this structure is reproduced in reverse time order in the echo. At present, we simply assume that both incident pulses are extremely sharp and therefore give rise, respectively, to two QS waves, and that the echo represents a third QS wave arising through some nonlinear mechanism. The wave vectors of the three disturbances are given by

$$\vec{k}_{-\tau} = -\vec{\nabla}\omega(t+\tau),$$
$$\vec{k}_0 = -\vec{\nabla}\omega t, \tag{3}$$
$$\vec{k}_\tau = -\vec{\nabla}\omega(t-\tau).$$

One immediately notes that all times

$$2\vec{k}_0 = \vec{k}_\tau + \vec{k}_{-\tau}. \tag{4}$$

This is the familiar matching condition for a 4-magnon interaction. At $t=0$, $\vec{k}_0=0$, $\vec{k}_{-\tau}=-\vec{k}_\tau$, and (4) reduces to the familiar relation associated with the Suhl instability in a uniform medium in which the uniform precession interacts with a pair of opposite spin waves with the result that the latter grow exponentially at the expense of the former. The transfer of energy among spin waves depends upon the relation among their phases. A fixed-phase relation results in a unidirectional energy flow which makes it possible for small perturbations to grow indefinitely with time. Equation (4) is the first of two conditions for maintaining a fixed-phase relation among the three QS waves for a 4-magnon interaction. The second condition, $2\omega_0 = \omega_\tau + \omega_{-\tau}$, is also satisfied provided exchange is absent or negligible, since in that case all three frequencies equal $\omega(\vec{r})$ at every point. The echo process is a manifestation of the instability of the QS wave a_0 associated with pulse 2. The instability is triggered by the small QS wave, $a_{-\tau}$, that was earlier produced by pulse 1. To start with, only two disturbances are present, namely, $a_{-\tau}$ and a_0. Under proper conditions, a third wave, a_τ, is generated by nonlinear coupling. If we assume that the nonlinear coupling among the waves is identical, or at least similar to that which exists among spin waves in an homogeneous medium, then in analogy to the Suhl instability, $a_{-\tau}$ and a_τ proceed to grow exponentially at the expense of the large disturbance, a_0. At $t=\tau$, the third wave, a_τ, evolves into a uniform precession and the emission of an amplified echo takes place. The evolution of the three waves is presented in Fig. 3.

The role of the exchange interaction has been neglected so far. However, the inclusion of exchange is essential in order to provide a mechanism for energy transfer between adjacent points in the medium without which instability must be ruled out. At first sight, it would seem that purely dipolar forces, on account of their long range, could play this role. However, the group velocity of purely magnetostatic volume waves is zero and hence no energy is propagated (see Sec. V).[9]

The inclusion of exchange into our model adds a dispersion term, Λk^2, to the local frequency, where Λ is an appropriately normalized exchange coefficient. The frequency of a QS wave at a point r is therefore given by

$$\omega_T(\vec{r}, t) = \omega(\vec{r}) + \Lambda k_T^2(t). \tag{5}$$

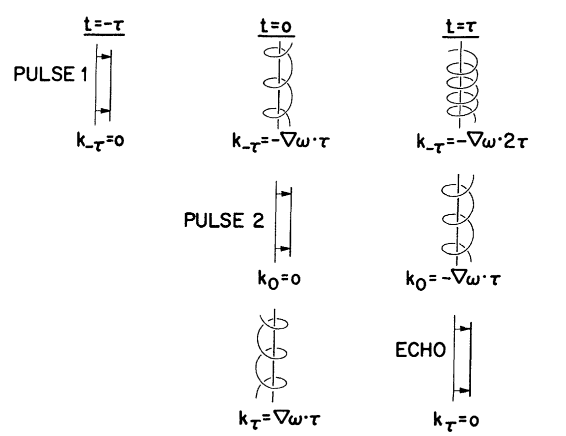

FIG. 3. Evolution of a system of three coupled QS waves, corresponding to two incident pulses and the echo pulse. Wave vectors obey the relation $2k_0 = k_{-\tau} + k_\tau$.

If $\vec{\nabla}\omega = \text{const}$, then the addition of this term does not change the wave vector $\vec{k}_T(t)$ of a QS wave. Indeed, if we put $\vec{k}_T(t) = -\vec{\nabla}\int\omega_T dt$ we find that this equation is already satisfied by $\vec{k}_T = -\vec{\nabla}\omega(t-T)$, since the latter equation also implies $\vec{\nabla}k_T^2 = 0$. Equation (4) remains unaffected. On the other hand, using the definition (5) and the expression (3), one finds a frequency mismatch

$$2\omega_0 - \omega_{-\tau} - \omega_\tau = -2\omega_E(\tau),$$

where

$$\omega_E(\tau) = \Lambda(\nabla\omega\tau)^2 = \Lambda k_{-\tau}^2(0). \tag{6}$$

This mismatch increases with increasing pulse separation, τ, making conditions for instability less favorable. At sufficiently large values of τ, one should expect the instability and with it, the amplified echo, to disappear. In sum, one is led to expect an initial exponential increase of the echo with increasing τ, followed first by leveling off and then a rapid decrease as the phase match deteriorates. This corresponds essentially to the observed experimental pattern.

III. SIMPLE MATHEMATICAL MODEL

The preceding qualitative discussion suggests that QS waves retain a number of properties associated with plane spin waves in a uniform field. In order to establish this relationship on a theoretical basis, it is most convenient to use a canonical Hamiltonian formalism. This method is mathematically powerful and leads most directly to the desired results, but its esoteric nature makes it somewhat inaccessible. It has been therefore relegated to the Appendix, and we shall confine ourselves here to the solution of the simplest mathematical equation which exhibits the principal physical features of our problem.

We consider a one-dimensional problem in which all physical parameters are assumed to vary only along the x coordinate and introduce a complex variable $a(x, t)$ to describe the precessing component of the magnetization.

We assume for the variable $a(x, t)$ an equation of the form

$$\frac{\partial a}{\partial t} = i(\omega' x)a - i\Lambda\frac{\partial^2 a}{\partial x^2} - iq\left|a^2\right|a. \tag{7}$$

The expression $\omega' x$ represents the local frequency which is assumed to vary linearly with x, with a coefficient of proportionality $\omega' = d\omega/dx = \text{const}$. The second term on the right represents the exchange interaction. In choosing a form for the third nonlinear term, one must recall that only those nonlinear terms whose time dependence is close to the natural resonance frequency have any long term effect on the motion. For example, a term proportional to a^2 excites the second harmonic and is therefore ineffectual. The term $|a^2|a$ is the lowest nonlinear term to occur at a fundamental frequency. Terms of this form can arise from the dipolar interaction or from the crystalline anisotropy. Nonlinearities in the exchange which would also introduce products of the spatial derivatives of $a(x, t)$ are not included since these are insignificant in the regime of interest, as we shall see below.

Equation (7) has the great merit of possessing exact solutions which are of the form

$$a_T(x, t) = A \exp i\phi(x, t-T), \tag{8}$$

where the constant A can be tacitly assumed, without loss in generality, to be real and where

$$\phi(x, t) = \omega' xt + \tfrac{1}{3}\Lambda\omega'^2 t^3 - qA^2 t. \tag{9}$$

The solution (8) represents a QS wave. At $t = T$, $\phi = 0$ and the precession is uniform. The spatial variation of ϕ can be made more explicit if we introduce

$$k_T(t) = \frac{-\partial\phi(x, t-T)}{\partial x} = -\omega'(t-T) \tag{10}$$

and put

$$\phi(x, t-T) = (\tfrac{1}{3}\Lambda k_T^2 - qA^2)(t-T) - k_T x.$$

In this form k_T appears clearly in the role of a wave vector. The temporal part of ϕ is composed of a dispersion term caused by exchange and a nonlinear frequency shift.

Consider now the response of the system to the pulse sequence of Fig. 1. A weak pulse is incident at $t = -\tau$ and is followed by a strong pulse at $t = 0$. In view of the smallness of pulse 1, we shall aim at the linearization of the equations with respect to that pulse. Suppose that pulses 1 and 2 excite, respectively, uniform precessions of magnitude ϵ and A. At $t = 0$, the first disturbance will have evolved into a wave $\epsilon\exp(i\omega'\tau x)$ (apart from an unimportant position-independent phase). At $t = 0$, after the incidence of pulse 2, one sets initial conditions as

$$a(x, 0) = A + \epsilon\exp(i\omega'\tau x). \tag{11}$$

The objective is to calculate the "echo amplitude" at $t = \tau$. This is defined simply as the average of a over the sample,

$$a_{\text{echo}} = (1/L)\int_0^L a\, dx, \tag{12}$$

and represents the uniform-precession component in the total disturbance. In practice we shall take the limit of (12) as $L \to \infty$.

The zeroth-order solution, obtained by setting $\epsilon = 0$, is given by $a = Ae^{i\phi}$, where ϕ is defined in (9). Before linearizing the equation, it is convenient to transform first to an "interaction representation" by putting

$$a = b\exp i\phi(x, t).$$

Substitution into (7) gives

$$\frac{\partial b}{\partial t} = -i\Lambda\left(\frac{\partial^2 b}{\partial x^2} + 2i\omega' t\frac{\partial b}{\partial x}\right) - iq(bb^* - A^2)b.$$

In this representation, the zeroth-order solution is given by the constant A, and we can proceed to linearize the equation by substituting $b = A + \epsilon c$ and retaining only linear terms in ϵ. The resulting equation is

$$\frac{\partial c}{\partial t} = -i\Lambda\left(\frac{\partial^2 c}{\partial x^2} + 2i\omega' t\frac{\partial c}{\partial x}\right) - iqA^2(c + c^*) \tag{13}$$

and the initial conditions become

$$c(x, 0) = \exp(i\omega'\tau x). \tag{14}$$

These initial conditions are periodic in x, with a period

$$P = 2\pi/\omega'\tau.$$

Since (13) is invariant with respect to a translation in x, $c(x, t)$ retains this periodicity at all times and may be expanded in a Fourier series:

$$c = \textstyle\sum_m u_m e^{-im\omega'\tau x}, \tag{15}$$

with u_m, in turn, given by

$$u_m = (1/P)\int_0^P c\, e^{im\omega'\tau x}\, dx. \tag{16}$$

The initial conditions in accordance with (14) are given by

$$u_{-1}(0) = 1, \quad u_m(0) = 0 \quad \text{for } m \neq -1.$$

Substitution of (15) into (13) and utilization of the orthogonality relation

$$\int_0^P e^{i(m-m')\omega'\tau x}\, dx = P\delta_{mm'},$$

results in a system of equations in which each u_m is coupled only to the corresponding u_{-m}. Because of the initial conditions, we need to consider only the equations for $m = \pm 1$. These are

$$\begin{aligned} \dot{u}_{-1} &= i(\Lambda\omega'^2\tau^2 + 2\Lambda\omega'^2\tau t - qA^2)u_{-1} - iqA^2 u_1^*, \\ \dot{u}_1 &= i(\Lambda\omega'^2\tau^2 - 2\Lambda\omega'^2\tau t - qA^2)u_1 - iqA^2 u_{-1}^*. \end{aligned} \tag{17}$$

By retracing the sequence of transformations one can put the solution $a(x, t)$ in the form

$$a(x, t) \cong e^{i\psi}[Ae^{i\omega' t} + \epsilon u_{-1}e^{i\omega' x(t+\tau)} + \epsilon u_1 e^{i\omega' x(t-\tau)}],$$

where $\psi = \frac{1}{3}\Lambda\omega'^2 t^3 - qA^2 t$. Hence, according to (12)

$$|a_{\text{echo}}(\tau)| = \epsilon|u_1|.$$

u_{-1} and u_1 thus represent the amplitudes of QS waves associated with the initial disturbance and the echo, respectively.

Equation (17) possesses exact solutions of the form

$$u_1 e^{i\theta} = -(iqA^2/\alpha)\sinh\alpha t,$$
$$u_{-1} e^{-i\theta} = \cosh\alpha t + \frac{i(\Lambda\omega'^2\tau^2 - qA^2)}{\alpha}\sinh\alpha t, \tag{18}$$

where α and θ are given by

$$\alpha^2 = \Lambda\omega'^2\tau^2(2qA^2 - \Lambda\omega'^2\tau^2),$$
$$\theta = \Lambda\omega'^2\tau t^2. \tag{19}$$

Hence, $|a_{\text{echo}}(\tau)| = (\epsilon qA^2/|\alpha|)|\sinh\alpha\tau|$. (20)

The solution depends strongly on the parameter α. For real α it is exponential in character, for imaginary α, periodic. If one defines a critical $\tau = \tau_c$ such that $\tau_c^2 = 2qA^2/\Lambda\omega'^2$, then for $\tau < \tau_c$, α is real, provided q and Λ are of identical sign. For $\tau > \tau_c$, α is imaginary. The expressions qA^2 and $\Lambda\omega'^2\tau^2$ represent, respectively, the nonlinear frequency shift, and the dispersive shift caused by exchange. α attains its maximum value when these two contributions are equal and goes to zero when the second is double the first. The functional dependence of the echo on τ therefore exhibits a pattern predicted in the qualitative model, and observed experimentally which consists of an initial exponential rise, followed, after a leveling-off period, by a rapid decrease to zero.

The effect of relaxation can be taken into account by including on the right-hand side of Eq. (7) a phenomenological term, $-\nu a$. One follows the same procedure to arrive at Eqs. (17) for u_{-1} and u_1 except that A must be replaced by $Ae^{-\nu t}$ in those equations. The echo is represented by $\epsilon|u_1|e^{-2\nu\tau}$. By a series of transformations the equations can be combined into a Bessel equation of nonintegral order, but there is little advantage in giving the solution explicitly. What concerns us most is the condition for instability, given by $\Lambda(2qA^2e^{-\nu t} - \Lambda\omega'^2\tau^2) > 0$. It indicates that for large t the solutions always become stable. In fact, because of the exponential dependence on νt, echoes will not increase with τ much beyond $\nu\tau \sim 1$, and are negligible for $\nu\tau \gg 1$.

IV. PHYSICAL CONSIDERATIONS

A. Physical Regime

Equation (20) in combination with (19) enables us to arrive at estimates for the physical parameters. This will, in turn, permit us to consider the effect of relaxing some of the assumptions underlying the calculated model.

It is convenient to define certain characteristic parameters. We define a characteristic wave vector, $k_c = \omega'\tau$, an exchange frequency shift, $\omega_E = \Lambda k_c^2$, and a nonlinear frequency shift, $\omega_{NL} = qA^2$. For given ω', Λ, and ω_{NL}, the exponential coefficient α attains its maximum value for τ such that $\alpha = \omega_E = \omega_{NL}$, and $|a_{\text{echo}}|$ then equals $\epsilon\sinh\alpha\tau$, where ϵ corresponds to the input excitation produced by pulse 1.

The time scale in echo experiments is dominated by relaxation. In YIG the relaxation times are of the order of 1 μsec and peak echoes are typically observed at $\tau = 5\times10^{-7}$ sec. The highest measured amplitude amplification factors are of the order 4×10^2. This corresponds to $\alpha \sim 10^7$ sec^{-1}. Hence, under these conditions also $\omega_E \sim \omega_{NL} \sim 10^7$ sec^{-1}. Since the typical operating frequency is $\omega \sim 10^{11}$, both ω_E and ω_{NL} can be legitimately regarded as very small. A characterization of the regime can be given in terms of the three dimensionless parameters $\omega_{NL}\tau = qA^2\tau$, $\omega_E\tau = \Lambda\omega'^2\tau^3$, and $\omega\tau$. The first two are of the order of unity or slightly larger while the third is of the order 10^3–10^5. From $\omega_E = \Lambda k_c^2$, we obtain for $\Lambda \sim 0.1$ cm^2 sec^{-1}, $k_c \sim 10^4$ cm^{-1}. In turn, from $k_c = \omega'\tau$ we get $\omega' \sim 2\times10^{10}$ sec^{-1} cm^{-1} which corresponds to a magnetic field variation of 1000 Oe/cm, a value consistent with experiment.

A group velocity can be formally defined as $\partial\omega/\partial k$. For a given experiment an average characteristic group velocity is thus given by $v_g = \Lambda k_c \sim 10^3$ cm/sec. During the interval τ energy travels a distance roughly given by $v_g\tau \sim 10^{-3}$ cm, or a distance comparable to a characteristic wavelength, $\lambda_c = 2\pi/k_c$. Interactions within the medium are therefore confined to extremely small neighborhoods during the time of an experiment. Summarizing:

(a) The characteristic wavelength of the disturbances is most of the time very small compared to sample size and to the relative field inhomogeneity (that is, $k \gg \nabla H/H$). Surface effects are therefore unimportant except during the very short periods of time when the precession is nearly uniform.

(b) Dynamic exchange fields are very small compared to the magnetic fields. Under conditions of high echo amplification they are roughly equal to the third-order nonlinear terms in the dipolar magnetic fields. Nonlinearities in the exchange itself are therefore unimportant.

(c) The interaction is semilocalized, and energy is transferred only over very short distances during the process.

The last conclusion is of particular importance in connection with the signal processing properties of an echo amplifier. Each position in the sample corresponds to a particular local frequency and the amplitude of the local precession is proportional to the corresponding Fourier component of the exciting pulses. Because of the local nature of the interaction each Fourier component interacts only with an extremely narrow frequency band surrounding it. To a very good approximation, the echo amplifier can therefore be considered as acting on each Fourier component independently of all others.

In the mathematical model we assumed infinite boundaries, a uniform field gradient, and uniform excitation of magnetization by the incident pulses. These assumptions can now be relaxed to a considerable extent.

B. Boundary Conditions

A fundamental difficulty in matching finite boundary conditions to QS waves as given, for example, by (8) is that the exchange energy of the excited system increases with increasing k and therefore varies with time. This creates no difficulty in an infinite sample, but in a finite sample any flow of energy through the system must be interrupted at the boundaries.

The distortion of an ideal QS wave because of the mismatched boundary can be described as a disturbance propagating inward from the boundaries. In the short time of the experiment only the components of the disturbance that are associated with high group velocities, and therefore with wavelengths extremely short compared to those of the QS waves, can penetrate deeply into the interior. One should also bear in mind that the exchange energy in the QS waves is only a minute fraction of the total energy and therefore a minor adjustment of amplitudes should suffice in order to balance it. It thus seems unlikely that boundary conditions will greatly interfere with the interactions inside the sample.

C. Nonuniform Field Gradient or Nonuniform Excitation

Because of the semilocalized character of the interaction one can extend the model to fields with a nonuniform gradient. Provided

$$\left(\frac{d^2\omega}{\partial x^2}\right)\Big/\left(\frac{d\omega}{\partial x}\right) \ll k_c$$

one is justified in applying a "local approximation," which consists in assuming at any position x a solution of the form (18) with $\omega' = \omega'(x)$. The total echo is then obtained as an average over the sample. Since the exponential coefficient α is a function of $\omega'\tau$, amplification will peak at different values of τ in different parts of the sample. The *phases* of the echo signal will also vary with position, although this variation is smaller than might at first seem to be the case. One should note that in the absence of exchange the phase-matching relation (4) holds *exactly* even when ω' is not uniform. The phase variation of the echo over the sample therefore arises entirely from the nonuniformity of the frequency dispersion term $\Lambda\omega'^2t^2$. The contribution of this term over a time τ to the total phase is approximately given by $\Lambda\omega'^2\tau^3$. In the regime defined above this corresponds to about one period of revolution or less. For moderate nonuniformity of the field gradient the echo phase variation over the sample can be only a small fraction of this value, and therefore not very significant. We therefore find that a nonuniform field gradient will result in echoes whose Fourier components are modified in amplitude and, to a lesser degree, in phase.

Unequal excitation of the sample by the electromagnetic pulses (more specifically by pulse 2 which dominates the amplification process) can be treated in essentially the same way. The "local approximation" applies only when spatial variation is relatively smooth. If this is not the case then one must consider some additional effects discussed in Sec. IV D.

D. Single Pulse Effects

In Sec. III we have solved Eq. (7) for initial conditions given by Eq. (11), which can be put in the form

$$a(x, 0) = A + \epsilon e^{-ik_c x}. \tag{21}$$

Strong amplification occurs for k_c such that $\Lambda k_c^2 \sim qA^2$. For k_c much larger or much smaller than this value, the echo is small. The disturbance described by Eq. (21) was assumed to have been caused by a weak pulse at $t = -\tau$ followed by a strong pulse at $t = 0$. Suppose now, that only a single pulse is incident at $t = 0$, but that this pulse is not perfectly uniform over the sample. A Fourier expansion at $t = 0$ will contain in addition to a uniform component also components corresponding to various values of the wavenumber k. Each Fourier component represents a QS wave which evolves into uniform precession at a time $t = k/\omega'$. Those waves whose k values fall within the amplification range are strongly amplified resulting in the emission of sharp spikes of radiation at the appropriate times. Such spikes are occasionally observed experimentally under conditions favoring irregularities in ω' or in the coupling of the electromagnetic field to the sample.

V. THREE-DIMENSIONAL CASE

The analysis of three-dimensional spin-wave propagation in a nonuniform field is so difficult that we cannot go beyond some heuristic arguments for the existence of QS waves with interaction patterns similar to those described above.

As in Sec. II, we initially ignore exchange. We also ignore all surface or boundary effects. The spin waves which we consider are so-called "volume magnetostatic modes" and we would like briefly to review the properties of these waves in a uniform medium. The dispersion relation, that is, the functional relationship between ω and k, is anisotropic. However ω does not depend on the magnitude of k but only on its direction relative to the magnetic field. In other words, all spin waves propagating in one direction have the same frequency, and so do all one-dimensional disturbances (that is, disturbances which vary only along one direction) which result from the superposition of these waves. In addition, such disturbances remain stationary in position since the group velocity $\partial\omega/\partial k$ equals zero. In particular, a disturbance that at a given time is confined to a narrow neighborhood of a plane and is uniform along the plane will remain thus confined. This behavior is, of course, a direct consequence of the fact that the dipolar (or demagnetizing) field lines all end within the disturbance itself. In spite of the long-range character of dipolar forces, their action in certain types of disturbances is thus strictly local.[10]

Suppose that we now depart from field uniformity by allowing variation perpendicular to the plane of the disturbance while maintaining a constant field along the plane of the disturbance. Precession within the plane is unaffected by the inhomogeneity and continues to occur at a frequency determined by the local value of the field and by the orientation of the plane relative to the field. More generally, a disturbance with parallel plane wave fronts perpendicular to the field gradient will remain in this geometric configuration, with each plane precessing at its proper local frequency. Such configurations fall within the one-dimensional case discussed in Sec. IV.

It is possible to conceive of more general families of surfaces with the same property. Let a surface be represented by $z=z(x,y)$, where x, y, and z are the surface coordinate. One can define a *formal* local frequency by putting

$$\omega_f(x,y,z)=\omega\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\vec{\mathrm{H}}(x,y,z)\right),$$

using the functional form of the dispersion equation for a uniform medium (still neglecting exchange). A solution of the differential equation $\omega_f(x,y,z)=\text{const}$ is a surface along which the formal local frequency is a constant (such a solution is usually not unique). One can construct a continuous nonintersecting family of such surfaces and define, with respect to this family, a formal local frequency $\omega_f(x,y,z)$ everywhere in the sample.

Let us now define a QS wave relative to such a surface family by considering a disturbance which is constant along each surface. If we assume that the characteristic wave vector k_c is very large compared to the surface curvature, then a QS wave can be locally approximated by a plane spin wave with a local frequency given by the formal frequency ω_f. A disturbance corresponding to a QS wave can be set up by assigning to each surface a phase ϕ proportional to the frequency, putting

$$\phi(r)=c\,\omega_f(r),$$

with uniform amplitude over the sample. If this equation holds at $t=0$, then at subsequent times

$$\phi=c\,\omega_f+\omega_f t=(c+t)\omega_f\,.$$

The phase therefore remains constant along each surface and proportional to ω_f. In particular, at $t=-c$, $\phi(r)=0$ and the disturbance reduces to a uniform precession.

In theory it thus seems possible to construct QS waves with all the desirable properties. In practice, things are unfortunately not that simple. The uniform precession represents a singularity where wave fronts are no longer defined. Starting with a uniform precession, it is not clear in advance into what kind of a configuration it will evolve. In an infinite sample with uniform ∇H and uniform excitation, symmetry arguments favor the exclusive generation of the one-dimensional (planar) configuration. In finite samples, the evolution during the initial period following uniform precession is strongly influenced by surface magnetostatic interactions, and although the disturbance remains only a short time in this ambiguous regime the effect on the subsequent evolution may be crucial. Since the difficulties of a rigorous calculation seem insurmountable at this point, the general problem of three-dimensional excitations must remain open.

VI. CONCLUSION

The introduction of the concept of QS waves makes it possible to apply ideas familiar from the study of nonlinear interactions in a uniform magnetic field to a ferromagnet in an inhomogeneous field.

The mathematical analysis is strictly applicable to the case of a uniform field gradient and sharp incident pulses, but leads itself readily to generalizations. The principal processes are semilocalized in character, that is, they depend primarily on the local value of the physical parameters as it can be shown that effective interaction among separate points in the sample is limited to extremely small distances. This property makes it possible to extend the analysis to arbitrary smoothly varying magnetic fields.

In particular, the theory provides a model for the phenomenon of amplified echoes, predicts its main features, and defines the regime under which it occurs.

APPENDIX

The similarity between the behavior of one-dimensional QS waves in an inhomogeneous field and that of planar spin waves in a uniform field is best displayed in a canonical Hamiltonian formalism. For a detailed exposition of this formalism the reader is referred to Schlömann.[7] The precession of the magnetization is represented by a canonical variable $s(r, t)$ which is the classical equivalent of the spin deviation operator introduced by Holstein and Primakoff.[11] The equation of motion is given in the form

$$s = i\frac{\partial \mathcal{H}}{\partial s^*} \quad , \tag{A1}$$

where the Hamiltonian $\mathcal{H}$, which is given in frequency units, is proportional to the total energy of the spin system.

We start by putting

$$\mathcal{H} = \mathcal{H}_Z + \mathcal{H}' \, , \tag{A2}$$

where $\mathcal{H}_Z$ represents the Zeeman interaction, and $\mathcal{H}'$ the part of the Hamiltonian that is independent of the external field. $\mathcal{H}_Z$ is given by

$$\mathcal{H}_Z = \int s^*(\vec{r}, t)\omega(\vec{r})s(\vec{r}, t)\,d^3r \, , \tag{A3}$$

where $\omega(\vec{r})$ is the Zeeman frequency, and integration is normalized per unit volume. We shall assume that ω varies linearly with a coordinate ξ and put $\omega = \omega'\xi$.

In the standard manner, $s(r, t)$ is decomposed into a Fourier series

$$s(\vec{r}, t) = \sum_{\vec{k}} s_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{r}} \, . \tag{A4}$$

One can put

$$\begin{aligned}\mathcal{H}_Z &= \omega' \int s^*(\vec{r})\xi\, s(\vec{r})\,d^3r \\ &= \omega' \sum_{\vec{k}} s^*_{\vec{k}}\left(\frac{-i\partial}{\partial k_\xi}\right)s_k \, .\end{aligned} \tag{A5}$$

[Since k is a discrete parameter, $\partial/\partial k_\xi$ assumes its usual meaning only at the limit where s_k can be regarded as a continuous function of k. A more elegant formulation could be obtained by putting (A4) and (A5) in the form of integrals, but the fine points of normalization and functional differentiation would require lengthy discussion.] The equations of motion are now in the form

$$\frac{\partial s_{\vec{k}}(t)}{\partial t} = i\frac{\partial \mathcal{H}}{\partial s^*_{\vec{k}}} = \omega'\frac{\partial s_{\vec{k}}}{\partial k_\xi} + iF_{\vec{k}}(s_{\vec{k}'}) \, , \tag{A6}$$

where

$$F_k(s_{k'}) = \frac{\partial \mathcal{H}'}{\partial s^*_{\vec{k}}} \, ,$$

and where the argument $s_{\vec{k}'}$ stands for all spin-wave amplitudes. Since $F_{\vec{k}}$ is independent of the external field its functional form is identical to its well-known form in a uniform field.

The term $\omega'\partial s_{\vec{k}}/\partial k_\xi$ can be viewed as representing a transport term in k space, which causes the disturbance to flow towards lower values of k_ξ. By transforming to a new set of moving coordinates one can eliminate this term. We shall, from this point on, confine ourselves to spin waves propagating along the ξ direction, putting $k_\xi = k$. We now define

$$a_T(t) = s_k(t) \, , \tag{A7}$$

where k is regarded as a function of t and T given by

$$k(t, T) = -\,\omega'(t - T) \, .$$

a_T is thus the amplitude of a QS wave originating at $t = T$. One has

$$\dot{a}_T = \frac{\partial s_k}{\partial t} + \frac{\partial s_k}{\partial k}\,\frac{\partial k(t, T)}{\partial t} = iF_k(a_{T'}) \, , \tag{A8}$$

where $F_k(a_{T'})$ is obtained by substituting $a_{T'}$ with $T' = k'/\omega' + t$ for $s_{k'}$ in $F_k(s_{k'})$.

Formally, (A8) represents a set of equations for QS waves which is essentially identical to the set of equations for plane spin waves in a uniform field. However, these equations are of no practical use unless the linear part of F_k is diagonal in $s_{k'}$, that is, contains only an s_k term. Otherwise, individual QS waves as defined by (A7) do not by themselves constitute solutions of the linear problem, but are linearly coupled to each other. In general, F_n will be diagonal only if $\vec{H}$, $\vec{\nabla}H$, and $\vec{k}$ are along the same direction.

For $\vec{\nabla}H$ along an arbitrary direction ξ the procedure can be modified as follows. Again, let us confine ourselves to one-dimensional (planar) disturbances with wave fronts perpendicular to $\vec{\nabla}H$. The second-order dipolar component of the Hamiltonian can be written as $\mathcal{H}_{D2} = \int \frac{1}{4}\omega_M (s + s^*)^2\, d\xi$, where $\omega_M = 4\pi M$. The combined Hamiltonian $\mathcal{H}_{M2} = \mathcal{H}_Z + \mathcal{H}_{D2}$ can be diagonalized by a change of variables, $v = \lambda s + \mu s^*$, to give

$$\mathcal{H}_{M2} = \int v^*(\xi)\omega(\xi)v(\xi)\,d\xi$$

with

$$\omega^2(\xi) = \omega_Z(\xi)[\omega_Z(\xi) + \omega_M \sin^2\theta] \, ,$$

where θ is the angle between the ξ direction and H. If one now puts $\omega(\xi) = \omega'\xi$, the procedure leading from (A3) to (A8) can be repeated with v replacing s.

In this formulation F_k no longer contains any linear terms from the dipolar interaction. On the other hand, the transformation from s to v results in off-diagonal terms in the exchange interaction. However, in the regime of interest these can be neglected in view of their smallness relative to the total diagonal terms.

When studying nonlinear effects such as instabilities, which evolve slowly with time, one need consider only the so-called "slowly varying" terms in the Hamiltonian, i.e., those products of s_k's and s_k^*'s which contain equal numbers of either type. These terms affect motion at the fundamental frequency whereas all others excite harmonics of the motion.

In the k representation the nonlinear terms obey certain selection rules, e.g., a term $s_k s_l s_m^* s_n^*$ will have a nonvanishing coefficient only if $k+l-m-n=0$. These constraints lead directly to similar constraints among sets of interacting QS waves, e.g.,

$$T_1 + T_2 - T_3 - T_4 = 0 , \tag{A9}$$

and account for the rules governing the intervals between echo pulses.

As an example, consider the Hamiltonian

$$\mathcal{H} = \int \left(|\omega' \xi |s|^2 + \Lambda \left| \frac{\partial s}{\partial \xi} \right|^2 - \tfrac{1}{2} q |s|^4] \right) dx .$$

This Hamiltonian generates an equation of motion identical with Eq. (7) in Sec. III. A straightforward calculation yields the equation

$$\dot{a}_T = i\Lambda\omega'^2 (t-T)^2 a_T - iq \sum a_{T'} a_{T''} a^*_{T'''} \tag{A10}$$

as the explicit form of (A8), where summation is over all T', T'', and T''' such that $T' + T'' - T''' - T = 0$. Consider now the set of three coupled QS waves associated with the simplest echo process, namely, $a_{-\tau}$, a_0, and a_τ. Assuming, as we did in the main text, that $a_{-\tau}$ and a_τ are very small, we may neglect all except first-order terms in a_τ and $a_{-\tau}$ and arrive at three coupled equations,

$$\begin{aligned} \dot{a}_{-\tau} &= i\Lambda\omega'^2 (t+\tau)^2 a_{-\tau} - 2iq|a_0|^2 a_{-\tau} - iq a_0^2 a_\tau^* , \\ \dot{a}_0 &= i\Lambda\omega'^2 t^2 a_0 - iq|a_0|^2 a_0 , \\ \dot{a}_\tau &= i\Lambda\omega'^2 (t-\tau)^2 a_\tau - 2iq|a_0|^2 a_\tau - iq a_0^2 a_{-\tau}^* . \end{aligned} \tag{A11}$$

Putting

$$a_0 = Ae^{i\psi}, \quad a_{-\tau} = u_{-1} e^{i\psi}, \quad a_\tau = u_1 e^{i\psi},$$

where A is a real constant and Ψ is given by

$$\psi = \tfrac{1}{3} \Lambda\omega'^2 t^3 - iqA^2 t ,$$

one obtains a pair of coupled equations identical with Eqs. (17) derived in a more roundabout way in Sec. III.

The reduction of the infinite set of Eqs. (A10) to three coupled equations is only possible when $a_{-\tau}$ is very small. In general, $a_{-\tau}$ and a_0 will couple to a large number of QS waves with $T = n\tau$, resulting in multiple, equally spaced echoes. From (A9) it is also clear that one can couple three incident pulses with $T_1 = 0$, $T_3 = \tau$, $T_4 = \tau'$, to an echo at $T_2 = \tau + \tau'$, the so-called three-pulse echo. These four pulses couple also to higher-order echoes with $T = n\tau + m\tau'$, where n and m are suitable positive and negative integers.

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[8]This holds in a convention where ∇H is along the direction of H. With ∇H opposite to H the helix would be left handed.

[9]Magnetostatic coupling can still occur as a result of boundary effects but would tend to be small at high k values.

[10]These local or purely "longitudinal" solutions owe their existence to the equation $\nabla \cdot (H + 4\pi M) = 0$ which enable one to separate $4\pi M$ into a divergenceless component and a component equal to $-H$.

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