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[10]Note that Eq. (4.7) of Ref. 5 contains an error. It should read

$$I_1 + I_1' = \frac{1}{2\pi}\left(\frac{\partial H_c(T)}{\partial T}\right)^2_{T\to T_c}$$

instead of $I_1 - I_1'$ on the left-hand side of this equation. As a matter of fact, one gets

$$\left(\frac{H_{c2,\,\mathrm{obs}}}{H_{c2,\,\mathrm{BCS}}}\right)_{T\to T_c} = \frac{H_{c,\mathrm{obs}}{}^2\,\Delta_{\mathrm{BCS}}{}^2}{H_{c,\mathrm{BCS}}{}^2\,\Delta_{\mathrm{obs}}{}^2}\Bigg|_{T\to T_c} .$$

Using the numerical values quoted below Eq. (33) of the present paper, we get for lead $(H_{c2,obs}/H_{c2,BCS})_{T\to T_c} = 1.3$.

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PHYSICAL REVIEW B VOLUME 2, NUMBER 1 1 JULY 1970

Superconductive Properties of the Excitonic Insulator*

R. H. Parmenter and W. R. Henson

Department of Physics, University of Arizona, Tucson, Arizona 85721

(Received 14 July 1969)

Analysis of a simple model of the excitonic insulator shows that the ordered phase exhibits electrical superconductivity whenever the conduction-band mass differs from the valence-band mass. *Interband* scattering of electrons by the magnetic vector potential plays an essential role. States of finite electric persistent current are demonstrated explicitly. The excitonic insulator is a system where diagonal versus off-diagonal long-range order is a function of one's bookkeeping.

I. INTRODUCTION

It is commonly believed that the excitonic insulator,[1] a hypothetical many-particle cooperative thermodynamic phase involving valence-band holes and conduction-band electrons, has the electrical properties of an insulator.[2] In this paper, we show that this is not the case for a simple model; rather the model has the electrical properties of a superconductor. Our model consists of a single spherical valence band $(\hbar^2/2m_a)(k_F^2 - k^2)$ and a single spherical overlapping conduction band $(\hbar^2/2m_b)(k^2 - k_F^2)$, energies being measured relative to that of the Fermi surface $k = k_F$. We show that whenever $m_b \neq m_a$, there is superconductivity.[3] Crucial to this demonstration is the inclusion of both *intraband* scattering and *interband* scattering of electrons by the magnetic vector potential. The latter kind of scattering has not been considered in the past.

Note that a hole at $-\vec{k}$ in an otherwise filled valence band can be consistently thought of as an excitation of momentum $+\hbar\vec{k}$, mass $+m_a$, energy $(\hbar^2/2m_a)(k^2 - k_F^2)$, and electric charge $+e$ ($-e$ being the electronic charge). We designate by $c^\dagger_{k\downarrow}$, $c_{k\downarrow}$ the creation and destruction operators associated with such an excitation of momentum $\hbar\vec{k}$. (Thus $c^\dagger_{k\downarrow}$ removes an electron from one-electron state $-\vec{k}$ in the valence band.) We designate by $c^\dagger_{k\uparrow}$, $c_{k\uparrow}$ the creation and destruction operators associated with an electron of momentum $\hbar\vec{k}$ in the conduction band.

In the absence of external electric and magnetic fields, the Hamiltonian for the excitonic insulator is

$$H_0 = \sum_{k,\sigma} \epsilon_{k\sigma}\, c^\dagger_{k\sigma} c_{k\sigma} - \sum_{k,k'} V_{kk'}\, c^\dagger_{-k'\downarrow} c^\dagger_{k'\uparrow} c_{k\uparrow} c_{-k\downarrow} , \tag{1.1}$$

where

$$\epsilon_{k\sigma} = (\hbar^2/2m_\sigma)(k^2 - k_F^0), \quad m_\downarrow = m_a, \quad m_\uparrow = m_b . \tag{1.2}$$

$V_{kk'}$ is the matrix element for the attractive Coulomb scattering between holes and electrons. The ground-state wave function of the ordered phase has the form

$$\Psi_0 = \prod_k [(1 - h_k)^{1/2} + h_k^{1/2}\, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}]\,\Phi_0, \tag{1.3}$$

Φ_0 being the state of filled valence band and empty conduction band. In the interest of simplicity, we make the nonessential assumption of constant $V_{kk'} = V$. Minimizing the expectation value of the free energy of the system with respect to the parameters h_k, we find

$$h_k = \tfrac{1}{2}[1 - (\epsilon_k/E_k)] , \tag{1.4}$$

where $\epsilon_k = \tfrac{1}{2}(\epsilon_{k\uparrow} + \epsilon_{k\downarrow})$, (1.5)

$$E_k = (\epsilon_k^2 + \epsilon_0^2)^{1/2} . \tag{1.6}$$

The order parameter ϵ_0 obeys the BCS gap equation[4]

$$1 = \tfrac{1}{2} V \textstyle\sum_k E_k^{-1} (1 - f_{k\uparrow} - f_{k\downarrow}) . \tag{1.7}$$

Here $f_{k\sigma}$ is the thermodynamic Fermi factor associated with quasiparticle excitation $(\vec{k}, \sigma)$ having excitation energy $E_{k\sigma}$, given by

$$E_{k\uparrow} = E_k + \bar{\epsilon}_k , \quad E_{k\downarrow} = E_k - \bar{\epsilon}_k , \tag{1.8}$$

where $\bar{\epsilon}_k = \tfrac{1}{2}(\epsilon_{k\uparrow} - \epsilon_{k\downarrow})$. (1.9)

It will later be useful to write $\bar{\epsilon}_k$ in the form

$$\bar{\epsilon}_k = \alpha\epsilon_k , \quad \alpha = (m_a - m_b)/(m_a + m_b) . \tag{1.10}$$

The destruction operators associated with the two types of quasiparticle excitation have the form

$$\begin{aligned} \alpha_{k\uparrow} &= (1 - h_k)^{1/2} c_{k\uparrow} - h_k^{1/2} c^\dagger_{-k\downarrow} , \\ \alpha_{-k\downarrow} &= (1 - h_k)^{1/2} c_{-k\downarrow} + h_k^{1/2} c^\dagger_{k\uparrow} . \end{aligned} \tag{1.11}$$

The inverse relations are

$$\begin{aligned} c_{k\uparrow} &= (1 - h_k)^{1/2} \alpha_{k\uparrow} + h_k^{1/2} \alpha^\dagger_{-k\downarrow} , \\ c_{k\downarrow} &= (1 - h_k)^{1/2} \alpha_{k\downarrow} - h_k^{1/2} \alpha^\dagger_{-k\uparrow} . \end{aligned} \tag{1.12}$$

II. ELECTROMAGNETIC RESPONSE

In the presence of weak perturbing electric and magnetic fields (Coulomb gauge, with vanishing electric potential), the perturbing Hamiltonian is

$$H = H' + H'' , \tag{2.1}$$

$$H' = -(2\pi)^{3/2}\left(\frac{\hbar}{2c}\right) \sum_{k,q,\sigma} \left(\frac{e_\sigma}{m_\sigma}\right) (2\vec{k} + \vec{q}) \cdot \vec{A}_q c^\dagger_{k+q,\sigma} c_{k,\sigma} , \tag{2.2}$$

$$\begin{aligned} H'' = -(2\pi)^{3/2}(\hbar/2c) \ &[\textstyle\sum_\sigma (e_\sigma/m_\sigma)] \\ &\times \sum_{k,q} [(\vec{M}_{-k} + \vec{M}_{-k-q}) \, c^\dagger_{-k,\uparrow} c^\dagger_{k+q,\downarrow} \\ &+ (\vec{M}^*_k + \vec{M}^*_{k-q}) \, c_{-k,\downarrow} c_{k-q,\uparrow}] \cdot \vec{A}_q . \end{aligned} \tag{2.3}$$

Here $\vec{A}_q = (2\pi)^{-3/2} \int \vec{A}(\vec{r}) \, e^{-i\vec{q}\cdot\vec{r}} d^3r$ (2.4)

is the $\vec{q}$th Fourier component of the magnetic vector potential $\vec{A}(\vec{r})$. $\vec{M}_k$ is the one-electron matrix element of $(\vec{p}/\hbar)$ between *one-electron states* $\vec{k}$ in the valence band and $\vec{k}$ in the conduction band. H' represents the contribution of *intraband* scattering of electrons by the vector potential to the Hamiltonian; H'' represents the contribution of *interband* scattering of electrons.[5] Making use of the fact that $\vec{q}\cdot\vec{A}_q$ vanishes, we can rewrite H' as

$$\begin{aligned} H' = \tfrac{1}{2}(m_b^{-1} + m_a^{-1}) \, (2\pi)^{3/2} (\hbar e/c) \\ \times \sum_{k,q} \vec{k} \cdot \vec{A}_q \{c^\dagger_{k+q,\uparrow} c_{k,\uparrow} + c^\dagger_{-k,\downarrow} c_{-k-q,\downarrow}\} \\ + \tfrac{1}{2}(m_b^{-1} - m_a^{-1}) \, (2\pi)^{3/2} (\hbar e/c) \\ \times \sum_{k,q} \vec{k} \cdot \vec{A}_q \{c^\dagger_{k+q,\uparrow} c_{k,\uparrow} - c^\dagger_{-k,\downarrow} c_{-k-q,\downarrow}\} . \end{aligned} \tag{2.5}$$

Since momentum is odd under time reversal, we have

$$\vec{M}_{-k} = -\vec{M}_k . \tag{2.6}$$

This allows us to rewrite H'' as

$$\begin{aligned} H'' = \tfrac{1}{4}(m_b^{-1} - m_a^{-1}) \, (2\pi)^{3/2} (\hbar e/c) \\ \times \sum_{k,q} \vec{A}_q \cdot \{[(\vec{M}_k + \vec{M}_{k+q}) + (\vec{M}^*_k + \vec{M}^*_{k+q})] \\ \times [c^\dagger_{k+q,\uparrow} c^\dagger_{-k,\uparrow} + c_{-k-q,\uparrow} c_{k,\downarrow}] \\ + [(\vec{M}_k + \vec{M}_{k+q}) - (\vec{M}^*_k + \vec{M}^*_{k+q})] \\ \times [c^\dagger_{k+q,\downarrow} c^\dagger_{-k,\uparrow} - c_{-k-q,\uparrow} c_{k,\downarrow}]\} . \end{aligned} \tag{2.7}$$

In both Eqs. (2.5) and (2.7), we have grouped together the terms that interfere coherently in the ordered phase. As far as H' is concerned, in BCS terminology, the factors containing a plus sign are type I perturbations, the factors containing a minus sign are type II perturbations. Superconductivity theory has no analog of the terms appearing in H''. It is possible to anticipate the results of the calculation by the following observation. In Eq. (2.5), any two coherently interfering terms correspond to the particles of both signs of charge getting the *same* momentum transfer. In Eq. (2.7), any two coherently interfering terms correspond to the particles of both signs of charge getting *opposite* momentum transfer, in the limit $\vec{q}\to 0$. The former case leads to a *vanishing* increment of electric current; the latter case to a *finite* increment. This allows us to infer that the superconductive response is associated with H'', not with H'.

We define the paramagnetic current operator

$$\mathcal{J}_{qP} = -(2\pi)^{-3} c(\delta H/\delta \vec{A}_q) . \tag{2.8}$$

Obviously, we can write

$$\begin{aligned} \mathcal{J}_{qP} &= \mathcal{J}'_{qP} + \mathcal{J}''_{qP} , \\ \mathcal{J}'_{qP} &= -(2\pi)^{-3} c(\delta H'/\delta \vec{A}_q) , \end{aligned} \tag{2.9}$$

$$\mathcal{J}''_{qP} = -(2\pi)^{-3}c(\delta H''/\delta \vec{A}_q) .$$

We get

$$\mathcal{J}'_{qP} = -\tfrac{1}{4}(m_b^{-1}+m_a^{-1})(2\pi)^{-3/2}(\hbar e) \times \sum_k (2\vec{k}+\vec{q})[c^\dagger_{k+q,\uparrow}c_{k,\uparrow}+c^\dagger_{-k,\downarrow}c_{-k-q,\downarrow}] - \tfrac{1}{4}(m_b^{-1}-m_a^{-1})(2\pi)^{-3/2}(\hbar e) \times \sum_k (2\vec{k}+\vec{q})[c^\dagger_{k+q,\uparrow}c_{k,\uparrow}-c^\dagger_{-k,\downarrow}c_{-k-q,\downarrow}] , \tag{2.10}$$

$$\mathcal{J}''_{qP} = -\tfrac{1}{4}(m_b^{-1}-m_a^{-1})(2\pi)^{-3/2}(\hbar e) \times \sum_k \{[(\vec{M}_k+\vec{M}_{k+q})+(\vec{M}^*_k+\vec{M}^*_{k+q})] \times [c^\dagger_{k+q,\downarrow}c^\dagger_{-k,\uparrow}+c_{-k-q,\uparrow}c_{k,\downarrow}] + [(\vec{M}_k+\vec{M}_{k+q})-(\vec{M}^*_k+\vec{M}^*_{k+q})] \times [c^\dagger_{k+q,\downarrow}c^\dagger_{-k,\uparrow}-c_{-k-q,\uparrow}c_{k,\downarrow}]\} . \tag{2.11}$$

The electric current density $\vec{J}(\vec{r})$ resulting from $\vec{A}(\vec{r})$ can be broken up into a paramagnetic contribution and a diamagnetic contribution. The $\vec{q}$th Fourier component of the diamagnetic contribution is

$$\vec{J}_{qD} = -(ne^2/c)(m_b^{-1}+m_a^{-1})\vec{A}_q , \tag{2.12}$$

n being the density of carriers of either sign. The $\vec{q}$th Fourier component of the paramagnetic contribution is

$$\vec{J}_{qP} = \vec{J}'_{qP}+\vec{J}''_{qP} , \tag{2.13}$$

where

$$\vec{J}'_{qP} = -2\sum_i (W_i-W_0)^{-1}\langle 0|\mathcal{J}'_{qP}|i\rangle\langle i|H'|0\rangle , \tag{2.14}$$

$$\vec{J}''_{qP} = -2\sum_i (W_i-W_0)^{-1}\langle 0|\mathcal{J}''_{qP}|i\rangle\langle i|H''|0\rangle . \tag{2.15}$$

Here $|0\rangle$ is the ground state (at a given temperature) and $|i\rangle$ is an excited state of excitation energy (W_i-W_0). In the normal phase ($\epsilon_0=0$), we may approximate the paramagnetic contribution $\vec{J}_{qPN}$ by the relation

$$\vec{J}_{qPN} = -\vec{J}_{qD} . \tag{2.16}$$

The relation is exact at $\vec{q}=0$. At finite $\vec{q}$, it ignores the weak *net* diamagnetism associated with the normal phase. In the ordered phase ($\epsilon_0\neq 0$), we are interested in

$$\vec{J}_q = \vec{J}_{qP}+\vec{J}_{qD} \tag{2.17}$$

in the limit as $\vec{q}\to 0$. If lim $\vec{J}_q$ as $q\to 0$ is finite, then the ordered phase is superconductive. It is convenient to combine Eqs. (2.16) and (2.17),

$$\vec{J}_q = \vec{J}_{qP}-\vec{J}_{qPN} . \tag{2.18}$$

The summands of Eqs. (2.14) and (2.15) appropriate to the ordered phase differ from the corresponding summands for the normal phase only when the excited state i is associated with points in $\vec{k}$ space close to the Fermi surface. Thus all slowly varying functions of the magnitude of $\vec{k}$ appearing in these summands may be replaced by their values at the Fermi surface, as long as we use Eq. (2.18) in evaluating $\vec{J}_q$ for the ordered phase.

If we define

$$\mathcal{L}^{(1)}_{k,l} = +\sum_i (W_i-W_0)^{-1}|\langle i|c^\dagger_{l,\uparrow}c_{k,\uparrow}+c^\dagger_{-k,\downarrow}c_{-l,\downarrow}|0\rangle|^2$$

$$\mathcal{L}^{(2)}_{k,l} = +\sum_i (W_i-W_0)^{-1}|\langle i|c^\dagger_{l,\uparrow}c_{k,\uparrow}-c^\dagger_{-k,\downarrow}c_{-l,\downarrow}|0\rangle|^2 , \tag{2.19}$$

$$\mathcal{L}^{(3)}_{k,l} = +\sum_i (W_i-W_0)^{-1}\langle i|c^\dagger_{l,\uparrow}c_{k,\uparrow}+c^\dagger_{-k,\downarrow}c_{-l,\downarrow}|0\rangle \times\langle i|c^\dagger_{l,\uparrow}c_{k,\uparrow}-c^\dagger_{-k,\downarrow}c_{-l,\downarrow}|0\rangle ;$$

and

$$\vec{J}^{(1)}_{qP} = \tfrac{1}{4}(m_b^{-1}+m_a^{-1})^2(\hbar^2e^2/c) \times\sum_k (2\vec{k}+\vec{q})(\vec{k}\cdot\vec{A}_q)\mathcal{L}^{(1)}_{k,k+q} ,$$

$$\vec{J}^{(2)}_{qP} = \tfrac{1}{4}(m_b^{-1}-m_a^{-1})^2(\hbar^2e^2/c) \times\sum_k (2\vec{k}+\vec{q})(\vec{k}\cdot\vec{A}_q)\mathcal{L}^{(2)}_{k,k+q} , \tag{2.20}$$

$$\vec{J}^{(3)}_{qP} = \tfrac{1}{2}(m_b^{-2}-m_a^{-2})(\hbar^2e^2/c) \times\sum_k (2\vec{k}+\vec{q})(\vec{k}\cdot\vec{A}_q)\mathcal{L}^{(3)}_{k,k+q} ;$$

then we can write

$$\vec{J}'_{qP} = \vec{J}^{(1)}_{qP}+\vec{J}^{(2)}_{qP}+\vec{J}^{(3)}_{qP} . \tag{2.21}$$

Similarly, if we define

$$\mathcal{L}^{(4)}_{k,l} = +\sum_i (W_i-W_0)^{-1}|\langle i|c^\dagger_{l,\downarrow}c^\dagger_{-k,\uparrow}+c_{-l,\uparrow}c_{k,\downarrow}|0\rangle|^2 ,$$

$$\mathcal{L}^{(5)}_{k,l} = +\sum_i (W_i-W_0)^{-1}|\langle i|c^\dagger_{l,\downarrow}c^\dagger_{-k,\uparrow}-c_{-l,\uparrow}c_{k,\downarrow}|0\rangle|^2 , \tag{2.22}$$

$$\mathcal{L}^{(6)}_{k,l} = +\sum_i (W_i-W_0)^{-1}\langle i|c^\dagger_{l,\downarrow}c^\dagger_{-k,\uparrow}+c_{-l,\uparrow}c_{k,\downarrow}|0\rangle \times\langle i|c^\dagger_{l,\downarrow}c^\dagger_{-k,\uparrow}-c_{-l,\uparrow}c_{k,\downarrow}|0\rangle ; \quad \text{and}$$

$$\vec{J}^{(4)}_{qP} = \tfrac{1}{8}(m_b^{-1}-m_a^{-1})^2(\hbar^2e^2/c)\sum_k \mathcal{L}^{(4)}_{k,k+q} \times[(\vec{M}_k+\vec{M}_{k+q})+(\vec{M}^*_k+\vec{M}^*_{k+q})] \times\{[(\vec{M}_k+\vec{M}_{k+q})+(\vec{M}^*_k+\vec{M}^*_{k+q})]\cdot\vec{A}_q\} ,$$

$$\vec{J}^{(5)}_{qP} = \tfrac{1}{8}(m_b^{-1} - m_a^{-1})^2(\hbar^2 e^2/c)\sum_k \mathfrak{L}^{(5)}_{k,k+q} \quad (2.23)$$
$$\times[(\vec{M}_k + \vec{M}_{k+q}) - (\vec{M}^*_k + \vec{M}^*_{k+q})]$$
$$\times\{[(\vec{M}^*_k + \vec{M}^*_{k+q}) - (\vec{M}_k + \vec{M}_{k+q})]\cdot\vec{A}_q\},$$
$$\vec{J}^{(6)}_{qP} = \tfrac{1}{4}(m_b^{-1} - m_a^{-1})^2(\hbar^2 e^2/c)\sum_k \mathfrak{L}^{(6)}_{k,k+q}$$
$$\times[(\vec{M}_k + \vec{M}_{k+q})\,(\vec{M}^*_k + \vec{M}^*_{k+q})\cdot\vec{A}_q$$
$$-(\vec{M}^*_k + \vec{M}^*_{k+q})\,(\vec{M}_k + \vec{M}_{k+q})\cdot\vec{A}_q];$$

then we can write

$$\vec{J}''_{qP} = \vec{J}^{(4)}_{qP} + \vec{J}^{(5)}_{qP} + \vec{J}^{(6)}_{qP}\,. \quad (2.24)$$

Next, we need to evaluate the $\mathfrak{L}$'s. With the aid of Eq. (1.12), we get

$$(c^\dagger_{l,\uparrow}c_{k,\uparrow} \pm c^\dagger_{-k,\downarrow}c_{-l,\downarrow}) = h_k(1\pm 1)\,\delta_{k,l}$$
$$+\{[h_k(1-h_l)]^{1/2} \pm [h_l(1-h_k)]^{1/2}\}$$
$$\times(\alpha^\dagger_{l,\uparrow}\alpha^\dagger_{-k,\downarrow} \pm \alpha_{-l,\downarrow}\alpha_{k,\uparrow})$$
$$+\{[(1-h_k)\,(1-h_l)]^{1/2} \mp [h_k h_l]^{1/2}\}$$
$$\times(\alpha^\dagger_{l,\uparrow}\alpha_{k,\uparrow} \pm \alpha^\dagger_{-k,\downarrow}\alpha_{-l,\downarrow})\,, \quad (2.25)$$

$$(c^\dagger_{l,\downarrow}c^\dagger_{-k,\uparrow} \pm c_{-l,\uparrow}c_{k,\downarrow}) = -[h_k(1-h_k)]^{1/2}(1\pm 1)\,\delta_{k,l}$$
$$+\{[h_k(1-h_l)]^{1/2} \pm [h_l(1-h_k)]^{1/2}\}$$
$$\times(\alpha^\dagger_{l,\downarrow}\alpha_{k,\downarrow} \pm \alpha^\dagger_{-k,\uparrow}\alpha_{-l,\uparrow})$$
$$+\{[(1-h_k)\,(1-h_l)]^{1/2} \mp [h_k h_l]^{1/2}\}$$
$$\times(\alpha^\dagger_{l,\downarrow}\alpha^\dagger_{-k,\uparrow} \pm \alpha_{-l,\uparrow}\alpha_{k,\downarrow})\,. \quad (2.26)$$

From Eqs. (1.4) and (1.6), we have

$$\{[(1-h_k)\,(1-h_l)]^{1/2} \mp [h_k h_l]^{1/2}\}^2$$
$$= \tfrac{1}{2} + \tfrac{1}{2}(1-2h_k)\,(1-2h_l) \mp 2\,[h_k(1-h_k)\,h_l(1-h_l)]^{1/2}$$
$$= \tfrac{1}{2}\,[1 + (\epsilon_k\epsilon_l \mp \epsilon_0^2/E_k E_l)]\,, \quad (2.27)$$
$$\{[h_k(1-h_l)]^{1/2} \pm [h_l(1-h_k)]^{1/2}\}^2$$
$$= \tfrac{1}{2} - \tfrac{1}{2}(1-2h_k)(1-2h_l) \pm 2\,[h_k(1-h_k)\,h_l\,(1-h_l)]^{1/2}$$
$$= \tfrac{1}{2}\,[1 - (\epsilon_k\epsilon_l \mp \epsilon_0^2/E_k E_l)]\,, \quad (2.28)$$
$$\{[(1-h_k)\,(1-h_l)]^{1/2} + [h_k h_l]^{1/2}\}$$
$$\times\{[(1-h_k)\,(1-h_l)]^{1/2} - [h_k h_l]^{1/2}\}$$
$$= \tfrac{1}{2}\,[(1-2h_k) + (1-2h_l)]$$
$$= \tfrac{1}{2}(\epsilon_k/E_k + \epsilon_l/E_l)\,, \quad (2.29)$$
$$\{[h_k(1-h_l)]^{1/2} + [h_l(1-h_k)]^{1/2}\}'$$
$$-[h_l(1-h_k)]^{1/2}\}\quad\{[h_k(1-h_l)]^{1/2}$$
$$= -\tfrac{1}{2}[(1-2h_k) - (1-2h_l)]$$
$$= -\tfrac{1}{2}(\epsilon_k/E_k - \epsilon_l/E_l)\,. \quad (2.30)$$

We therefore get

$$\mathfrak{L}^{(1)}_{k,l} = -\tfrac{1}{2}\left[1 - \left(\frac{\epsilon_k\epsilon_l - \epsilon_0^2}{E_k E_l}\right)\right]\left(\frac{f_{k\uparrow}f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}}\right) - \left[\frac{(1-f_{k\downarrow})\,(1-f_{l\uparrow})}{(E_{k\downarrow}+E_{l\uparrow})}\right]$$
$$-\tfrac{1}{2}\left[1 + \left(\frac{\epsilon_k\epsilon_l - \epsilon_0^2}{E_k E_l}\right)\right]\left[\frac{f_{k\uparrow}(1-f_{l\uparrow})}{(E_{k\uparrow}-E_{l\uparrow})} + \frac{f_{l\downarrow}(1-f_{k\downarrow})}{(E_{l\downarrow}-E_{k\downarrow})}\right]\,, \quad (2.31)$$

$$\mathfrak{L}^{(2)}_{k,l} = -\tfrac{1}{2}\left[1 - \left(\frac{\epsilon_k\epsilon_l + \epsilon_0^2}{E_k E_l}\right)\right]\left[\left(\frac{f_{k\uparrow}f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}}\right) - \frac{(1-f_{k\downarrow})\,(1-f_{l\uparrow})}{(E_{k\downarrow}+E_{l\uparrow})}\right]$$
$$-\tfrac{1}{2}\left[1 + \left(\frac{\epsilon_k\epsilon_l + \epsilon_0^2}{E_k E_l}\right)\right]\left[\frac{f_{k\uparrow}(1-f_{l\uparrow})}{(E_{k\uparrow}-E_{l\uparrow})} + \frac{f_{l\downarrow}(1-f_{k\downarrow})}{(E_{l\downarrow}-E_{k\downarrow})}\right]\,, \quad (2.32)$$

$$\mathfrak{L}^{(3)}_{k,l} = -\tfrac{1}{2}\left(\frac{\epsilon_k}{E_k} - \frac{\epsilon_l}{E_l}\right)\left[\frac{f_{k\uparrow}f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}} + \frac{(1-f_{k\downarrow})\,(1-f_{l\uparrow})}{(E_{k\downarrow}+E_{l\uparrow})}\right]$$
$$-\tfrac{1}{2}\left(\frac{\epsilon_k}{E_k} + \frac{\epsilon_l}{E_l}\right)\left[\frac{f_{k\uparrow}(1-f_{l\uparrow})}{(E_{k\uparrow}-E_{l\uparrow})} - \frac{f_{l\downarrow}(1-f_{k\downarrow})}{(E_{l\downarrow}-E_{k\downarrow})}\right]\,, \quad (2.33)$$

$$\mathfrak{L}^{(4)}_{k,l} = -\tfrac{1}{2}\left[1 + \left(\frac{\epsilon_k\epsilon_l - \epsilon_0^2}{E_k E_l}\right)\right]\left[\left(\frac{f_{k\downarrow}f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right) - \frac{(1-f_{k\uparrow})\,(1-f_{l\downarrow})}{(E_{k\uparrow}+E_{l\downarrow})}\right]$$
$$-\tfrac{1}{2}\left[1 - \left(\frac{\epsilon_k\epsilon_l - \epsilon_0^2}{E_k E_l}\right)\right]\left[\frac{f_{k\downarrow}(1-f_{l\downarrow})}{(E_{k\downarrow}-E_{l\downarrow})} + \frac{f_{l\uparrow}(1-f_{k\uparrow})}{(E_{l\uparrow}-E_{k\uparrow})}\right]\,, \quad (2.34)$$

$$\mathfrak{L}^{(5)}_{k,l} = -\tfrac{1}{2}\left[1 + \left(\frac{\epsilon_k\epsilon_l + \epsilon_0^2}{E_k E_l}\right)\right]\left[\left(\frac{f_{k\downarrow}f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right) - \frac{(1-f_{k\uparrow})\,(1-f_{l\downarrow})}{(E_{k\uparrow}+E_{l\downarrow})}\right]$$
$$-\tfrac{1}{2}\left[1 - \left(\frac{\epsilon_k\epsilon_l + \epsilon_0^2}{E_k E_l}\right)\right]\left[\frac{f_{k\downarrow}(1-f_{l\downarrow})}{(E_{k\downarrow}-E_{l\downarrow})} + \frac{f_{l\uparrow}(1-f_{k\uparrow})}{(E_{l\uparrow}-E_{k\uparrow})}\right]\,, \quad (2.35)$$

$$\mathfrak{L}^{(6)}_{k,l} = +\tfrac{1}{2}\left(\frac{\epsilon_k}{E_k} + \frac{\epsilon_l}{E_l}\right)\left[\left(\frac{f_{k\downarrow}f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right) + \frac{(1-f_{k\uparrow})\,(1-f_{l\downarrow})}{(E_{k\uparrow}+E_{l\downarrow})}\right]$$
$$+\tfrac{1}{2}\left(\frac{\epsilon_k}{E_k} - \frac{\epsilon_l}{E_l}\right)\left[\frac{f_{k\downarrow}(1-f_{l\downarrow})}{(E_{k\downarrow}-E_{l\downarrow})} - \frac{f_{l\uparrow}(1-f_{k\uparrow})}{(E_{l\uparrow}-E_{k\uparrow})}\right]\,. \quad (2.36)$$

Let $\mathcal{S}$ be the operator which interchanges isospin indices. Obviously, $\vec{J}'_{qP}$ and $\vec{J}''_{qP}$ are invariant to such an operation, provided that, at the same time, we interchange m_a and m_b and interchange $(\vec{M}_k + \vec{M}_{k+q})$ and its complex conjugate. This means that, in the equations for $\vec{J}^{(\nu)}_{qP}$, we can replace $\mathfrak{L}^{(\nu)}_{k,l}$ by $L^{(\nu)}_{k,l}$ without altering the values of $\vec{J}'_{qP}$ and $\vec{J}''_{qP}$. Here we are defining

$$L^{(\nu)}_{k,l} = \tfrac{1}{2}\,[\mathfrak{L}^{(\nu)}_{k,l} + \mathcal{S}\,\mathfrak{L}^{(\nu)}_{k,l}]\,, \qquad \nu = 1,\ 2,\ 4,\ 5 \quad (2.37)$$

and $L^{(\nu)}_{k,l} = \tfrac{1}{2}\,[\mathfrak{L}^{(\nu)}_{k,l} - \ \mathcal{S}^{(\nu)}_{k,l}]$, $\quad \nu = 3,\ 6\,.$ (2.38)

We get

$$L^{(1)}_{k,l} = +\tfrac{1}{4}\left[1 - \left(\frac{\epsilon_k\epsilon_l - \epsilon_0^2}{E_k E_l}\right)\right]$$
$$\times\left[\left(\frac{1-f_{k\uparrow}-f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}}\right) + \left[\left(\frac{1-f_{k\downarrow}-f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right)\right]\right.$$

$$+\tfrac{1}{4}\left[1+\left(\frac{\epsilon_k\epsilon_l-\epsilon_0^2}{E_kE_l}\right)\right]\left[\left(\frac{f_{l\uparrow}-f_{k\uparrow}}{E_{k\uparrow}-E_{l\uparrow}}\right)+\left(\frac{f_{l\downarrow}-f_{k\downarrow}}{E_{k\downarrow}-E_{l\downarrow}}\right)\right], \tag{2.39}$$

$$L_{k,l}^{(2)}=+\tfrac{1}{4}\left[1-\left(\frac{\epsilon_k\epsilon_l+\epsilon_0^2}{E_kE_l}\right)\right]$$
$$\times\left[\left(\frac{1-f_{k\uparrow}-f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}}\right)+\left(\frac{1-f_{k\downarrow}-f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right)\right]$$
$$+\tfrac{1}{4}\left[1+\left(\frac{\epsilon_k\epsilon_l+\epsilon_0^2}{E_kE_l}\right)\right]\left[\left(\frac{f_{l\uparrow}-f_{k\uparrow}}{E_{k\uparrow}-E_{l\uparrow}}\right)+\left(\frac{f_{l\downarrow}-f_{k\downarrow}}{E_{k\downarrow}-E_{l\downarrow}}\right)\right], \tag{2.40}$$

$$L_{k,l}^{(3)}=+\tfrac{1}{4}\left[\frac{\epsilon_k}{E_k}-\frac{\epsilon_l}{E_l}\right]$$
$$\times\left[\left(\frac{1-f_{k\uparrow}-f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}}\right)-\left(\frac{1-f_{k\downarrow}-f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right)\right]$$
$$+\tfrac{1}{4}\left[\frac{\epsilon_k}{E_k}+\frac{\epsilon_l}{E_l}\right]\left[\left(\frac{f_{l\uparrow}-f_{k\uparrow}}{E_{k\uparrow}-E_{l\uparrow}}\right)-\left(\frac{f_{l\downarrow}-f_{k\downarrow}}{E_{k\downarrow}-E_{l\downarrow}}\right)\right], \tag{2.41}$$

$$L_{k,l}^{(4)}=+\tfrac{1}{4}\left[1+\left(\frac{\epsilon_k\epsilon_l-\epsilon_0^2}{E_kE_l}\right)\right]$$
$$\times\left[\left(\frac{1-f_{k\uparrow}-f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}}\right)+\left(\frac{1-f_{k\downarrow}-f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right)\right]$$
$$+\tfrac{1}{4}\left[1-\frac{\epsilon_k\epsilon_l-\epsilon_0^2}{E_kE_l}\right]\left[\left(\frac{f_{l\uparrow}-f_{k\uparrow}}{E_{k\uparrow}-E_{l\uparrow}}\right)+\left(\frac{f_{l\downarrow}-f_{k\downarrow}}{E_{k\downarrow}-E_{l\downarrow}}\right)\right], \tag{2.42}$$

$$L_{k,l}^{(5)}=+\tfrac{1}{4}\left[1+\left(\frac{\epsilon_k\epsilon_l+\epsilon_0^2}{E_kE_l}\right)\right]$$
$$\times\left[\left(\frac{1-f_{k\uparrow}-f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}}\right)+\left(\frac{1-f_{k\downarrow}-f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right)\right]$$
$$+\tfrac{1}{4}\left[1-\left(\frac{\epsilon_k\epsilon_l+\epsilon_0^2}{E_kE_l}\right)\right]\left[\left(\frac{f_{l\uparrow}-f_{k\uparrow}}{E_{k\uparrow}-E_{l\uparrow}}\right)+\left(\frac{f_{l\downarrow}-f_{k\downarrow}}{E_{k\downarrow}-E_{l\downarrow}}\right)\right], \tag{2.43}$$

$$L_{k,l}^{(6)}=+\tfrac{1}{4}\left[\frac{\epsilon_k}{E_k}+\frac{\epsilon_l}{E_l}\right]$$
$$\times\left[\left(\frac{1-f_{k\uparrow}-f_{l\downarrow}}{E_{k\uparrow}+E_{l\downarrow}}\right)-\left(\frac{1-f_{k\downarrow}-f_{l\uparrow}}{E_{k\downarrow}+E_{l\uparrow}}\right)\right]$$
$$+\tfrac{1}{4}\left[\frac{\epsilon_k}{E_k}-\frac{\epsilon_l}{E_l}\right]\left[\left(\frac{f_{l\uparrow}-f_{k\uparrow}}{E_{k\uparrow}-E_{l\uparrow}}\right)-\left(\frac{f_{l\downarrow}-f_{k\downarrow}}{E_{k\downarrow}-E_{l\downarrow}}\right)\right]. \tag{2.44}$$

Setting $\vec{l}=\vec{k}$, we get

$$L_{k,k}^{(1)}=\tfrac{1}{2}\left[\left(\frac{\epsilon_0}{E_k}\right)^2\left(\frac{1}{E_k}\right)+\left(\frac{\epsilon_k}{E_k}\right)^2\frac{d}{dE_k}\right](1-f_{k\uparrow}-f_{k\downarrow}), \tag{2.45}$$

$$L_{k,k}^{(2)}=\tfrac{1}{2}\frac{d}{dE_k}(1-f_{k\uparrow}-f_{k\downarrow}), \tag{2.46}$$

$$L_{k,k}^{(3)}=\tfrac{1}{2}\left(\frac{\epsilon_k}{E_k}\right)\frac{d}{dE_k}(f_{k\downarrow}-f_{k\uparrow}), \tag{2.47}$$

$$L_{k,k}^{(4)}=\tfrac{1}{2}\left[\left(\frac{\epsilon_k}{E_k}\right)^2\frac{1}{E_k}+\left(\frac{\epsilon_0}{E_k}\right)^2\frac{d}{dE_k}\right](1-f_{k\uparrow}-f_{k\downarrow}), \tag{2.48}$$

$$L_{k,k}^{(5)}=\tfrac{1}{2}(1/E_k)(1-f_{k\uparrow}-f_{k\downarrow}), \tag{2.49}$$

$$L_{k,k}^{(6)}=0. \tag{2.50}$$

It is apparent that $L_{k,k}^{(\nu)}$ is an *even* function of ϵ_k, for all ν. Note that

$$L_{k,k}^{(4)}=L_{k,k}^{(5)}+L_{k,k}^{(2)}-L_{k,k}^{(1)}. \tag{2.51}$$

We next insert these results into the various $\vec{J}_{qP}^{(\nu)}$ in order to calculate $\vec{J}'_{qP}$ and $\vec{J}''_{qP}$ in the limit $\vec{q}\to 0$. After averaging over all orientations of $\vec{k}$, we replace the k sums by the equivalent energy integral,

$$\textstyle\sum_k=N(0)\int_{-\hbar\omega}^{+\hbar\omega}d\epsilon_k, \tag{2.52}$$

where $N(0)=6n(\hbar k_F)^{-2}(m_b^{-1}+m_a^{-1})^{-1}$. (2.53)

The attractive Coulomb interaction $-V_{kk'}=-V$ is assumed finite over the region of k space given by

$$|\epsilon_k|\le\hbar\omega. \tag{2.54}$$

(Outside this region there is no pairing, and thus no difference between the ordered and the normal phases.) In the vicinity of the Fermi surface, we assume that $\vec{M}_k$ has the form

$$\vec{M}_k=\vec{k}M, \tag{2.55}$$

M being a complex number independent of $\vec{k}$. In performing the energy integrals, we assume the magnitude of $\vec{k}$ can be replaced by k_F. Since $L_{k,k}^{(1)}$, $L_{k,k}^{(2)}$, and $L_{k,k}^{(3)}$ are negligibly small at $|\epsilon_k|=h\omega$, we replace $\pm h\omega$ by $\pm\infty$ in integrals involving these L's. We get

$$\vec{J}'_{0P}=\lim_{q\to 0}\vec{J}'_{qP}=(ne^2/c)(m_b^{-1}+m_a^{-1})\vec{A}_0$$
$$\times\left[\int_{-\infty}^{\infty}L_{k,k}^{(1)}d\epsilon_k+\alpha^2\int_{-\infty}^{\infty}L_{k,k}^{(2)}d\epsilon_k+2\alpha\int_{-\infty}^{+\infty}L_{k,k}^{(3)}d\epsilon_k\right], \tag{2.56}$$

$$\vec{J}''_{0P}=\lim_{q\to 0}\vec{J}''_{qP}=(ne^2/c)(m_b^{-1}+m_a^{-1})\alpha^2\vec{A}_0$$
$$\times\left[(M+M^*)^2\int_{-\hbar\omega}^{\hbar\omega}L_{k,k}^{(4)}d\epsilon_k-(M-M^*)^2\int_{-\hbar\omega}^{\hbar\omega}L_{k,k}^{(5)}d\epsilon_k\right]. \tag{2.57}$$

The expressions inside the brackets in these last two equations can be considerably simplified. Since

$$\left(\frac{\epsilon_k}{E_k}\frac{d}{dE_k}\right)(f_{k\downarrow}-f_{k\uparrow})+\alpha\frac{d}{dE_k}(1-f_{k\uparrow}-f_{k\downarrow})=\frac{d}{d\epsilon_k}(f_{k\downarrow}-f_{k\uparrow}), \tag{2.58}$$

it follows that

$$\int_{-\infty}^{\infty} L_{k,k}^{(3)}\,d\epsilon_k + \alpha\int_{-\infty}^{\infty} L_{k,k}^{(2)}\,d\epsilon_k$$
$$= \tfrac{1}{2}\int_{-\infty}^{\infty}\left[\left(\frac{\epsilon_k}{E_k}\,\frac{d}{dE_k}\right)(f_{k\downarrow}-f_{k\uparrow}) + \alpha\,\frac{d}{dE_k}\left[(1-f_{k\uparrow}-f_{k\downarrow})\right]d\epsilon_k\right.$$
$$= \tfrac{1}{2}\int_{-\infty}^{\infty}\frac{d}{d\epsilon_k}\,(f_{k\downarrow}-f_{k\uparrow})\,d\epsilon_k = 0 \;. \qquad (2.59)$$

Thus, $\int_{-\infty}^{\infty} L_{k,k}^{(3)}\,d\epsilon_k = -\alpha\int_{-\infty}^{\infty} L_{k,k}^{(2)}\,d\epsilon_k \;.$ (2.60)

Consider the identity

$$1 = \tfrac{1}{2}\int_{-\infty}^{\infty}\frac{d}{d\epsilon_k}\left[\left(\frac{\epsilon_k}{E_k}\right)(1-f_{k\uparrow}-f_{k\downarrow})\right]d\epsilon_k$$
$$= \tfrac{1}{2}\int_{-\infty}^{\infty}(1-f_{k\uparrow}-f_{k\downarrow})\frac{d}{d\epsilon_k}\left(\frac{\epsilon_k}{E_k}\right)d\epsilon_k + \tfrac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\epsilon_k}{E_k}\right)\frac{d}{d\epsilon_k}\,(1-f_{k\uparrow}-f_{k\downarrow})\,d\epsilon_k$$
$$= \tfrac{1}{2}\int_{-\infty}^{\infty}\left[\left(\frac{\epsilon_k}{E_k}\right)^2\frac{1}{E_k} + \left(\frac{\epsilon_k}{E_k}\right)^2\frac{d}{dE_k}\right](1-f_{k\uparrow}-f_{k\downarrow})\,d\epsilon_k + \tfrac{1}{2}\alpha\int_{-\infty}^{\infty}\left(\frac{\epsilon_k}{E_k}\right)\frac{d}{dE_k}\,(f_{k\downarrow}-f_{k\uparrow})\,d\epsilon_k$$
$$= \int_{\infty}^{\infty} L_{k,k}^{(1)}\,d\epsilon_k + \alpha\int_{-\infty}^{\infty} L_{k,k}^{(3)}\,d\epsilon_k \;. \qquad (2.61)$$

Thus,

$$\int_{-\infty}^{\infty} L_{k,k}^{(1)}\,d\epsilon_k = 1 - \alpha\int_{-\infty}^{\infty} L_{k,k}^{(3)}\,d\epsilon_k = 1 + \alpha^2\int_{-\infty}^{\infty} L_{k,k}^{(2)}\,d\epsilon_k \;. \qquad (2.62)$$

The expression inside the brackets of Eq. (2.56) is therefore unity, and

$$\vec{J}'_{0P} = (ne^2/c)\,(m_b^{-1}+m_a^{-1})\,\vec{A}_0 \qquad (2.63)$$

is the same for both the ordered and the normal phases. As anticipated, we see that $\vec{J}'_{0P}$, the current resulting from *intraband* scattering by the vector potential, contributes *nothing* to the total $\vec{J}_0$ calculated from Eq. (2.18).

Equation (1.7) can be rewritten

$$1 = N(0)\,V\int_{-\hbar\omega}^{\hbar\omega} L_{k,k}^{(5)}\,d\epsilon_k \;. \qquad (2.64)$$

Thus, in the ordered phase,

$$\int_{-\hbar\omega}^{\hbar\omega} L_{k,k}^{(5)}\,d\epsilon_k = [N(0)V]^{-1} \;, \qquad (2.65)$$

and similarly, in the normal phase

$$\lim_{\epsilon_0\to 0}\int_{-\hbar\omega}^{\hbar\omega} L_{k,k}^{(5)}\,d\epsilon_k = [N(0)V]^{-1} + \ln(T_c/T) \;, \qquad (2.66)$$

where T is the temperature of the system and T_c is the ordering temperature. For the normal phase, we have

$$\lim_{\epsilon_0\to 0}\int_{-\infty}^{\infty} L_{k,k}^{(2)}\,d\epsilon_k = 1 \;. \qquad (2.67)$$

We can rewrite Eq. (2.57) as

$$\vec{J}''_{0P} = (ne^2/c)\,(m_b^{-1}+m_a^{-1})\,\alpha^2\vec{A}_0\Big[-(M+M^*)^2 + 4MM^*\int_{-\hbar\omega}^{\hbar\omega} L_{k,k}^{(5)}\,d\epsilon_k + (M+M^*)^2(1-\alpha^2)\int_{-\infty}^{\infty} L_{k,k}^{(2)}\,d\epsilon_k\Big] \;. \qquad (2.68)$$

Our final result is

$$\vec{J}_0 = \left[1 - \lim_{\epsilon_0\to 0}\right]\;\vec{J}''_{0P} = -K_0\vec{A}_0 \;, \qquad (2.69)$$

where we are defining

$$K_0 = (ne^2/c)\,(m_b^{-1}+m_a^{-1})\,\alpha^2\{4MM^*\ln(T_c/T) + (M+M^*)^2(1-\alpha^2)\,[1-\textstyle\int_{-\infty}^{\infty} L_{k,k}^{(2)}d\epsilon_k]\} \;. \qquad (2.70)$$

Note that $K_0 > 0$ if $m_b \neq m_a$ and $T < T_c$. This demonstrates a superconductive response in the ordered phase.

III. PERSISTENT CURRENTS

The time derivative of Eq. (2.69) implies that we can transform the ground state of zero current into a state of finite electric current by applying a transient electric field. We will show that such a state is *metastable,* in the sense of having a *positive* quasiparticle excitation spectrum. For simplicity we restrict the discussion to samples whose cross-sectional dimensions are small enough to allow the magnetic field resulting from the persistent current to be negligibly small. This means that the current density is *uniform* in space.

Following a procedure very similar to that used in superconductivity theory,[6] we minimize the free energy F of the system subject to the constraint that

$$\vec{J}_E = -\hbar e(m_b^{-1}+m_a^{-1})\textstyle\sum_k \vec{k}\,h_k(1-f_{k,\uparrow}-f_{-k,\downarrow}) \qquad (3.1)$$

be finite. $\vec{J}_E$ is the electric current density due to exciton pairs in the presence of quasiparticle excitations. In other words, although $\vec{J}_E$ results from exciton pairs, $\vec{J}_E$ is modified by the presence of excitations through the factor $(1-f_{k,\uparrow}-f_{-k,\downarrow})$. The *total* supercurrent density can be written

$$\vec{J} = \vec{J}_E + \vec{J}_Q \;, \qquad (3.2)$$

where

$$\vec{J}_Q = -\hbar e\textstyle\sum_k \vec{k}(m_b^{-1}f_{k,\uparrow} + m_a^{-1}f_{-k,\downarrow}) \;. \qquad (3.3)$$

In addition to the constraint that $\vec{J}_E$ be finite, we also want the constraint that the total electric charge vanish. The exciton pairs can give rise to no electric charge, but the quasiparticles may. In general, the net charge of the system is

$$Q=-e\sum_k(f_{k,\uparrow}-f_{-k,\downarrow}) \ . \tag{3.4}$$

Rather than constrain Q, it is easier to constrain the current

$$\vec{J}_R\equiv -\hbar e(m_b^{-1}+m_a^{-1})\sum_k\vec{k}(f_{k,\uparrow}-f_{-k,\downarrow}) \ . \tag{3.5}$$

As we shall see presently, constraining $\vec{J}_R$ to vanish will lead to vanishing Q. Thus we define

$$F'\equiv F+(\hbar/e)\vec{K}\cdot(\vec{J}_E+\gamma\vec{J}_R) \ , \tag{3.6}$$

$-(\hbar/e)\vec{K}$ and $-(\hbar/e)\gamma\vec{K}$ being the vector Lagrange multiplier associated with the constraints that $\vec{J}_E$ be finite while $\vec{J}_R$ vanish. Because of the symmetry of the problem, we can take these two multipliers to be parallel. The ordinary free energy F is

$$\begin{aligned}F=&\sum_k\epsilon_k[2h_k(1-f_{k,\uparrow}-f_{-k,\downarrow})\\&+(f_{k,\uparrow}+f_{-k,\downarrow})+\alpha(f_{k,\uparrow}-f_{-k,\downarrow})]\\&-\sum_{k,k'}V_{k,k'}[h_k(1-h_k)\,h_{k'}(1-h_{k'})]^{1/2}\\&\times(1-f_{k,\uparrow}-f_{-k,\downarrow})\\&\times(1-f_{k',\uparrow}-f_{-k',\downarrow})\\&+k_BT\sum_{k,\sigma}[f_{k,\sigma}\ln f_{k,\sigma}+(1-f_{k,\sigma})\ln(1-f_{k,\sigma})],\end{aligned} \tag{3.7}$$

while the additional terms in F' are

$$\begin{aligned}&(\hbar/e)\vec{K}\cdot(\vec{J}_E+\gamma\vec{J}_R)\\&=-2\hbar\vec{v}_0\cdot\sum_k\vec{k}[h_k(1-f_{k,\uparrow}-f_{-k,\downarrow})+\gamma(f_{k,\uparrow}-f_{-k,\downarrow})].\end{aligned} \tag{3.8}$$

Here we have defined

$$\vec{v}_0\equiv\tfrac{1}{2}(m_b^{-1}+m_a^{-1})\,\hbar\vec{K} \ . \tag{3.9}$$

We shall also define

$$\bar{\epsilon}_k\equiv\epsilon_k-\hbar\vec{k}\cdot\vec{v}_0 \ . \tag{3.10}$$

Note that $\bar{\epsilon}_k$ is a spherically symmetric function of $(\vec{k}-\vec{K})$. Since $f_{k,\sigma}$ is appreciable in size only near the Fermi surface, we can make the replacement

$$\hbar\vec{k}\cdot\vec{v}_0=p_F\,v_0\mu_k \tag{3.11}$$

in all terms proportional to $f_{k,\sigma}$. Here p_F is the Fermi momentum and μ_k is the cosine of the angle between $\vec{k}$ and $\vec{v}_0$.

Minimizing F' with respect to h_k, we get

$$h_k=\tfrac{1}{2}[1-\bar{\epsilon}_k(\bar{\epsilon}_k^2+\bar{\epsilon}_0^2)^{-1/2}] \ , \tag{3.12}$$

where ϵ_0 satisfies the gap equation

$$\begin{aligned}1=&\tfrac{1}{4}N(0)V\int_{-1}^{1}d\mu_k\int_{-\hbar\omega}^{\hbar\omega}d\bar{\epsilon}_k(\bar{\epsilon}_k^2+\epsilon_0^2)^{-1/2}\\&\times(1-f_{k,\uparrow}-f_{-k,\downarrow}) \ .\end{aligned} \tag{3.13}$$

In (3.13), the correct limits on the $\bar{\epsilon}_k$ integral, namely, $(\pm\hbar\omega-p_Fv_0\mu_k)$, have been approximated by $\pm\hbar\omega$. This leads to negligible error for all values of v_0 of interest here.

Minimizing F' with respect to $f_{k,\uparrow}$ and $f_{-k,\downarrow}$, we find that $f_{k,\uparrow}$ and $f_{-k,\downarrow}$ are Fermi factors associated, respectively, with energies

$$\begin{aligned}E_{k,\uparrow}&=(\bar{\epsilon}_k^2+\epsilon_0^2)^{1/2}+\alpha\,\bar{\epsilon}_k+[1+(\alpha-2\gamma)]\,p_Fv_0\mu_k,\\E_{k,\downarrow}&=(\bar{\epsilon}_k^2+\epsilon_0^2)^{1/2}-\alpha\,\bar{\epsilon}_k+[1-(\alpha-2\gamma)]p_Fv_0\mu_k.\end{aligned} \tag{3.14}$$

We now set

$$\gamma=\tfrac{1}{2}\alpha \ , \tag{3.15}$$

thereby ensuring that $(f_{k,\uparrow}+f_{-k,\downarrow})$ is an *even* function of $\bar{\epsilon}_k$ and that $(f_{k,\uparrow}-f_{-k,\downarrow})$ is an *odd* function of $\bar{\epsilon}_k$. The oddness of $(f_{k,\uparrow}-f_{-k,\downarrow})$ forces both $\vec{J}_R$ and Q to vanish.

It is convenient to rewrite (3.2) as

$$\vec{J}=\vec{J}_1+\vec{J}_2 \ , \tag{3.16}$$

where

$$\vec{J}_1\equiv-\hbar e(m_b^{-1}+m_a^{-1})\sum_k\vec{k}h_k, \tag{3.17}$$

$$\begin{aligned}\vec{J}_2\equiv&-\tfrac{1}{2}\hbar e(m_b^{-1}+m_a^{-1})\sum_k\vec{k}[(1-2h_k)(f_{k,\uparrow}+f_{-k,\downarrow})\\&+\alpha(f_{k,\uparrow}-f_{-k,\downarrow})] \ .\end{aligned} \tag{3.18}$$

Since the summand of $\vec{J}_2$ is *odd* in $\bar{\epsilon}_k$, we see that $\vec{J}_2$ vanishes. Since h_k is a spherically symmetric function of $(\vec{k}-\vec{K})$, it follows that

$$\sum_k(\vec{k}-\vec{K})\,h_k=0 \ , \tag{3.19}$$

Thus

$$\vec{J}=\vec{J}_1=-\hbar e(m_b^{-1}+m_a^{-1})\sum_k\vec{K}h_k=-2ne\vec{v}_0 \ . \tag{3.20}$$

Although the situation here is analogous to that of persistent currents in a superconductor,[6,7] there is one important difference. In the superconductor, $\vec{J}$ is a nonlinear function of $\vec{v}_0$ which reaches a maximum at some value of $\vec{v}_0$. This maximum is the critical current density. In the excitonic insulator, $\vec{J}$ has no maximum as a function of $\vec{v}_0$. Rather it appears that the critical current density will be set by that value of $\vec{v}_0$ for which

$$F(\vec{v}_0)=F_n \ , \tag{3.21}$$

F_n being the free energy of the normal phase.

Here we have formed states of finite electric current by pairing $(\vec{k}-\vec{K},\uparrow)$ with $(-\vec{k}+\vec{K},\downarrow)$. It has been suggested by Kozlov and Maksimov[8] that a state of zero electric current but finite energy transport could be formed by pairing $(\vec{k}+\vec{K},\uparrow)$ with $(-\vec{k}+\vec{K},\downarrow)$. Although this is correct, such a state appears to be *physically inaccessible* in the sense that there is no force which can generate it when applied to the ground state. Thus such a metastable state will play no role in ordinary physical phenomena.

IV. LONG-RANGE ORDER

Jérome *et al.*[1] have buttressed their argument that the excitonic insulator cannot superconduct by pointing out that the ordered phase does not have *off-diagonal long-range order* (ODLRO).[9] Rather, it has *diagonal long-range order* (DLRO). Both types of long-range order are defined in terms of the two-particle density matrix, the latter being the thermodynamic average of the product

$$\psi^\dagger(\vec{r}_2)\, \psi^\dagger(\vec{r}_1)\, \psi(\vec{r}_1{}')\, \psi(\vec{r}_2{}') \ . \qquad (4.1)$$

We have ODLRO when the density matrix remains finite as $|\vec{r}_1 - \vec{r}_1{}'| \to \infty$, provided $\vec{r}_1 \approx \vec{r}_2$ and $\vec{r}_1{}' \approx \vec{r}_2{}'$. In contrast, we have DLRO when the density matrix stays finite as $|\vec{r}_1 - \vec{r}_2| \to \infty$, provided $\vec{r}_1 \approx \vec{r}_1{}'$, $\vec{r}_2 \approx \vec{r}_2{}'$. In the above product, $\psi(\vec{r})$ is the conventional one-electron wave operator, which can be written as

$$\psi(\vec{r}) = \psi_\uparrow(\vec{r}) + \psi_\downarrow^\dagger(\vec{r}) \ . \qquad (4.2)$$

Here $\psi_\uparrow(\vec{r})$ is the wave operator associated with *electrons in the conduction band;* $\psi_\downarrow(\vec{r})$ is the wave operator associated with *holes in the valence band.* In other words, $\psi_\uparrow$ is a linear combination of conduction-band electron destruction operators; $\psi_\downarrow$ is a linear combination of valence-band hole destruction operators.

Let us define a slightly different form of two-particle density matrix,[10] namely, the thermodynamic average of the product

$$\psi_\uparrow^\dagger(\vec{r}_2)\, \psi_\downarrow^\dagger(\vec{r}_1)\, \psi_\downarrow(\vec{r}_1{}')\, \psi_\uparrow(\vec{r}_2{}') \ . \qquad (4.3)$$

It is easy to check that, in the ordered phase, this latter form has ODLRO rather than DLRO. The conclusion is that diagonal versus off-diagonal long-range order is a function of one's bookkeeping. The excitonic insulator is a system where Yang's criterion[9] seems ambiguous.

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*Work supported by the U. S. Office of Naval Research, under Contract No. N00014-67-A-0209-0007.

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[2]J. Zittartz, Phys. Rev. 165, 605 (1968).

[3]Since both Jérome *et al.* and Zittartz actually completed their respective calculations of electrical response only for the special case $m_b = m_a$, there is no conflict between these results and those of the present paper.

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[9]C. N. Yang [Rev. Mod. Phys. 34, 694 (1962)] has postulated that superfluid systems are characterized by ODLRO.

[10]The writers are indebted to Michael Schick for suggesting this form.