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Study of two-dimensional electrons in a magnetic field. II. Intermediate field

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Taking into consideration the first-order-exchange and ring diagrams, we present a theory of two-dimensional (2-D) electrons in a magnetic field under the de Haas–van Alphen condition. The Fermi momentum or the chemical potential of the system oscillates with the magnetic field, its interaction term being characterized by a factor $(e^2/p_0)^{3/4} = (2^{1/2}r_s)^{3/4}$. Due to a 2-D peculiarity, the susceptibility oscillates as in the ideal case without a constant phase $-\pi/4$ characteristic of the three-dimensional case. A relation between the amplitude of the oscillating susceptibility and the field and temperature is derived. The energy variation is like $a^4\cos(\pi\epsilon_0/a^2)$, where a^2 represents the field energy and ϵ_0 is the Fermi energy for the ideal case, i.e., $2\pi n$, in the units $\hbar = 1$ and $2m = 1$, where n is the number density. The amplitude of the energy oscillation increases with the field strength squared. A new specific-heat formula is also presented.

I. INTRODUCTION

Since the early works of Fowler, Fang, Howard, and Stiles on magnetic conduction and of Fang and Stiles on the g factor,[1] the many-body properties of two-dimensional (2-D) electrons in a magnetic field have attracted considerable attention.[2] Stimulated by these works, we have developed in a previous paper, hereafter to be called I, a general statistical-mechanical theory of 2-D electrons in a weak magnetic field and evaluated the magnetic energy, susceptibility, and other quantities as functions of r_s, the density parameter.[3] Recognizing the roles played by electron spin and orbital motion in the magnetic response of the system, we have succeeded in deriving the effective g factor and effective mass from the paramagnetic and diamagnetic susceptibilities, respectively.

When plotted against r_s the paramagnetic susceptibility was found to vary almost linearly. This means that the effective g factor squared is also approximately linear in r_s dependence. Note that in this respect the 2-D susceptibility defined similarly to the three-dimensional (3-D) case is not dimensionless but is proportional to the Bohr magneton squared. In addition, the paramagnetic susceptibility is proportional to the square of the g factor. The diamagnetic susceptibility is smaller than the paramagnetic susceptibility. In fact, the ideal-gas susceptibilities maintain the same one-to-three ratio as in the 3-D case. However, the diamagnetic susceptibility shows a nonlinear increase when plotted against r_s. Therefore, the effective mass is also nonlinear. Only for a limited domain of r_s is it a linear function of r_s. The approximate character of this dependence is clear also because the susceptibility is proportional to the square of the Bohr magneton.

We have developed the weak-field theory for absolute zero. The susceptibilities have been derived from the grand-partition function in consideration of the exchange and ring contributions. In view of our purpose, these contributions have been evaluated to order a^4 in magnetic field, where a^2 represents the field energy, and to order e^4 in interaction. Concerning the latter, we have neglected the second-order-exchange contribution. As can be seen from the case of zero field,[4] this contribution is very difficult. Even for the 3-D case which is somewhat easier to treat mathematically, we have not been able to obtain an explicit result. Apart from this neglection, our theory is rigorous and is a natural extension of the theory of the correlation energy for zero field. In fact, we have obtained the correlation energy shift due to a magnetic field.

It is the purpose of the present paper to extend the weak-field approach of I to the intermediate-field case which corresponds to the de Haas–van Alphen (dHvA) oscillations. Under the so-called dHvA condition, the field energy is less than the Fermi energy but larger than or comparable to the thermal energy. This condition makes the theoretical approach extremely difficult in contrast to the weak-field case. We have recently succeeded in treating the 3-D case[5] and to extend the Lifshitz and Kosevich ideal-gas theory[6] to the case with Coulomb interaction. Among several interesting correlation effects, we have found that the extra field dependence of the amplitude due to interaction can be represented by an exponential factor, giving a molecular basis to the phenomenological Dingle exponential reduction factor.

We shall employ a grand-ensemble approach and the units in which $\hbar = 1$ and $2m = 1$, where m is the electron mass. The grand-partition function shall be constructed systematically in the interaction parameter e^2. Hence, in Sec. II we start with the ideal-gas case. We shall construct the free-electron propagator rigorously and then the ideal grand-partition function. The propagator will be used for the subsequent interacting cases also.

In Sec. III, the contribution to the grand-partition function from the first-order-exchange graphs will be given. For weak fields, we have shown that this contribution and the first ring contribution should be combined in order to eliminate a divergence. Therefore, expecting the same requirement we shall give only the general form for the first-order exchange contribution.

In Sec. IV, the ring-diagram contribution will be considered. The important step here is to evaluate the eigenvalues of the propagator under the dHvA condition. Note that the eigenvalues as functions of integers j and momentum q give the dielectric function as a function of frequency and momentum, as shown generally elsewhere.[7] The eigenvalues turn out to be oscillating with the magnetic field. It is important to note that the effect of spins on the eigenvalues is factored out. This is an indication that the electron spins are not the primary source of the oscillations. It is the orbital motion of the electrons that is important for the dHvA phenomenon.

After considering the exchange and ring contributions separately, we shall, in Sec. V, combine the two and obtain an explicit result. In Sec. VI, we shall evaluate the Fermi momentum which enters grand-ensemble theory as a parameter. This renormalization of the Fermi momentum follows the golden rule of grand-ensemble theory. Note that the Fermi momentum p_F squared represents the chemical potential. Therefore, in Sec. VI we are actually giving an oscillating chemical potential. Note in the result for the Fermi momentum that an interaction parameter characteristic of two dimensions enters. This parameter is nonlinear in e^2/p_0, which is a dimensionless combination of the Bohr radius represented by $1/e^2$ in our units and the average electron distance represented by $1/p_0$; p_0 being the ideal-gas Fermi momentum.

The oscillating susceptibility will be obtained explicitly in Sec. VII. In deriving the susceptibility, we have considered the interaction parameter e^2/p_0 to be small and retained only the lowest-order terms. We shall then find that the interaction parameter is cancelled out from the final susceptibility expression. That is, the susceptibility within the above approximation looks like an ideal-gas formula. This is a 2-D peculiarity combined with the approximation.

Section VIII will give the internal energy explicitly. From this, the specific heat will be obtained in Sec. IX. Since the dHvA phenomenon is essentially characteristic of free electrons, our emphasis in these sections will be in deriving oscillating functions. For this reason, the interaction effect on the internal energy will not be pursued to a higher order in which a coupling of the oscillating terms and the interaction terms appears. That is, we consider the oscillating terms and interaction terms to be small and neglect their cross products. As we shall see, the interaction effect appears as a reduction in the internal energy within the approximation. Since this reduction is independent of temperature, the specific heat will not have an interaction term. That is, we shall report an ideal oscillating specific heat. This is the zeroth-order result. We shall report on the interaction effects on the specific heat in a later article. Finally, in Sec. X we shall give several relevant comments concerning the oscillating susceptibility and the g factor, etc. In 1961, Kohn discussed the independence of the cyclotron resonance and dHvA frequencies on short-range interactions.[8] We shall be interested in the derivation of an explicit and reliable formula for the dHvA oscillations with emphasis on the effects of Coulomb interactions on the amplitude and phase of the oscillations.

II. IDEAL-GAS CONTRIBUTION

The free-electron propagator in reciprocal temperature and coordinate space is constructed by

$$K_0(\vec{r}_2\beta_2;\vec{r}_1\beta_1) = \sum_{n,\sigma} e^{-(\beta_2-\beta_1)\epsilon_{n\sigma}}\, \psi_{n\sigma}(\vec{r}_2)\psi^*_{n\sigma}(\vec{r}_1)\,, \quad (2.1)$$

where σ is the spin variable, $\beta = 1/kT$. The eigenfunctions and eigenvalues are known. We find

$$K_0(\vec{r}_2\beta_2,\vec{r}_1\beta_1) = \frac{a^2}{2\pi} K_s(s)\frac{\exp[-\frac{1}{4}a^2\coth(sa^2)(x^2+y^2)]}{2\sinh(sa^2)}\, e^{i\phi}\,,$$

where

$$\begin{aligned} &s=\beta_2-\beta_1\,,\quad \vec{x}=\vec{x}_2-\vec{x}_1\,,\\ &\phi=\tfrac{1}{2}a^2(y_1+y_2)(x_2-x_1)\,, \qquad (2.2)\\ &K_s(s)=\exp(-s\tfrac{1}{2}ga^2)\uparrow\rangle\langle\uparrow| + \exp(s\tfrac{1}{2}ga^2)\downarrow\rangle\langle\downarrow|\,,\\ &a^2=eH/c=\omega_0/2\,, \end{aligned}$$

g is Landé's factor, a^2 represents the magnetic field energy, and $\omega_0 = 2eH/c$.

The free-electron contribution to the grand-partition function is given by

$$\ln \Xi_0 = \sum_{l=1}^{\infty} \frac{z^l}{l} (-)^{l+1} A_l \,, \tag{2.3}$$

where

$$A_l = \frac{a^2}{2\pi} \operatorname{Tr}\left(K_s(l\beta)e^{i\phi} \times \frac{\exp[-\frac{1}{4}a^2\coth(l\beta a^2)(x^2+y^2)]}{2\sinh(l\beta a^2)}\right). \tag{2.4}$$

Hence,

$$\ln \Xi_0 = -\frac{Aa^2}{2\pi}\sum_{l=1}^{\infty}(-z)^l \frac{1}{l}\frac{\cosh(\frac{1}{2}gl\beta a^2)}{\sinh(l\beta a^2)}\,, \tag{2.5}$$

where A is the surface area.

The right-hand side sum can be expressed in terms of the Mellin transform:

$$\sum_{l=1}^{\infty}(-)^{l+1}\frac{x^l f(l)}{l} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \frac{\pi}{s\sin\pi s}\, x^s f(s)\,, \quad 0<c<1. \tag{2.6}$$

We find

$$\ln\Xi_0 = (Aa^2/2\pi)I_0\,, \tag{2.7}$$

$$I_0 = \frac{\pi}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \frac{e^{\eta s}}{s\sin\pi s}\,\frac{\cosh(\frac{1}{2}gs\alpha)}{\sinh(s\alpha)}\,, \quad 0<c<1\,, \tag{2.8}$$

where

$$\eta = \beta\mu = \beta p_F^2\,, \quad z = e^{\eta}\,, \quad \alpha = \beta a^2\,. \tag{2.9}$$

The contour shall be closed to the left in the complex s space so that the residues at $s\alpha = il\pi$, $(l=-\infty)$ are picked up. Let us denote the residues as I_{01} for $l=0$ and I_{02} for $l\neq 0$. The residues I_{03} due to $\sin\pi s$ are negligible for large η because they vary as $e^{-\eta}$ as shown in Appendix A. We have

$$I_0 = I_{01} + I_{02} + I_{03}\,. \tag{2.10}$$

I_{01} is the contribution from $s=0$. It is given by

$$I_{01} = (\eta^2/2\alpha)\{1+(\pi^2/3\eta^2)+\gamma^2[(\tfrac{1}{2}g)^2 - \tfrac{1}{3}]\}\,, \tag{2.11}$$

where $\gamma = a^2/p_F^2$. I_{02} and I_{03} are the contributions from the residues at $s\alpha = il\pi$ and $s=l$, respectively:

$$\left.\begin{matrix} I_{02} \\ I_{03}\end{matrix}\right\} = \frac{\pi}{2\pi i}\sum_{l\neq 0}\oint \frac{e^{s\eta}\cosh(\frac{1}{2}gs\alpha)}{s\sin(\pi s)\sinh(\alpha s)}\, ds\,. \tag{2.12}$$

By changing variable s such that $s = x/\alpha + il\pi/\alpha$ for the first integral I_{02} and $s = x - l$ for the second integral I_{03} we find

$$I_{02} = 2\sum_{l=1}^{\infty}(-)^{l+l}\frac{\cos(l\pi/\gamma)\cos(\frac{1}{2}gl\pi)}{l\sinh(\pi^2 l/\alpha)}\,,$$

$$I_{03} = e^{-\eta}\sum_{1}^{\infty}(-)^l e^{-(l-1)\eta}\frac{\cosh(\frac{1}{2}g\alpha l)}{l\sinh(\alpha l)}\,. \tag{2.13}$$

Combining these results and neglecting I_{03} which is small, we arrive at

$$\ln\Xi_0 = \frac{A\beta p_F^4}{4\pi}\left\{1+(\pi^2/3\eta^2)+[(\tfrac{1}{2}g)^2-\tfrac{1}{3}]\gamma^2 + \frac{4\alpha}{\eta^2}\sum_{l=1}^{\infty}(-)^{l+1}\frac{\cos(l\pi/\gamma)\cos(\frac{1}{2}gl\pi)}{l\sinh(\pi^2 l/\alpha)}\right\}. \tag{2.14}$$

III. FIRST-ORDER-EXCHANGE CONTRIBUTION

The first-order-exchange graphs contribute the following[5]:

$$\ln\Xi_{1x} = \sum_{l=2}^{\infty}(-)^l z^l B_l\,, \tag{3.1}$$

where the cluster coefficients are

$$B_l = \beta\int d\vec{r}_1 d\vec{r}_2 \sum_{t=1}^{l-1} K_0(\vec{r}_1,(l-t)\beta;\vec{r}_2,0) \times K_0(\vec{r}_2, t\beta;\vec{r}_1,0)\tfrac{1}{2}\phi(|\vec{r}_1-\vec{r}_2|)\,. \tag{3.2}$$

We find

$$\ln\Xi_{1x} = \frac{\beta A}{2}(a^2/2\pi)^2\frac{\pi^{3/2}e^2}{a}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{(-z)^{s+t}\cosh(l\beta a^2\frac{1}{2}g)}{[\sinh(s\beta a^2)\sinh(t\beta a^2)\sinh(l\beta a^2)]^{1/2}}\,, \tag{3.3}$$

where $l = s+t$.

IV. RING-DIAGRAM CONTRIBUTION

The ring-diagram contribution is given by

$$\ln\Xi_r = \frac{A}{2(2\pi)^2}\int d\vec{q}\sum_j\{u(q)\lambda_j(q) - \ln[1+u(q)\lambda_j(q)]\}\,, \tag{4.1}$$

where

$$\lambda_j(q) = \frac{a^2}{2\pi}\sum_{l=1}^{\infty} z^l(-)^{l+1}\frac{\cosh(\frac{1}{2}gl\beta a^2)}{\sinh(l\beta a^2)}\sum_{s+t=l}\int_0^{\beta} d\alpha\exp\left(-\frac{2\pi ij\alpha}{\beta}\right) \times\exp\left(-q^2\frac{\sinh[(s\beta+\alpha)a^2]\sinh[(t\beta-\alpha)^2]}{a^2\sinh(l\beta a^2)}\right). \tag{4.2}$$

In the summations over s and t are subject to $s \geqslant 0$, $t \geqslant 1$. In order to evaluate the grand-partition function the exponential factor of the eigenvalues has to be simplified. For this purpose, let us define

$$I(s,t) = \int_0^\beta d\alpha \exp(-2\pi i j\alpha/\beta) E(q,\alpha;s,t), \quad (4.3)$$

where

$$E(q,\alpha:s,t) \equiv E = \exp\left(-\frac{q^2}{a^2}\frac{\sinh[(s\beta+\alpha)a^2]\sinh[(t\beta-\alpha)a^2]}{\sinh(l\beta a^2)}\right). \quad (4.4)$$

(i) $s \geqslant 1$, $t \geqslant 2$

$$\sinh[(s\beta+\alpha)a^2] \sim \tfrac{1}{2}\exp(s\beta+\alpha)a^2,$$

$$\sinh[(t\beta-\alpha)a^2] \sim \tfrac{1}{2}\exp(t\beta-\alpha)a^2,$$

since $l = s+t$ and due to the α integration, $I(s,t)$ can be neglected. (ii) $s=0$, $t=1$

$$E \approx \begin{cases} \exp(-q^2|\alpha), & \alpha \sim 0, \\ \exp[-(\beta-\alpha)q^2], & \alpha \sim \beta. \end{cases} \quad (4.5)$$

For other α values

$$E \approx \exp(-q^2/2a^2). \quad (4.6)$$

(iii) $s \geqslant 1$, $t=1$

$$E \approx \exp\left(-\frac{1-\exp[-2(\beta-\alpha)a^2]}{2a^2}q^2\right). \quad (4.7)$$

(iv) $s=0$, $t \geqslant 2 (t=l)$

$$E \approx \exp\left(-\frac{1-\exp(-2\alpha a^2)}{2a^2}q^2\right). \quad (4.8)$$

Hence, if $\alpha \to 0$, $E \approx \exp(-\alpha q^2)$, otherwise

$$E \approx \exp(-q^2/2a^2). \quad (4.9)$$

The integral over α of Eq. (4.3) consists of the extreme regions in which $|\alpha \approx 0$ or $\alpha \approx \beta$ and the intermediate region. Accordingly, we have

$$\left(\int_0^{\beta'} + \int_{\beta'}^{\beta-\beta'} + \int_{\beta-\beta'}^{\beta}\right) d\alpha \; .$$

We note that E is symmetric about $\alpha = \frac{1}{2}\beta$ and after a short calculation arrive at

$$\lambda_j(q) = \frac{a^2 X}{\pi}\left[-\frac{\beta}{2\pi j}\sin\left(\frac{2\pi j\beta'}{\beta}\right)\exp\left(\frac{-q^2}{2a^2}\right) + \int_0^{\beta'} d\alpha\cos\left(\frac{2\pi j\alpha}{\beta}\right)\exp(-\alpha q^2)\right] \quad (4.10)$$

$$= \frac{a^2X}{\pi}\left\{\frac{q^2}{q^4+(2\pi j/\beta)^2}\left[1-\exp(-\beta' q^2)\cos\left(\frac{2\pi j\beta'}{\beta}\right)\right] - \frac{\beta}{2\pi j}\sin\left(\frac{2\pi j\beta'}{\beta}\right)\exp\left(\frac{-q^2}{2a^2}\right)\right.$$

$$\left. + \exp(-\beta' q^2)\frac{2\pi j/\beta}{q^4+(2\pi j/\beta)^2}\sinh\left(\frac{2\pi j\beta'}{\beta}\right)\right\}. \quad (4.11)$$

Naturally, this result depends on β'. Since $E \approx \exp(-q^2/2a^2)$ when αa^2 is not small, we estimate that $\alpha \approx 1/2a^2$ gives the point beyond which $E \approx \exp(-\alpha q^2)$. Therefore, we estimate β' to be $1/2a^2$. Then, Eq. (4.11) becomes

$$\lambda_j(q) = \frac{a^2X}{\pi}\frac{q^2}{q^4+(2\pi j/\beta)^2}\left\{1-\left(\exp\frac{-q^2}{2a^2}\right)\left[\cos\left(\frac{\pi j}{\beta a^2}\right) + \frac{q^2\beta}{2\pi j}\sin\left(\frac{\pi j}{\beta a^2}\right)\right]\right\}, \quad (4.12)$$

where X includes a spin factor and is defined below. The eigenvalues are oscillatory through this factor X and also through the second term in the square bracket. The factor X is characteristic of a single free electron, while the momentum-dependent factors represent the coupling of the effective units (torons in the sense that arbitrary number of electrons with exchanges can be the units in grand-ensemble theory). As a function of momentum, q (the eigenvalues are zero for $q=0$) reaches a maximum and then decreases. This result may be improved if $\beta' = c/a^2$ is used at the expense of introducing an adjustable constant c. Under the dHvA condition, $\beta a^2 \gg 1$, and because j^2 in the denominators is dominant for large j, we can simplify Eq. (4.12) as follows:

$$\lambda_j(q) = \frac{a^2}{\pi}X\frac{q^2}{q^4+(2\pi j/\beta)^2}\left[1-\left(1+\frac{q^2}{2a^2}\right)e^{-q^2/2a^2}\right], \quad (4.13)$$

where

$$X=\sum_{l=1}^{\infty}(-)^{l+1}z^{l}\frac{\cosh(\frac{1}{2}gl\beta a^{2})}{\sinh(l\beta a^{2})}$$
$$=\frac{\pi}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\,\frac{e^{s\eta}}{\sin\pi s}\,\frac{\cosh(\frac{1}{2}gs\alpha)}{\sinh(s\alpha)},\quad 0<c<1. \tag{4.14}$$

In the last integral, $\alpha=\beta a^{2}$. The contribution from the pole at the origin is η/α. The contributions from the poles at $\alpha s=il\pi$ are given by

$$\frac{\pi}{2\pi i}\oint\frac{dx}{\alpha}\,\frac{e^{x/\gamma+il\pi/\gamma}\cos(\frac{1}{2}l\pi g)}{i\sinh[(l\pi^{2}/\alpha)(-)^{l}x]}$$
$$=\frac{\pi}{\alpha}\,\frac{(-)^{l}e^{il\pi/\gamma}\cos(\frac{1}{2}l\pi g)}{i\sinh(l\pi^{2}/\alpha)}.$$

Similar integrals appeared in the 3-D case.[5] We arrive at

$$X=\frac{\eta}{\alpha}\left(1+\frac{2\pi}{\eta}\sum_{l=1}^{\infty}(-)^{l}\,\frac{\sin(l\pi/\gamma)(\cos\frac{1}{2}gl\pi)}{\sinh(l\pi^{2}/\alpha)}\right). \tag{4.15}$$

The second term represents a typical dHvA oscillating function. As we see. The spin factor g is included only in the sign factor, showing its minor ro le in the oscillations.

V. COMBINATION OF THE CONTRIBUTIONS OF THE FIRST-ORDER-EXCHANGE AND RING DIAGRAMS

For weak fields, we have seen elsewhere that the divergence in a part of the ring-diagram contribution is cancelled by the first-order-exchange diagram. For our present case of strong field, it is difficult to see the same because the eigenvalues of the ring diagrams are very involved. Therefore, without looking into this point concerning cancellation let us evaluate the contribution combining the ring and exchange contributions. We note that the first ring contributes as follows:

$$\ln\Xi_{r1}=\frac{A}{2(2\pi)^{2}}\int d\vec{q}\sum_{j}u(q)\lambda_{j}(q)$$
$$=\frac{A}{2(2\pi)^{2}}\int d\vec{q}\,u(q)\,\frac{a^{2}}{2\pi}\sum_{l=1}^{\infty}(-)^{l}z^{l+1}\frac{\cosh(\frac{1}{2}gl\beta a^{2})}{\sinh(l\beta a^{2})}$$
$$\times\sum_{s+t=l}\int_{0}^{\beta}d\alpha\,e^{-2\pi ij\alpha/\beta}\exp\left(-\frac{\sinh[(s\beta+\alpha)a^{2}]\sinh[(t\beta-\alpha)a^{2}]}{a^{2}\sinh(l\beta a^{2})}\,q^{2}\right). \tag{5.1}$$

Using the relation

$$\sum_{j=-\infty}^{\infty}e^{-2\pi ij\alpha/\beta}=\beta\delta(\alpha+n\beta),\quad (n\text{:interger})$$

we find

$$\ln\Xi_{r1}=\frac{A\beta}{2(2\pi)^{2}}\frac{a^{2}}{2\pi}\int d\vec{q}\,u(q)\sum_{l=1}^{\infty}(-)^{l+1}z^{l}\frac{\cosh(\frac{1}{2}gl\beta a^{2})}{\sinh(l\beta a^{2})}\sum_{s+t=l}\frac{1}{2}\left[\exp\left(-\frac{\sinh[(s+1)\beta a^{2}]\sinh[(t-1)\beta a^{2}]}{a^{2}\sinh(l\beta a^{2})}\,q^{2}\right)\right.$$
$$\left.+\exp\left(-\frac{\sinh(s\beta a^{2})\sinh(t\beta a^{2})}{a^{2}\sinh(l\beta a^{2})}\,q^{2}\right)\right]. \tag{5.2}$$

Let us now separate the terms for $t-1=0$ and $s=0$ which are special in the double sum. The rest will then be symmetric with respect to t and s, and we obtain

$$\ln\Xi_{r1}=\frac{A\beta a^{2}}{2(2\pi)^{3}}\int d\vec{q}\,u(q)\sum_{l=1}^{\infty}(-)^{l+1}z^{l}\frac{\cosh(\frac{1}{2}ql\beta a^{2})}{\sinh(l\beta a^{2})}$$
$$+\frac{A\beta a^{2}}{2(2\pi)^{3}}\int d\vec{q}\,u(q)\sum_{l=2}^{\infty}(-)^{l+1}z^{l}\frac{\cosh(\frac{1}{2}gl\beta a^{2})}{\sinh(l\beta a)^{2}}$$
$$\times\sum_{\substack{s+t=l\\ s>1\\ t>1}}\exp\left(-\frac{q^{2}}{a^{2}}\,\frac{\sinh(s\beta a^{2})\sinh(t\beta a^{2})}{\sinh(l\beta a^{2})}\right). \tag{5.3}$$

The first term on the right-hand side is equal to

$$\frac{A\beta a^{2}}{2(2\pi)^{3}}\int d\vec{q}\,u(q)X,$$

where X has been defined by Eq. (4.14). The second term, upon integrating over $\vec{q}$, gives $-\ln\Xi_{1x}$. Hence, the combination of the first-order-exchange and ring diagrams is

$$\ln\Xi_{1x}+\ln\Xi_{r}=\frac{A\beta a^{2}}{2(2\pi)^{3}}\int d\vec{q}\,u(q)X-\frac{A}{2(2\pi)^{2}}$$
$$\times\int d\vec{q}\int_{0}^{1}\sum_{j}d\xi\,\frac{u(q)\lambda_{j}(q)}{1+\xi u(q)\lambda_{j}(q)}$$
$$=M_{1}+M_{2}, \tag{5.4}$$

where M_1 and M_2 represent, respectively, the first and second terms.

In order to evaluate these terms, let us write the eigenvalues of Eq. (4.13) as

$$\lambda_j(q)=\frac{a^2X}{\pi}\,\frac{q^2}{q^4+(2\pi j/\beta)^2}\,K(q,a^2)\,, \tag{5.5}$$

with

$$K(q,a^2)=1-(1-q^2/2a^2)e^{-q^2/2a^2}\;. \tag{5.6}$$

The sum over j can be replaced by integration over

$$x=2\pi j/\beta. \tag{5.7}$$

Hence,

$$M_2=-\frac{A\beta}{2(2\pi)^3}\int d\vec{q}\int_0^1 d\xi\int_{-\infty}^{\infty}dx\,\frac{(a^2X/\pi)u(q)K(q,a^2)q^2}{q^4+\xi(a^2X/\pi)K(q,a^2)u(q)q^2+x^2} \tag{5.8}$$

$$=-\frac{A\beta\pi}{2(2\pi)^3}\int d\vec{q}\int_0^1 d\xi\,\frac{(a^2X/\pi)K(q,a^2)u(q)q^2}{[q^4+\xi(a^2X/\pi)K(q,a^2)u(q)q^2]^{1/2}}\;. \tag{5.9}$$

Performing the ξ integration, we arrive at

$$M_2=-\frac{A\beta}{2(2\pi)^2}\int d\vec{q}\,[(q^4+F)^{1/2}-q^2]\,, \tag{5.10}$$

where

$$F=(a^2X/\pi)K(q,a^2)u(q)q^2=2e^2a^2XqK(q,a^2)\,. \tag{5.11}$$

On the other hand, the first term M_1 is

$$M_1=\frac{A\beta}{2(2\pi)^2}\int d\vec{q}\,\frac{1}{2}\,(2a^2e^2X)\,\frac{1}{q}\;. \tag{5.12}$$

Hence, we obtain

$$M_1+M_2=\frac{A\beta}{2(2\pi)}\int d\vec{q}\left(\frac{1}{2q}(2a^2e^2X)+q^2-[q^4+2e^2a^2XqK(q,a^2)]^{1/2}\right)$$

$$=\frac{A\beta(a^2e^2X)^{4/3}}{4\pi}\int_0^{\infty}dx\left[1+x^3-x^3\left(1+\frac{2}{x^3}K((a^2e^2X)^{1/3}x,a^2)\right)^{1/2}\right]\;. \tag{5.13}$$

Note that

$$K((a^2e^2X)^{1/3}x,a^2)=1-\left(1+\frac{(a^2e^2X)^{2/3}}{2a^2}x^2\right)\exp\left(-\frac{(a^2e^2X)^{2/3}x^2}{2a^2}\right) \tag{5.14}$$

and

$$\frac{(a^2e^2X)^{2/3}}{a^2}=s^{2/3}\frac{p_F^2}{a^2}\left[1+\frac{2\pi}{\eta}\sum_{l=1}^{\infty}(-)^l\,\frac{\sinh(l\pi/\gamma)\cos(\frac{1}{2}gl\pi)}{\sinh(l\pi^2/\alpha)}\right], \tag{5.15}$$

where

$$s=e^2/p_F\,,\quad \alpha=\beta a^2\,,\quad \gamma=a^2/p_F^2\,,\quad \eta=\beta p_F^2\;. \tag{5.16}$$

In Eq. (5.15) the second oscillations term is small due to the large factor η in the denominator. We can now write

$$M_1+M_2=\frac{A\beta}{4\pi}\,s^{4/3}p_F^4I(\theta)\times\left(1+\frac{2\pi}{\eta}\sum_{l=1}^{\infty}(-)^l\,\frac{\sin(l\pi/\gamma)\cos(\frac{1}{2}gl\pi)}{\sinh(l\pi^2/\alpha)}\right), \tag{5.17}$$

where

$$I(\theta)=\int_0^{\infty}dz\left[1+z^3-z^3\left(1+\frac{2}{z^3}\,[1-(1+\theta z)e^{-\theta z^2}]\right)^{1/2}\right]$$

$$\theta=\frac{s^{2/3}p_F^2}{a^2}\left(1+\frac{2\pi}{\eta}\sum_{l=1}^{\infty}(-1)^l\,\frac{\sin(l\pi/\gamma)\cos(\frac{1}{2}l\pi)}{\sinh(l\pi^2/\alpha)}\right)\;. \tag{5.18}$$

In the first approximation, $\exp(-\theta z^2)$ may be neglected because $\theta\gg 1$. Then,

$$I(\theta)=\int_0^{\infty}dz\,[1+z^3-z^3(1+2/z^3)^{1/2}]$$

$$=0.8149\equiv I\;. \tag{5.19}$$

Our final expression is then

$$\ln\Xi_{1x}+\ln\Xi_r \simeq \frac{A\beta}{4\pi} s^{4/3} p_F^4 I \left(1+\frac{2\pi}{\eta}\sum_{l=1}^{\infty}(-)^l \frac{\sin(l\pi/\gamma)\cos(\frac{1}{2}gl\pi)}{\sinh(l\pi^2/\alpha)}\right) . \tag{5.20}$$

Combining all the contributions, the total grand-partition function is given by

$$\begin{aligned}\ln\Xi &= \ln\Xi_0+\ln\Xi_{1x}+\ln\Xi_r \\ &= \frac{A\beta p_F^4}{4\pi}\left\{1+\frac{\pi^2}{3\eta^2}+[(\tfrac{1}{2}g)^2-\tfrac{1}{3}]\gamma^2+4\frac{\alpha}{\eta^2}\sum_{l=1}^{\infty}(-)^{l+1}\frac{\cos(l\pi/\gamma)\cos(\frac{1}{2}gl\pi)}{l\sinh(\pi^2 l/\alpha)}\right. \\ &\qquad \left. + Is^{4/3}\left[1+\frac{2\pi}{\eta}\sum_{l=1}^{\infty}(-)^l\frac{\sin(l\pi/\gamma)\cos(\frac{1}{2}gl\pi)}{\sinh(l\pi^2/\alpha)}\right]^{4/3}+\cdots\right\} .\end{aligned} \tag{5.21}$$

In this result, the last term is characterized by a factor $Is^{4/3}$ which represents the correlation effect. As we see, this term is nonlinearly coupled with the oscillating term in the bracket. Therefore, we shall introduce a linearization in order to solve the number density relation for the determination of the Fermi momentum p_F. This step of linearization is unfortunate because we must assume that the oscillating term is small.

The nonlinear character of the interaction effect can be important. We shall further investigate this point in a later article.

VI. FERMI MOMENTUM

From the total grand-partition function we can evaluate the Fermi momentum p_F as a function of density. This is achieved by making use of the number density relation and neglecting small terms such as η^{-2} and $(\eta\alpha)^{-1}$. After three times of interation, we arrive at

$$\begin{aligned}p_F = p_0\Bigg\{1 &+ \frac{\pi}{\eta_0}\sum_{l=1}^{\infty}\frac{(-)^{l+1}\sin(l\pi/\gamma_0)\cos(\frac{1}{2}gl\pi)}{\sinh(l\pi^2/\alpha)} - \frac{1}{3}Is_0^{4/3} \\ &\times\left[1-\frac{\pi}{\eta_0}\sum_{l=1}^{\infty}(-)^{l+1}\frac{\sin(l\pi/\gamma_0)\cos(\frac{1}{2}gl\pi)}{\sin(l\pi^2/\alpha)}+\cdots\right]+\cdots\Bigg\},\end{aligned} \tag{6.1}$$

$p_0=(2\pi n)^{1/2}$, (n: density).

Here, s_0, η_0, and γ_0 are what we obtain from s, η, and γ by replacing p_F by p_0, the ideal-gas Fermi momentum. The above result shows that the Fermi momentum in the actual interacting system oscillates, although in a small magnitude. Such an oscillating p_F has been found for 3-D by us. It causes modulations of the usual dHvA oscillations because the period is dependent on p_F.

VII. SUSCEPTIBILITY

The susceptibility is given by

$$\chi = \frac{1}{Aa^2\beta}\frac{e^2}{c^2}\frac{\partial\ln\Xi}{\partial a^2}\bigg|_{z,A,\beta} . \tag{7.1}$$

Note here that the differentiation is at constant $z=\exp\beta p_F^2$.

Because of the small oscillating part of p_F, we shall have relations such as

$$\cos(l\pi/\gamma)\cong\cos(l\pi/\gamma_0)+\tfrac{2}{3}Is_0^{4/3}(l\pi/\gamma_0)\sin(l\pi/\gamma_0) ,$$

$$\sin(l\pi/\gamma)\cong\sin(l\pi/\gamma_0)-\tfrac{2}{3}Is_0^{4/3}(l\pi/\gamma_0)\cos(l\pi/\gamma_0) , \tag{7.2}$$

where $s_0=e^2/p_0$.

For $\alpha=\beta a^2>1$, we can introduce an approximation

$$1/\sinh(l\pi^2/\alpha)\sim 2e^{-l\pi^2/\alpha} .$$

As indicated in Eq. (7.2) there appear $\cos(l\pi/\gamma_0)$ and $\sin(l\pi/\gamma_0)$. Of these, we find that the coefficient of the latter is much larger than that of the former. The terms, with I, are cancelled out. Therefore, the susceptibility is given approximately by

$$\begin{aligned}\chi^{\text{osc}} = \chi_0(\tfrac{1}{4}g^2-\tfrac{1}{3}) &+ \chi_0\frac{2}{\alpha}\sum_{l=1}^{\infty}(-)^{l+1}\frac{\cos(\frac{1}{2}gl\pi)}{l\sinh(l\pi^2/\alpha)} \\ &\times\{[1+(l\pi^2/\alpha)\coth(l\pi^2/\alpha)]\cos(l\pi/\gamma_0)+(l\pi/\gamma_0)\sin(l\pi/\gamma_0)\} ,\end{aligned} \tag{7.3}$$

where

$$\chi_0 = (1/2\pi)(e^2/c^2) . \tag{7.4}$$

The most dominant oscillating term corresponding to $l = 1$ is

$$\chi^{\rm osc} = \chi_0 \frac{2}{\alpha} \frac{\cos(\frac{1}{2}g\pi)}{\sinh(\pi^2/\alpha)} \frac{\pi\eta_0}{\alpha} \sin\frac{\pi}{\gamma_0} \equiv \chi^{\rm osc} \sin\frac{\pi}{\gamma_0} , \tag{7.5}$$

where $\chi_M^{\rm osc}$ is the amplitude. In the regular units, we then find a relation

$$\frac{\chi_M^{\rm osc} \hbar^2 H^2 \sinh(\pi^2 kT/\mu_B H)}{2m\epsilon_F kT \mid \cos\frac{1}{2} g\pi \mid} = 1 , \tag{7.6}$$

where $\epsilon_F = p_F^2/2m$.

VIII. INTERNAL ENERGY

The internal energy U is obtained from

$$U = -(\partial \ln\Xi/\partial\beta)_\eta . \tag{8.1}$$

The calculation is straightforward. After differentation, the energy should be expressed as a function of density, temperature, and field. The result is

$$U = \frac{Ap_0^4}{4\pi}\left[1 + \frac{\pi^2}{3\eta_0^2} - \frac{a^4}{p_0^4}\left(\frac{g^2}{4} - \frac{1}{3}\right) - \frac{4\pi^2}{\eta_0^2} \times \sum_l (-)^{l+1} \cos\frac{l\pi}{\gamma_0} \frac{\cos(\frac{1}{2}gl\pi)}{\sinh(l\pi^2/\alpha)} \times \coth(\pi^2 l/\alpha) - Is_0^{4/3}\right] . \tag{8.2}$$

Hence, the oscillating energy is

$$U^{\rm osc} = -\frac{\pi A}{\beta^2}\sum_{l=1}^{\infty} (-)^{l+1} \frac{\cos(l\pi/\gamma_0)\cos(\frac{1}{2}gl\pi)}{\sinh(l\pi^2/\alpha)} \times \coth(\pi^2 l/\alpha) . \tag{8.3}$$

For large α and $l \sim 1$, this is approximately

$$(U/A) \sim -(a^4/\pi^2)\cos(\pi\epsilon_0/a^2)\cos(\tfrac{1}{2} q\pi) ,$$

where $\epsilon_0 = p_0^2$. The energy varies as $\cos(\pi\epsilon_0/a^2)$. As a function of $1/H$, the amplitude of oscillation decreases rather strongly.

IX. SPECIFIC HEAT

The specific heat at constant area is found to be

$$c_V = \frac{\pi^2}{3}\frac{k^2 T}{\epsilon_0}\left(1 - 12\sum_{l=1}^{\infty} (-)^{l+1} \frac{\cos(\frac{1}{2}gl\pi)\cos(l\pi/\gamma_0)}{\sinh(\pi^2 l/\alpha)} \times \left\{\coth(\pi^2 l/\alpha) + \frac{1}{2}\frac{\pi^2 l}{\alpha}\left[1 - 2\coth^2(\pi^2 l/\alpha)\right]\right\} + \cdots\right) . \tag{9.1}$$

This result depends on $\alpha = \beta\mu_B H$.

If α is $1 \sim 3$, Eq. (9.1) may be approximated by

$$c_V = \frac{\pi^2}{3}\frac{k^2 T}{\epsilon_0}\left[1 - 12\frac{\cos(\frac{1}{2}g\pi)\cos(\pi/\gamma_0)}{\sinh(\pi^2/\alpha)}\left(1 - \frac{\pi^2}{2\alpha}\right)\right] . \tag{9.2}$$

On the other hand, for $\alpha \gg 1$, it is given by

$$c_V = \frac{\pi^2}{3}\frac{k^2 T}{\epsilon_0}\left(1 - \frac{2\pi^2}{\alpha}\sum_l (-)^{l+1} l \frac{\cos(\frac{1}{2}gl\pi)}{\sinh(\pi^2 l/\alpha)} \cos(l\pi/\gamma_0)\right) . \tag{9.3}$$

These expressions show that the specific heat oscillates as in the case of the susceptiblity.

X. CONCLUDING REMARKS

We have developed a statistical-mechanical many-body theory of a 2-D electron gas with Coulomb interaction in a magnetic field under the dHvA condition. Under this condition, the field energy is in between the Fermi energy and the thermal energy. Our theory is based only on first principles and is suitable to low temperature and high density. The principle of our calculation is similar to that for the correlation energy of 3-D electrons.

The Fermi momentum of the interacting 2-D electron gas differs from the ideal case for which

$$p_0 = (2\pi n)^{1/2} . \tag{10.1}$$

The Fermi momentum of the interacting system in the intermediate field is found to oscillate with a characteristic frequency parameter

$$\gamma_0 = a^2/p_0^2 . \tag{10.2}$$

As for its dependence on the interaction, it varies with $r_s^{3/4}$ where r_s is the familiar density parameter

$$r_s = (\pi a_0^2 n)^{-1/2} = 2^{1/2} s_0 = e^2/2^{1/2} p_0 . \tag{10.3}$$

The interaction causes a decrease in the amplitude of the oscillations. Assuming that s_0 is small we can use the expression

$$p_F = p_0 \exp(-\tfrac{1}{3} I s_0^{4/3}) \times g(\gamma_0)\,, \tag{10.4}$$

where $g(\gamma_0)$ is an oscillating function.

The susceptibility shows the characteristic 2-D de Haas–van Alphen phenomenon. The amplitude of the most dominant susceptibility oscillation is proportional to

$$\chi_0 \eta_0 / \alpha^2 \sim \chi_0 p_0^2 / a^4\,, \tag{10.5}$$

where χ_0 is the ideal susceptibility corresponding to the Pauli susceptibility for the 3-D case. Different from the 3-D case, the ideal susceptibility has a dimension due to the 2-D character of the system. The susceptibility oscillations are characterized by the amplitude which varies with the field as H^{-2}. We have given a relation between the amplitude of the most dominant oscillation and the field and temperature. An experimental test of such a relation is interesting.

The internal energy has also been derived. The amplitude of the main oscillation is approximately proportional to H^2 and the frequency is determined by a^2/p_0^2 when plotted against $1/H$. The oscillation is determined by the ratio of the Fermi energy to the field energy. The specific heat shows the same type of oscillations, as we expect. Under an adiabatic condition, the temperature of the system will oscillate. In the case of 3-D metallic electrons such oscillations have been observed.

So far, we have considered the low-temperature degenerate case. The opposite case of low density and high temperature can be investigated based on our general grand-partition function. Taking the first-order-exchange contribution into consideration, we find the quantum effects as follows:

$$\chi^{\text{para}} = (\mu_B^2 n/kT)\tfrac{1}{4} g^2 [1 - \tfrac{1}{4}\lambda^2 n (1 - \lambda/2^{1/2} a_0)]\,, \tag{10.6}$$

$$\chi^{\text{dia}} = -(\mu_B^2 n/k\,T)\tfrac{1}{3}[1 - \tfrac{1}{4}\lambda^2 n(1 - \lambda/2^{1/2} a_0)]\,. \tag{10.7}$$

If we introduce as effective g factor in the first equation above, we find a first quantum correction which is determined by the de Broglie thermal wavelength λ and density:

$$g^* \sim g(1 - \tfrac{1}{8}\lambda^2 n)\,. \tag{10.8}$$

For convenience, a conversion table from our units to the regular units is given in Table I.

It is very important to note that the oscillating susceptibility formula given by (7.3) has been derived in consideration of electron-electron interaction. That is, it is not an ideal-gas formula.

TABLE I. Conversion of units.

Quantities	Atomic units	Regular units
χ_0	$(1/2\pi)(e^2/c^2)$	$(4\pi m/h^2)\mu_B^2$
α	$\beta a^2 = \beta(e/c)H$	$\beta\mu_B H$
γ_0	α/η_0	$\mu_B H/(\hbar^2 p_0^2/2m)$

Yet, the formula does not contain an interaction term. This is a 2-D peculiarity in which the interaction terms cancel out in the high-density approximation. On the other hand, a similar treatment of a 3-D electron gas yields a factor[5] $\exp(-c r_s/a^2)$ which reduces the amplitude of the dHvA oscillations. This reduction is related to the decrease of the Fermi momentum in the direction of a magnetic field. The 2-D electrons do not have motion in this direction when the field is perpendicular to their surface. Therefore, as far as many-electron interaction is concerned, there is no such reduction factor for two dimensions. If experimentally the amplitude is found to show an extra density dependence, it is quite possibly due to some other effect than electron-electron interaction.

We also remark that in Eq. (7.3), the Landé g factor appears merely as a sign factor. This is because the dHvA oscillations are essentially due to electron orbital motion. The same is true in the 3-D case.

Therefore, in consideration of these fundamental features of the 2-D dHvA phenomena, we may not introduce an effective g factor to represent many-electron interactions.

The amplitude χ_M^{osc} of the basic oscillation defined by Eq. (7.5) is approximately

$$\chi_M^{\text{osc}} \sim 4\chi_0(kT/a^2)\exp(-kT\pi^2/a^2)\,. \tag{10.9}$$

The exponential factor shows that the thermal energy causes a kind of level broadening. Note in the 3-D exponential reduction factor which we have just mentioned that the interaction parameter r_t appears in place of kT.

Because of the form of Eq. (10.9), the amplitude of the oscillating susceptibility will be a maximum at

$$kT = a^2/\pi^2 = \mu_B H/\pi^2\,.$$

This temperature is of order 0.1 K for H of order 10^4 G. On the other hand, as a function of density the amplitude increases monotonically. Unless the oscillating part, i.e., $\sin \pi/\gamma_0$, is introduced, there will be no maximum.

Finally, we remark that Eq. (7.3) for the oscillating susceptibility does not have a constant phase

$-\frac{1}{4}\pi$ in the oscillating terms. This phase characterizes the 3-D case and therefore its disappearance in the 2-D case should be noted. We hope that this and other 2-D oscillating properties be tested by experiments in the future.

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APPENDIX

Contribution of Eq. (2.8) from the residues at $\pi s = -l\pi$.

$$I_{03} = \frac{\pi}{2\pi i} \sum \oint ds \, \frac{e^{s\eta} \cosh(\frac{1}{2} g s \alpha)}{s \sin\pi s \sinh(\alpha s)} .$$

Using $s = x - l, (l = 1, 2, 3, \ldots)$, we find

$$I_{03} = \sum_{l=1}^{\infty} (-)^l e^{-l\eta} \frac{\cosh(\frac{1}{2} g\alpha l)}{\sinh(\alpha l) l} \frac{1}{2\pi i} \oint dx \, \frac{e^{x\eta}}{x}$$
$$= e^{-\eta} \sum_{l=1}^{\infty} (-)^l e^{-(l-1)\eta} \frac{\cosh(\frac{1}{2} g\alpha l)}{\sinh(\alpha l) l} .$$

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