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Galvanomagnetic Properties of a Single-Crystal Sphere by the Induced-Torque Method. I. General Theory

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The boundary-value problem for the torque induced by a rotating magnetic field of constant strength on a single-crystal sphere with an arbitrary conductivity tensor is solved. The high-field behavior of the torque and its relation to the Fermi-surface topology is discussed.

I. INTRODUCTION

Experiments designed to measure galvanomagnetic properties of pure metals by the induced-torque method have been performed recently.[1] A spherical metallic single crystal is placed in a magnetic field. The sphere is suspended by a wire along the y axis (see Fig. 1). The magnetic field, which is held constant in strength, rotates at a constant frequency Ω (typically $\Omega \approx 0.01\ \text{sec}^{-1}$) about the y axis, being all the time perpendicular to it. If we assume that at a given instant, $\vec{B}$ is in the z direction, then

$$\frac{d\vec{B}}{dt} = \Omega B \hat{x} \ . \tag{1.1}$$

A current is created in the sample, inducing a torque along the axis of rotation. Measurement of the torque as a function of magnetic induction strength B and of sample orientation provides information about the conductivity tensor, and hence about the topology of the Fermi surface. If the conductivity is isotropic, the eddy currents will flow around the sample in the yz plane, creating a magnetic moment in the x direction and therefore a torque $\vec{N} = \vec{M} \times \vec{B}$.

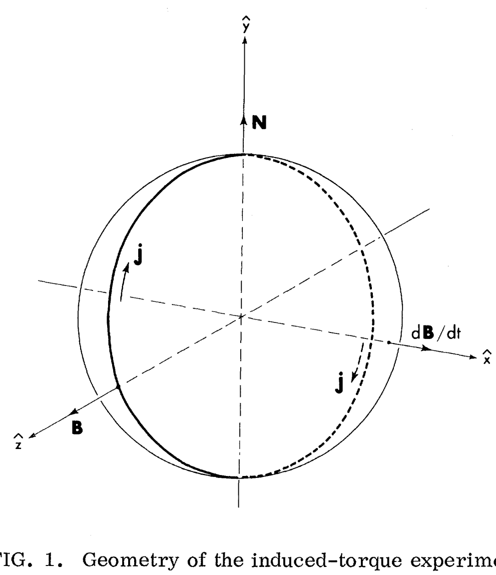

FIG. 1. Geometry of the induced-torque experiment and definition of coordinate system.

The solution of this problem for isotropic field-independent conductivity σ_0 is given in the classical textbook of Landau and Lifschitz[2]; the resulting torque is

$$N_y = (2\pi R^5/15c^2)\Omega B^2 \sigma_0 \ , \tag{1.2}$$

where R is the radius of the sphere. This solution is valid for $\Omega \to 0$.

An exact calculation of the torque in the limit $\Omega \to 0$ for an arbitrary conductivity tensor is presented in Sec. II. We find there[3] that the torque is given by (1.2), but with σ_0 replaced by $\bar{\sigma}$, an effective conductivity which depends on the components of the conductivity tensor $\overleftrightarrow{\sigma}$ in a fairly complicated way. The high-field behavior of the induced torque depends critically on the topology of the Fermi surface; in particular, if open orbits are present, their effects will dominate the torque at high fields. Thus the induced-torque method provides a convenient way of isolating the effect of open orbits.

In Sec. III we examine the relation between the asymptotic behavior of the torque and the topology of the Fermi surface; this method for the detection of open orbits is often easier and less ambiguous than ordinary magnetoresistance methods. The method also provides an accurate way of determining properties related to magnetic breakdown.[1,4,5]

II. CALCULATION OF INDUCED TORQUE

Consider a homogeneous spherical sample (radius R) with resistivity tensor $\overleftrightarrow{\rho}$ in a magnetic field $\vec{B}$. If $\vec{B}$ is rotating with frequency $\vec{\Omega}$, we have $\mathring{\vec{B}} = \vec{\Omega} \times \vec{B}$. We will regard $\vec{\Omega}$ as a small parameter and consider all quantities only to first order in $\vec{\Omega}$ (it turns out that any attempt to go to higher order in Ω involves such complications as variation of resistivity with magnetic field direction and problems related to the finite skin depth for $|\Omega| > 0$). The experiments described in Ref.

1 were carried out in the linear region, as was verified by varying Ω and observing a linear dependence of the torque.

The most important Maxwell equation for this problem is

$$\vec{\nabla}\times\vec{E} = -(1/c)\dot{\vec{B}} \ . \tag{2.1}$$

We will also use the continuity equation

$$\vec{\nabla}\cdot\vec{j} + \dot{\rho}_c = 0 \tag{2.2}$$

(here $\vec{j}$ is the current density and $\dot{\rho}_c$ the time derivative of the charge density), which follows from two other Maxwell equations. The rest of the information contained in the Maxwell equations can be shown to merely determine the first-order correction to the zero-order field $\vec{B}$, which does not enter our calculation. $\vec{E}$ and $\vec{j}$ are related by the constitutive relation

$$\vec{E} = \overleftrightarrow{\rho}\cdot\vec{j} \ , \tag{2.3}$$

where $\overleftrightarrow{\rho} = \overleftrightarrow{\sigma}^{-1}$ is the resistivity tensor.

The continuity conditions on the fields at the boundary of the sphere are of no use, since the first-order external fields are nonzero and unknown. But there can be no radial current, so the boundary condition is

$$\vec{j}\cdot\vec{r}\,\big|_{|\vec{r}|=R} = 0 \ . \tag{2.4}$$

Unfortunately these four equations do not determine the first-order fields and currents uniquely. We can add any longitudinal electric field (and its corresponding charge density $4\pi\rho_c = \vec{\nabla}\cdot\vec{E}$) as long as it vanishes at the boundary or satisfies (2.4) in some other way [the time dependence of ρ_c would then be determined by (2.2)]. This corresponds to the fact that these equations are consistent with any internal charge distribution, which depends on the previous history of the sample. Let us determine which charge distributions are compatible with the assumption that the magnetic field $\vec{B}$ rotates very slowly (and hence which may appear, to first order in the rotation frequency Ω). Since the charge density ρ_c is of first order, and all quantities are to vary on a time scale like $1/\Omega$, the derivative $\dot{\rho}_c$ must be a second-order quantity. The continuity equation (2.2) for the first-order current $\vec{j}$ then requires

$$\vec{\nabla}\cdot\vec{j} = 0 \ . \tag{2.5}$$

For an isotropic conductivity, this is the same as saying that stray charges are damped out by the conductivity in times much shorter than $1/\Omega$ for low Ω, so there can be no first-order charge density. But if the conductivity is anisotropic, it is no longer true that the charge density must vanish; it may be nonzero and still slowly varying in time as long as (2.5) is satisfied.

The four equations (2.1) and (2.3)–(2.5) determine the first-order fields uniquely. This can be shown by a slight variation of the standard uniqueness argument for the Laplace equation. Equation (2.1) shows that the difference $\vec{E}'$ between any two solutions for the electric field must be longitudinal, i.e., $\vec{E}' = \vec{\nabla}\Phi$. Then Stokes's theorem says

$$\int_{\text{surf}} d\vec{S}\cdot(\Phi\overleftrightarrow{\sigma}\cdot\vec{\nabla}\Phi) = \int_{\text{vol}} \vec{\nabla}\cdot(\Phi\overleftrightarrow{\sigma}\cdot\vec{\nabla}\Phi)\,dV \ .$$

If we denote the difference between the two currents by $\vec{j}' = \overleftrightarrow{\sigma}\cdot\vec{E}' = \overleftrightarrow{\sigma}\circ\vec{\nabla}\Phi$ and expand derivatives on the right, we get

$$\int d\vec{S}\cdot(\Phi\vec{j}') = \int \vec{\nabla}\Phi\cdot\overleftrightarrow{\sigma}\cdot\vec{\nabla}\Phi\,dV + \int \Phi(\vec{\nabla}\cdot\overleftrightarrow{\sigma}\cdot\vec{\nabla}\Phi)\,dV$$

$$= \int \vec{\nabla}\Phi'\overleftrightarrow{\sigma}\cdot\vec{\nabla}\Phi\,dV + \int \Phi(\vec{\nabla}\cdot\vec{j}')\,dV.$$

The left-hand side vanishes because of (2.4) and the rightmost term because of (2.5). Furthermore $\vec{\nabla}\Phi\cdot\overleftrightarrow{\sigma}\cdot\vec{\nabla}\Phi$ is non-negative because the conductivity tensor is always positive definite (by energy conservation). Since its integral vanishes, it must vanish at every point; hence $\vec{E}' = \vec{\nabla}\Phi = 0$ everywhere, and the solution is unique.

We now look for solutions of the form $\vec{j} = \overleftrightarrow{T}\cdot\vec{r}$, where $\overleftrightarrow{T}$ is an arbitrary tensor. (If such a solution exists, it is the unique one.) It is easy to show that the boundary condition (2.4) is equivalent to the requirement that $\overleftrightarrow{T}$ be antisymmetric. It is well known that this is in turn equivalent to the existence of a vector $\vec{t}$ such that $\overleftrightarrow{T}\cdot\vec{r} = \vec{t}\times\vec{r}$ for all $\vec{r}$, where

$$\overleftrightarrow{T} = \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix} \ .$$

From this form of $\vec{j}$,

$$\vec{j} = \vec{t}\times\vec{r} \ , \tag{2.6}$$

it follows easily that condition (2.5) is satisfied; the current simply circulates about the axis $\vec{t}$ and does not build up excess charge anywhere.

The only condition not yet satisfied is Maxwell's equation (2.1). This may be used to determine the current axis $\vec{t}$; putting in (2.3) for $\vec{E}$ and (2.6) for $\vec{j}$ we obtain

$$\vec{\nabla}\times(\overleftrightarrow{\rho}\circ\vec{t}\times\vec{r}) = -(1/c)\dot{\vec{B}} \ . \tag{2.7}$$

Upon straightforward expansion of the cross products, the left-hand side of (2.7) becomes $[\mathrm{Tr}\,\overleftrightarrow{\rho} - \overleftrightarrow{\rho}^{\dagger}]\vec{t}$, where $\overleftrightarrow{\rho}^{\dagger}$ is the transpose of the (real) tensor $\overleftrightarrow{\rho}$. Solving for $\vec{t}$, we get

$$\vec{t} = -(1/c)[\mathrm{Tr}(\overleftrightarrow{\rho}) - \overleftrightarrow{\rho}^{\dagger}]^{-1}\,\dot{\vec{B}} \ .$$

If we define an effective resistivity

$$\overleftrightarrow{\bar{\rho}} \equiv \tfrac{1}{2}(\mathrm{Tr}\overleftrightarrow{\rho} - \overleftrightarrow{\rho}^{\dagger})$$

$$= \tfrac{1}{2}\begin{pmatrix} \rho_{yy}+\rho_{zz} & -\rho_{yx} & -\rho_{zx} \\ -\rho_{xy} & \rho_{zz}+\rho_{xx} & -\rho_{zy} \\ -\rho_{xz} & -\rho_{yz} & \rho_{xx}+\rho_{yy} \end{pmatrix}, \quad (2.8)$$

then it is clear from (2.7) and conservation of energy that $\overleftrightarrow{\bar{\rho}}$ cannot be singular for any physical $\overleftrightarrow{\rho}$. We may then define an effective conductivity

$$\overleftrightarrow{\bar{\sigma}} = \overleftrightarrow{\bar{\rho}}^{-1} \quad (2.9)$$

and the equation for the current axis $\vec{t}$ becomes

$$\vec{t} = -(1/2c)\,\overleftrightarrow{\bar{\sigma}} \cdot \dot{\vec{B}}\,. \quad (2.10)$$

The factor of 2 has been inserted in (2.8) so that for an isotropic material $\overleftrightarrow{\bar{\sigma}} = \overleftrightarrow{\sigma}$.

It is interesting to calculate the charge density ρ_c in the sample:

$$\rho_c = \frac{1}{4\pi}\,\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi}\,\vec{\nabla}\cdot\overleftrightarrow{\rho}\cdot(\vec{t}\times\vec{r}) = \frac{1}{4\pi}\epsilon_{ijk}\rho_{ij}t_k\,. \quad (2.11)$$

This is constant throughout the sample (balanced by an opposite charge distributed over the surface). It vanishes if $\overleftrightarrow{\rho}$ is diagonal, and corresponds to off-diagonal conductivity components (produced, say, by the Hall effect if $\vec{B}$ has a component along $\vec{t}$) trying to push the eddy currents toward the surface, building up a charge there. In most cases considered in this paper there is no charge because $\vec{t}$ has no z component and $\rho_{xz} = \rho_{yz} = 0$ (no transverse-longitudinal mixing), so none of i, j, or k in (2.11) may be z without either ρ_{ij} vanishing.

The torque on a unit volume is given by the Lorentz force:

$$d\vec{N} = \vec{r}\times d\vec{F} = (1/c)\,\vec{r}\times(\vec{j}\times\vec{B})\,d^3\vec{r}$$
$$= (1/c)[\vec{j}(\vec{B}\cdot\vec{r}) - \vec{B}(\vec{r}\cdot\vec{j})]\,d^3\vec{r}$$
$$= (1/c)\vec{j}(\vec{B}\cdot\vec{r})\,d^3r\,. \quad (2.12)$$

The last step in (2.12) follows from Eq. (2.6). Inserting (2.6) for $\vec{j}$ and integrating over the sphere we obtain

$$\vec{N} = (1/c)\int(\vec{t}\times\vec{r})\vec{r}\cdot\vec{B}\,d^3r$$
$$= (1/c)\vec{t}\times\left[\int\vec{r}\,\vec{r}\,d^3r\right]\cdot\vec{B}\,. \quad (2.13)$$

The bracketed tensor is easily seen to be diagonal, with diagonal elements $\frac{4}{15}\pi R^5$, where R is the radius of the sphere. Thus

$$\vec{N} = (4\pi R^5/15c)\,\vec{t}\times\vec{B} = -(4\pi R^5/15c)\vec{B}\times\vec{t}\,. \quad (2.14)$$

Inserting $\vec{t}$ from (2.10), we have

$$\vec{N} = (2\pi R^5/15c^2)\vec{B}\times\overleftrightarrow{\bar{\sigma}}\cdot\dot{\vec{B}}. \quad (2.15)$$

This gives the torque on a spherical sample in a uniform magnetic field or arbitrary direction and rate of change, with $\overleftrightarrow{\bar{\sigma}}$ given by (2.9). For the geometry used in Ref. 7 $\vec{B} = B\hat{z}$, $\dot{\vec{B}} = \Omega B\hat{x}$, and N_y is measured

$$N_y = (2\pi R^5/15c^2)B^2\Omega\bar{\sigma}_{xx}\,. \quad (2.16)$$

We may write $\bar{\sigma}_{xx}$ explicitly from (2.8) and (2.9):

$$\bar{\sigma}_{xx} = 2[(\rho_{xx}+\rho_{zz})(\rho_{xx}+\rho_{yy}) - \rho_{yz}\rho_{zy}]\{[(\rho_{xx}+\rho_{zz})(\rho_{xx}+\rho_{yy}) - \rho_{yz}\rho_{zy}](\rho_{yy}+\rho_{zz}) - (\rho_{xx}+\rho_{zz})\rho_{xz}\rho_{zx}$$
$$- (\rho_{xx}+\rho_{yy})\rho_{xy}\rho_{yz} - \rho_{xy}\rho_{yz}\rho_{zx} - \rho_{xz}\rho_{zy}\rho_{yx}\}^{-1}\,. \quad (2.17)$$

If we make the simplifying assumption of no longitudinal-transverse mixing

$$\sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = 0\,, \quad (2.18)$$

we find

$$\bar{\sigma}_{xx} = \frac{2(\rho_{xx}+\rho_{zz})}{(\rho_{xx}+\rho_{zz})(\rho_{yy}+\rho_{zz}) + \rho_{xy}\rho_{yx}}\,. \quad (2.19)$$

The formula for the torque directly in terms of the conductivity $\overleftrightarrow{\sigma}$ is somewhat more complicated. However, for the very common cases in which either the longitudinal magnetoresistance ρ_{zz} can be neglected in comparison with the transverse one ρ_{xx} (high-field limit),

$$\rho_{zz} \ll \rho_{xx}\,, \quad (2.20)$$

or if the Hall resistivity ρ_{xy} can be neglected (low-field limit), then (2.19) can be approximated by

$$\bar{\sigma}_{xx} \cong 2/(\sigma_{yy}^{-1} + \sigma_{zz}^{-1})\,. \quad (2.21)$$

In the high-field limit, we might also add that, since

$$\sigma_{yy} \ll \sigma_{zz}\,, \quad (2.22)$$

we obtain

$$\bar{\sigma}_{xx} \cong 2\sigma_{yy}\,. \quad (2.23)$$

In Ref. 1, the experimental results for cadmium and gallium were interpreted using (2.16), but with $\bar{\sigma}_{xx}$ given by

$$\bar{\sigma}_{xx} = 2(\rho_{yy}+\rho_{zz})^{-1} \quad (2.24)$$

instead of, say, (2.20). For a compensated metal like cadmium this is nearly correct ($\sigma_{xy}/\sigma_{yy} \to 0$ at high fields, and is quite small even at low fields owing to the open orbits). However, the induced torque in copper was interpreted using the more exact result (2.19); in the uncompensated case, (2.24) is not applicable. This is because in computing $\overleftrightarrow{\rho}$ we use the Hall conductivity in an essential way, whereas it actually has very little effect on the torque about the y axis. It twists the eddy currents out of the yz plane and produces a torque about the x axis, but leaves the y axis torque unchanged.

III. ASYMPTOTIC BEHAVIOR

The asymptotic dependence of the resistivity tensor on the magnetic field is tabulated in Table I, together with the resulting effective conductivity $\overline{\sigma}_{xx}$ and the torque for the geometry of Fig. 1. The asymptotic forms are given for the standard three types of electronic structure and orbit topology: an uncompensated metal with all closed orbits, a compensated one, and a metal with open orbits along a direction which makes an angle α with the y axis in real space (x axis in reciprocal space).

The important result is that the torque always increases proportionally to B^2 in the presence of open orbits and saturates in their absence, irrespective of the state of compensation of the metal. The experimental work[1] has determined the similarity of the induced torque for compensated and uncompensated metals. The only exception is when the axis of suspension and the open orbit direction in real space are exactly perpendicular to one another. This is in marked contrast to the situation faced in making resistivity[6] measurements, where (as can be seen in Table I) the behavior of the resistivity depends critically on the compensation.

TABLE I. Asymptotic behavior of the induced torque and the galvanomagnetic tensors.

	All closed orbits		Open orbits[a]
	Uncompensated $n_e \neq n_h$	Compensated $n_e = n_h$	
ρ_{xx}	B^0	B^2	$a_1B^2\cos^2\alpha + a_2$
ρ_{yy}	B^0	B^2	$b_1B^2\sin^2\alpha + b_2$
ρ_{zz}	B^0	B^0	B^0
ρ_{xy}	B	B^2	B
$\overline{\sigma}_{xx}$	B^{-2}	B^{-2}	$c_1\cos^2\alpha + c_2B^{-2}$
Torque	B^0	B^0	$d_1B^2\cos^2\alpha + d_2$

[a]In k space the direction of open orbits makes an angle α with the x axis; in real space they are at an angle α from the y direction.

The advantages of the torque method are therefore (a) it distinguishes unequivocally the presence of open orbits from any other situation and (b) it does not require careful alignment of the sample to detect the open orbits.

For the divalent metals, in which we find either open orbits or compensated closed orbits, the advantages are thus enormous. In the particular case of zinc[5] the torque method has permitted the detection of magnetic breakdown between the first and third bands; such a phenomenon could not be detected with any other technique.

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