Study of a two-dimensional electron gas in a magnetic field. I. Weak field

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(Received 11 August 1977)

A many-body theory of a two-dimensional electron gas in a magnetic field is presented. The field dependence of the Fermi momentum shows a two-dimensional peculiarity. It consists of the spin-dependent paramagnetic part and the orbital diamagnetic part. The former increases more strongly with r_s than the latter. The paramagnetic and diamagnetic susceptibilities are found as functions of r_s. The field-dependent ground-state energy is evaluated. Its paramagnetic part decreases steadily with r_s, while the diamagnetic part shows a maximum. No such maximum has been found for the three-dimensional case in the same approximation which is valid for high density and low magnetic fields. From the susceptibilities, an effective g factor and an effective mass are derived. They are given approximately by $g^*/g = 1 + 0.87 \times 10^6 n^{-1/2}$ or more accurately by $g^{*2}/g^2 = 1 + 1.74 \times 10^6 n^{-1/2} - 0.708 \times 10^{12} n^{-1}$ and by $m^*/m = 1 + 3.21 \times 10^4 n^{-1/2}$, where n is the number density.

I. INTRODUCTION

Since the famous work of Fang and Stiles[1] on the extraordinary g factor, the many-body effects of two-dimensional (2-D) electrons on liquid helium or semiconductor interfaces have attracted much attention in recent years. Active theoretical and experimental works have been performed on the magnetoconductance,[2] dispersion relation,[3] valley splitting,[4] exchange and correlation energy,[5] electrodynamics,[6] cyclotoron resonance,[7] optical properties,[8] Wigner lattice formation,[9] effective mass, and g factor,[10] etc. In contrast to the metallic electrons, the 2-D electrons show relatively large correlation effects in a very wide density range. Hence, they provide us with a very important test ground for many-body theory.

Therefore, we have also studied the spatial correlation of these electrons and the correlation energy.[11] Concerning the latter, Rajagopal and Kimball have made an independent approach almost simultaneous to ours.[12] We have shown recently that the results of both approaches for the second-order ring contribution are in perfect agreement with each other, although the numerical coefficient which appeared in our Eq. (4 4) of Ref. 11 involved a typographical error: It should read −0.1728. However, Rajagopal and Kimball's second-order exchange contribution was different.

Concerning the g factor, Janak[13] was the first to attribute the extraordinary density variation to electron interaction. Following him, Suzuki and Kawamoto[15] considered the weak-field case and Ando and Uemura[14] treated the strong-field case.

The g factor has been evaluated by considering the spin-up and spin-down electrons separately. If their self-energies are given by Σ_+ and Σ_-, the effective g factor is obtained from $g^* = g + (\Sigma_- - \Sigma_+)/\mu_B H$. Here, the electrons in different spin states are considered to have the same dielectric function. Otherwise, their direct coupling does not enter the above derivation of g^*. Moreover, the dielectric function has often been replaced by its static limit.

In the past few years, we have studied the many body properties of a three-dimensional (3-D) electron gas in a weak,[16] intermediate,[17] and strong[18] magnetic field. The first case corresponds to constant susceptibility, the second to oscillating susceptibility, and the third to what is known as the quantum strong-field limit. These three cases must be treated separately because the strength of a magnetic field relative to the thermal energy and also the Fermi energy determine the mathematical treatments.

For all these cases we have constructed the propagator correctly. The eigenvalues of the propagator correspond to the dielectric function, as has been proven elsewhere.[19] Although we do not use the dielectric function formalism explicitly, it has been obtained for all the above three cases as a function of momentum, frequency, and also field. By an obvious transformation from $1/kT$ to frequency, we are able to convert the eigenvalues to the dielectric function.

In the presence of a magnetic field, the electrons change their thermal motion. It is very important to recognize this change in investigating many-body properties of the electrons. In our formalism the change is represented by the field dependence of the eigenvalues which in turn determine magnetic properties.

It is the purpose of the present paper to extend our 3-D weak-field theory to two dimensions and try to evaluate the field energy and the susceptibility as functions of r_s explicitly. To our knowledge, the magnetic correlation energy has not been evaluated analytically yet. The magnetic susceptibility will be derived from the grand-partition function based on the standard differentiation with respect to the field. We shall evaluate the total magnetic response of the 2-D system by this grand-ensemble method. As can be expected, the total response consists of two parts: paramagnetic and diamagnetic. These two parts can be separated from each other easily because the former depends on Landé's g factor, while the latter is independent. Indeed, the former is due to electron spins while the latter is due to electron orbital motion. In view of this difference and because we shall obtain the paramagnetic and diamagnetic susceptibilities, we shall determine the effective g^* and m^* explicitly as functions of r_s.

In Sec. II, we shall construct the free-electron propagator rigorously. For convenience, we shall use the units in which $\hbar=1$ and $2m=1$. From the propagator, the grand-partition function of the ideal 2-D electron gas will be obtained. Section III gives the grand-ensemble treatment of the first-order exchange graphs. We shall assume that the field energy is small and calculate the exchange contribution to order a^4, where a^2 represents the field energy. The field-independent part of this contribution is known and agrees with what we have obtained before.[11] The field-dependent part will consist of two terms, one being common to paramagnetic and diamagnetic responses and the other being intrinsic to diamagnetic response only. The latter represents the anisotropy due to the magnetic field. In Sec. IV, the ring-diagram contribution will be evaluated. The eigenvalues which represent the unit propagation in the rings are given rigorously first. We shall then evaluate the eigenvalues in powers of a^2. Note that the eigenvalues correspond to the dielectric function.[19] The field-dependent part of the eigenvalues for small field consists again of two parts, exactly as in the case of the exchange contribution. Hence, there will be two field-dependent terms in the grand-partition function which represent the contribution from the ring diagrams The first, which may be called the isotropic part, shall be given relatively easily and rigorously. Its contribution will be of order e^4 in interaction and a^4 in field strength. The second, which may be called an anisotropic part, includes a divergence which can be exactly cancelled by the corresponding term from the exchange contribution. This divergence is due to zero momentum transfer and remains in the ring-diagram contribution even after summing over all orders in the interaction parameter e^2. There is yet another divergence in each interaction order of the anisotropic part of the ring contribution. This divergence is due to the singularity of the eigenvalues at momentum $2p_F$. Its origin and elimination will be discussed in Sec. V.

We shall assemble all contributions in Sec. VI and give the total grand-partition function explicitly. Based on the golden rule of grand-ensemble theory, the Fermi momentum will be derived in Sec. VII. This quantity squared represents the chemical potential of the interacting system. In Sec. VIII, the total susceptibility will be derived as a function of r_s. The result will be a zero-temperature susceptibility formula. It contains the paramagnetic part and the diamagnetic part. Section IX will give the energy due to the magnetic field. Finally, in Sec. X, we shall show that the effective g factor and mass can be derived from the paramagnetic and diamagnetic susceptibilities, respectively. Although improvements on our susceptibilities are possible in principle, they are not what can be expected in practice in the immediate future.

II. FREE-ELECTRON PROPAGATOR

Let us consider N electrons in a uniform magnetic field. The Hamiltonian is

$$\mathcal{H}=\sum_i\left[\left(p_{ix}-\frac{e}{c}Hy_i\right)^2+p_{iy}^2-\tfrac{1}{2}g\mu_B\vec{\sigma}_i\cdot\vec{H}\right]+\sum_{i<j}\frac{e^2}{r_{ij}}\,, \tag{2.1}$$

where g is Landé's factor, H is the magnitude of the magnetic field and the σ's are spin variables which take on either 1 or -1.

The solutions of the Schrödinger problem for the case without Coulomb interaction are known:

$$\psi_{n,\sigma,p_x}=\frac{A_n}{(2\pi)^{1/2}}\exp[ip_xx-\tfrac{1}{2}a^2(y-y_0)^2]\times H_n(a(y-y_0))|\sigma\rangle \tag{2.2}$$

$$\epsilon_{n\sigma}=(n+\tfrac{1}{2})\omega_0\pm\tfrac{1}{4}g\omega_0,$$

where

$$A_n^2=\frac{a}{2^n\pi^{1/2}n!}\,,\quad y_0=\frac{p_x}{a^2}\,,\quad a^2=\frac{eH}{c}=\tfrac{1}{2}\omega_0. \tag{2.3}$$

The H_n's are the Hermite polynomials and $|\sigma\rangle$ represents spin states.

Therefore, the free-electron propagator can be constructed as follows:

$$K_0(\vec{r}_2\vec{r}_1,\beta_2-\beta_1)=\sum_{n,\sigma,p_x} e^{-(\beta_2-\beta_1)\epsilon_{n\sigma}}\psi_{n\sigma p_x}(\vec{r}_2)\psi^*_{n\sigma p_x}(\vec{r}_1)=\frac{a^2}{2\pi}K_s(s)\frac{\exp[-\frac{1}{4}a^2\coth(sa^2)(x^2+y^2)]}{2\sinh(sa^2)}e^{i\phi}, \tag{2.4}$$

where

$$K_s(s)=\exp(-s\tfrac{1}{2}ga^2)\,|\uparrow\rangle\langle\uparrow|\,\exp(s\tfrac{1}{2}ga^2)\,|\downarrow\rangle\langle\downarrow|,$$
$$s=\beta_2-\beta_1,\quad \phi=\tfrac{1}{2}a^2(y_1+y_2)(x_2-x_1),\quad \vec{x}=\vec{x}_2-\vec{x}_1. \tag{2.5}$$

Using the propagator, we shall evaluate systematically the grand-partition function. The free-electron contribution to the grand-partition function is given by

$$\ln\Xi_0=\sum_{l=1}^{\infty}\frac{z^l}{l}(-)^{l+1}A_l, \tag{2.6}$$

where

$$A_l=\tfrac{1}{2}a^2\,\mathrm{Tr}\left(K_s(l\beta)e^{i\phi}\frac{\exp[-\frac{1}{4}a^2\coth(l\beta a^2)(x^2+y^2)]}{2\sinh(l\beta a^2)}\right). \tag{2.7}$$

We obtain

$$\ln\Xi_0=-\frac{Aa^2}{2\pi}\sum_{l=1}^{\infty}(-z)^l\frac{1}{l}\frac{\cosh(\frac{1}{2}gl\beta a^2)}{\sinh(l\beta a^2)}, \tag{2.8}$$

where A is the surface area.

For weak fields the right-hand side of Eq. (2.8) can be expanded in powers of a^2. To order a^4, we find

$$\ln\Xi_0=\frac{A}{2\pi\beta}F_1(\eta)+\frac{A\beta a^4}{4\pi}(\tfrac{1}{4}g^2-\tfrac{1}{3})F_{-1}(\eta)+O(a^6), \tag{2.9}$$

where

$$F_s(\eta)=\frac{1}{\Gamma(s+1)}\int_0^\infty dx\frac{x^s}{e^{x-\eta}+1}. \tag{2.10}$$

Equation (2.9) is meaningful only for weak fields. The low-temperature limit should be taken after the field has been brought to zero. Since in general the case $g=2$ is important, we have introduced into Eq. (2.9) a factor of 4 together with g^2.

In Eq. (2.8), the magnetic field enters the hyperbolic cosine function, which is even, and also the hyperbolic sine function, which is odd. Due to the extra a^2 outside of the summation, the grand-partition function is an even function of the magnetic field—as it should be, because we expect the same statistical properties from the system even when the field is reversed. In Eq. (2.9) we have retained terms to order a^4. For the purpose of deriving the susceptibility, the terms to this order in field strength are needed.

The function $F_s(\eta)$ defined by Eq. (2.10) is characteristic of Fermi statistics. In the 3-D case, the field-independent term and the first field-dependent term proportional to a^4 in the free-electron grand-partition function are characterized by $F_{3/2}$ and $F_{-1/2}$, respectively. For the present 2-D case, the two F functions are

$$F_1(\eta)=\frac{\eta^2}{2}+\sum_{m=1}\frac{(-)^{m+1}\exp(-\eta m)}{m^2},$$
$$F_{-1}(\eta)=\sum_{l=1}(-)^{l+1}z^l,\quad z=\exp\eta=z/(1+z). \tag{2.11}$$

III. FIRST-ORDER-EXCHANGE GRAPHS

The contribution from the first-order-exchange graphs to the grand-partition function is given by

$$\begin{aligned}\ln\Xi_{1x}&=\frac{1}{2}\sum_{l=2}^{\infty}(-z)^l\sum_{j=1}^{l-1}\mathrm{Tr}\int d\beta'\int d\vec{r}'\int d\vec{r}''\,\phi(\vec{r}'-\vec{r}'')K_0(\vec{r}'\vec{r}'',j\beta)K_0(\vec{r}''\vec{r}',(l-j)\beta)\\&=\frac{A\beta a^2}{16\pi^3}\sum_{l=2}^{\infty}(-z)^l\frac{\cosh(\frac{1}{2}gl\beta a^2)}{\sinh(l\beta a^2)}\sum_{j=1}^{l-1}\int d\vec{q}\,u(q)\exp\left(-q^2\frac{\sinh(j\beta a^2)\sinh[(l-j)\beta a^2]}{a^2\sinh l\beta a^2}\right),\end{aligned} \tag{3.1}$$

where $u(q)$ is the Fourier transform of the Coulomb potential. We can expand the right-hand side in powers of a^2. After term-by-term integrations over q, we perform summations over j and l. To order a^4 we find the grand-partition function as follows:

$$\ln\Xi_{1x}=(\mathrm{I})+(\mathrm{II})+(\mathrm{III}), \tag{3.2}$$

where

$$(\mathrm{I})=\frac{A\beta e^2}{4\pi}\sum_{l=2}^{\infty}\frac{(-z)^l}{l\beta}\sum_{j=1}^{l-1}\int_0^\infty dq\,e^{-j(1-j/l)\beta q^2}, \tag{3.3}$$

$$(\mathrm{II})=\frac{A\beta e^2}{8\pi}\left(\frac{g^2}{4}-\frac{1}{3}\right)a^4\sum_{l=2}^{\infty}\frac{(-z)^l}{l\beta}(l\beta)^2\times\sum_{j=1}^{l-1}\int_0^\infty dq\,e^{-j(1-j/l)\beta q^2}, \tag{3.4}$$

$$(\mathrm{III}) = \frac{A\beta e^2}{4\pi} a^4 \sum_{l=2}^{\infty} \frac{(-z)^l}{l\beta}\left(\frac{\beta^3}{3l}\right)\sum_{j=1}^{l-1} j^2(l-j)^2 \times\int_0^{\infty} dq\, q^2 e^{-j(1-j/l)\beta q^2}. \quad (3.5)$$

The integrals over q are easy to obtain. The summations can be expressed in terms of

$$F_{-1/2}(\eta') = \sum_{t=1}^{\infty} (-)^{t+1}\frac{e^{\eta' t}}{t^{1/2}} = 2\left(\frac{\beta}{\pi}\right)^{1/2}\int_0^{\infty} dk\,\frac{1}{e^{-\eta'+\beta k^2}+1} = 2\left(\frac{\beta}{\pi}\right)^{1/2} |p_F^2 - q^2|^{1/2}, \quad (3.6)$$

where

$$\eta' = \eta - \beta q^2. \quad (3.7)$$

We find

$$(\mathrm{I}) = \frac{Ae^2}{4\pi}\int_0^{\infty} dq\,[F_{-1/2}(\eta')]^2 = \frac{2A\beta e^2}{3\pi^2}p_F^3. \quad (3.8)$$

Similarly,

$$(\mathrm{II}) = \frac{A\beta^2 e^2 a^4}{8\pi}\left(\frac{g^2}{4}-\frac{1}{3}\right)\frac{\partial^2}{\partial\eta^2}\int_0^{\infty} dq\,[F_{-1/2}(\eta')]^2 = \frac{A\beta e^2 a^4}{4\pi^2}\left(\frac{g^2}{4}-\frac{1}{3}\right)\frac{1}{p_F}. \quad (3.9)$$

The third integral shall be combined with the ring-diagram contribution as will be discussed in Sec. IV.

IV. RING DIAGRAMS

The ring-diagram contribution can be expressed in terms of the eigenvalues of the effective propagator. They are

$$\lambda_j(q) = \frac{a^2}{2\pi}\sum_{l=1}^{\infty}(-)^{l-1}z^l\,\frac{\cosh(\frac{1}{2}gl\beta a^2)}{\sinh(l\beta a^2)} \times\sum_{j=0}^{l-1}\int_0^{\beta}\exp\left(\frac{2\pi i j}{\beta}\alpha - \frac{q^2}{4\gamma}\right)d\alpha, \quad (4.1)$$

where

$$\frac{1}{4\gamma} = \frac{1}{a^2}\,\frac{\sinh[(\alpha+j\beta)a^2]\sinh\{[-\alpha+(l-j)\beta]a^2\}}{\sinh(l\beta a^2)}. \quad (4.2)$$

Performing the j sum and expanding in powers of a^2, we find

$$\lambda_j(q) = \lambda_j^{(0)}(q) + \lambda_j^{(1a)}(q) + \lambda_j^{(1b)}(q), \quad (4.3)$$

$$\lambda_j^{(0)}(q) = \frac{1}{2\pi}\sum_{l=1}^{\infty}\frac{(-)^{l-1}z^l}{l}\int_0^l \exp\left[2\pi i j x - x\left(1-\frac{x}{l}\right)\beta q^2\right]dx, \quad (4.4)$$

$$\lambda_j^{(1a)}(q) = \frac{\beta^2 a^4}{4\pi}\left(\frac{g^2}{4}-\frac{1}{3}\right)\sum_{l=1}^{\infty}(-)^{l-1}z^l l\int_0^l \exp\left[2\pi i j x - x\left(1-\frac{x}{l}\right)\beta q^2\right]dx, \quad (4.5)$$

$$\lambda_j^{(1b)}(q) = \frac{\beta^3 a^4 q^2}{6\pi}\sum_{l=1}^{\infty}(-)^{l-1}z^l\int_0^l x^2\left(1-\frac{x}{l}\right)^2 \exp\left[2\pi i j x - x\left(1-\frac{x}{l}\right)\beta q^2\right]dx. \quad (4.6)$$

Note that the two field-dependent parts of the eigenvalues are related to the field-independent eigenvalues as follows:

$$\lambda_j^{(1a)}(q) = a^4\left(\frac{g^2}{8}-\frac{1}{6}\right)\left(\frac{\partial^2}{\partial\epsilon_F^2}\lambda_j^{(0)}(q)\right)_{\beta,q}, \quad (4.7)$$

$$\lambda_j^{(1b)}(q) = \frac{q^2 a^4}{3}\left(\frac{\partial^3}{\partial\epsilon_F\partial(q^2)^2}\lambda_j^{(0)}(q)\right)_{\beta}, \quad (4.8)$$

where $\epsilon_F = p_F^2$ represents the Fermi energy of the electron gas.

The field-independent eigenvalues have been evaluated elsewhere.[11] We note that it is possible to express the eigenvalues as follows:

$$\lambda_j^{(0)}(q) \equiv \lambda^{(0)}(s,u) = (1/2\pi)F(s,u),$$
$$F(s,u) = 1 - (1/2s)[g_+(s,u)+g_-(s,u)], \quad (4.9)$$

where we have replaced the discrete integer j by a continuous variable u and introduced a dimensionless variable s such that

$$s = q/p_F,$$
$$u = (2\pi j/\beta p_F^2)(1/s). \quad (4.10)$$

It is easy to distinguish the variable u thus defined and the Coulomb interaction $u(s)$. The functions $g_\pm(s,u)$ are defined by

$$g_\pm(s,u) = [(s\pm iu)^2 - 4]^{1/2}. \quad (4.11)$$

In comparison with the real representation of the eigenvalues which we derived in a recent paper,[11] the above complex representation is compact and convenient.

Explicitly, the field-dependent eigenvalues are

$$\lambda_j^{(1a)}(q) \equiv \lambda^{(1a)}(s,u) = \frac{a^4}{6\pi p_F^4}\,(\tfrac{1}{4}3g^2 - 1)F_1(s,u),$$

$$F_1(s,u) = \frac{1}{s}\left(\frac{1}{g_+^3} + \frac{1}{g_-^3}\right), \tag{4.12}$$

$$\lambda_j^{(1b)}(q) = \lambda^{(1b)}(s,u) = \frac{a^4}{6\pi\epsilon_F^2}\,F_2(s,u), \tag{4.13}$$

$$F_2(s,u) = \frac{2}{s^3}\left(\frac{-s^2u^2 + i2s(s^2-3)u + (s^4 - 4s^2 + 6)}{g_+^5} + \text{c.c.}\right). \tag{4.14}$$

For small s the following approximation is reasonable:

$$F(s,u) = 1 - [u/(u^2+4)^{1/2}] \equiv R(u). \tag{4.15}$$

(As in the 3-D case, the first approximation function $R(u)$ is independent of s.) Note that in this small s region, $F(s,u)$ is independent of s. The notation $R(u)$ has been introduced in analogy with the 3-D case in which essentially the same approximation which is valid for small momentum transfer gives an s-independent R. From Eqs. (4.15), (4.7), and (4.8) we find

$$F_1(s,u) = -\frac{6u}{(u^2+4)^{5/2}} + O(s), \tag{4.16}$$

$$F_2(s,u) = \frac{24}{s^2}\left(\frac{5u}{(u^2+4)^{7/2}} - \frac{u}{(u^2+4)^{5/2}}\right) \equiv \frac{24}{s^2}\,R_B(u), \tag{4.17}$$

where $R_B(u)$ represents the terms in the large parentheses. In terms of these eigenvalues the ring-diagram contribution can be given by

$$\ln\Xi_r = \ln\Xi_r^{(0)} + \ln\Xi_r^{(1a)} + \ln\Xi_r^{(1b)} + O(a^8). \tag{4.18}$$

The first term is independent of the field, the second and third terms are proportional to the square of the field strength. These terms are given by

$$\ln\Xi_r^{(0)} = \frac{A\beta p_F^4}{8\pi^2}\sum_{n=2}^{\infty}\frac{1}{n}\int_0^{\infty} s^2\,ds\int_{-\infty}^{\infty} du[-u(s)\lambda^{(0)}(s,u)]^n, \tag{4.19}$$

$$\ln\Xi_r^{(1a)} = \frac{A\beta p_F^4}{8\pi^2}\times\sum_{n=2}^{\infty}\int_0^{\infty} s^2\,ds\int_{-\infty}^{\infty} du\,[-u(s)]^n \times[\lambda^{(0)}(s,u)]^{n-1}\lambda^{(1a)}(s,u), \tag{4.20}$$

$$\ln\Xi_r^{(1b)} = \frac{A\beta p_F^4}{8\pi^2}\times\sum_{n=2}^{\infty}\int_0^{\infty} s^2\,ds\int_{-\infty}^{\infty} du[-u(s)]^n \times[\lambda^{(0)}(s,u)]^{n-1}\lambda^{(1b)}(s,u). \tag{4.21}$$

The field-independent part of the ring-diagram contribution has been evaluated by us recently.[11] Therefore, we shall omit the discussion. In what follows we shall be concerned only with the field-dependent part. To order e^4, $\ln\Xi_r^{(1a)}$ is given by (see Appendix A)

$$\ln\Xi_r^{(1a)} = \frac{A\beta e^4a^4}{12\pi^2p_F^2}\left(\frac{3g^2}{4} - 1\right)\left(-\frac{\pi}{8} + O(e^2)\right) + O(a^8). \tag{4.22}$$

This result is correct to the indicated order. In the second integral $\ln\Xi_r^{(1b)}$, the expansion in powers of e^2 causes a divergence due to small momentum transfer so that we must perform the summation over n:

$$\sum_{n=2}^{\infty}[-u(s)\lambda^{(0)}(s,u)]^{n-1} = -1 + \frac{1}{1+u(s)\lambda^{(0)}(s,u)}, \tag{4.23}$$

where $\lambda^{(0)}(s,u)$ has been defined in Eq. (4.9).

In Eq. (4.21), even after taking into account all $(e^2/s)^n$ terms, the s integral is divergent at $s=0$. This divergence can be removed by combining the first-order exchange term $\ln\Xi_{1x}^{(1b)}$ with $\ln\Xi_r^{(1b)}$. It can be proved that we can combine $\ln\Xi_{1x}^{(1b)}$ with $\ln\Xi_r^{(1b)}$ by replacing $\sum_{n=2}^{\infty}$ by $\sum_{n=1}^{\infty}$ in Eq. (4.21). We will obtain

$$\ln\Xi_{r1x}^{(1b)} = \ln\Xi_{1x}^{(1b)} + \ln\Xi_r^{(1b)}$$
$$= -\frac{A\beta e^2a^4}{12\pi^2p_F}\int_{s_c}^{\infty} s\,ds\int_0^{\infty} du\,F_2(s,u)\sum_{n=1}^{\infty}\left(\frac{-e^2}{p_Fs}F(s,u)\right)^{n-1} - \frac{A\beta e^2a^4}{12\pi^2p_F}\int_0^{\infty} du\int_0^{s_c} ds\,\frac{24R_B(u)}{s+(e^2/p_F)R(u)}, \tag{4.24}$$

where s_c, a cutoff parameter, has been introduced in order to show the cancellation of a divergence clearly. For $s < s_c$ the summation over n is performed and $R(u)$ and $R_B(u)$ are introduced. These integrals can be evaluated analytically (see Appendixes B and C):

$$J_{1b}^{(n)} \equiv \int_{s_c}^{\infty} s\,ds\int_0^{\infty} du\,F_2(s,u)\,\frac{1}{s^{n-1}}\,[F(s,u)]^{n-1}, \tag{4.25}$$

$$J^{c}_{1b} \equiv \int_0^\infty du\, R_B(u) \int_0^{s_c} ds\, \frac{1}{s+(e^2/p_F)R(u)} . \quad (4.26)$$

Using these $J^{(n)}_{1b}$ and J^{c}_{1b} we can write $\ln \Xi^{(1b)}_{r1x}$ as follows:

$$\ln \Xi^{(1b)}_{r1x} = -\frac{A\beta e^2 a^4}{12\pi^2 p_F} \sum_{n=1}^{\infty} \left(\frac{-e^2}{p_F}\right)^{n-1} J^{(n)}_{1b} - \frac{2A\beta e^2 a^4}{\pi^2 p_F} J^{c}_{1b}, \quad (4.27)$$

where the last term is given by the integral of Eq. (4.26) which is

$$J^{c}_{1b} = \frac{1}{96} \ln\left(\frac{e^2}{s_c p_F}\right) - \frac{7\pi}{1536} - \frac{11}{1440} + \frac{1}{24}\left(\frac{9\pi}{128} - \frac{1}{4}\right)\frac{e^2}{s_c p_F} + O(e^4), \quad (4.28)$$

where in view of e^2 in that term, the terms of order e^4 are not evaluated. The coefficients of the first two terms in the sum of Eq. (4.27) are given by the next two equations.

$$J^{(1)}_{1b} = \tfrac{1}{4} \ln s_c - \tfrac{1}{4} - \tfrac{1}{2}\ln 2 + \phi. \quad (4.29)$$

The first term diverges when $s_c \to 0$ but this divergence is cancelled by the corresponding term in the first term of Eq. (4.28)

$$J^{(2)}_{1b} = (\tfrac{9}{128}\pi - \tfrac{1}{4})(1/s_c) - \tfrac{11}{192}\pi + \phi. \quad (4.30)$$

Again, we have retained only the terms of order e^4 and lower. The divergent first term here is cancelled by the fourth term in Eq. (4.28). The term ϕ is divergent in the following way:

$$\phi = \int^2 \frac{s\, ds}{2(4-s^2)^{3/2}} . \quad (4.31)$$

We will show in Sec. V that this divergence is artificial and can be eliminated. It is due to the singularity in $\lambda^{(1b)}(s,u)$ at $s=2$. It is actually caused by a premature zero-temperature limit taken in the eigenvalues before integration. In Sec. V, we will make a finite-temperature approach and take the zero-temperature limit after the s integration. The result thus obtained agrees exactly with what we have obtained, minus the above divergent terms. The result is

$$\ln \Xi^{(1b)}_{r1x} = \frac{A\beta a^4}{48\pi^2}\left[-(e^2/p_F)\ln(e^2/2p_F) + (\ln 2 + \tfrac{7}{16}\pi + \tfrac{26}{15})(e^2/p_F) + \tfrac{11}{48}\pi(e^4/p_F^2) + O(e^6)\right]. \quad (4.32)$$

V. FINITE-TEMPERATURE APPROACH

The divergence in Eq. (4.29) or Eq. (4.30) is due to the singularity in the eigenvalues $\lambda^{(0)}(s,y)$. In this section, we shall show that a finite-temperature approach eliminates this divergence because the eigenvalues are free from the singularity.

We note first

$$\lambda^{(0)}_j(q) = \frac{1}{2\pi q}\int_0^\infty f(p)\left(\frac{1}{(p_+^2-p^2)^{1/2}} + \frac{1}{(p_-^2-p^2)^{1/2}}\right) p\, dp, \quad (5.1)$$

where

$$p_\pm = \tfrac{1}{2}q \pm i(\pi j/\beta q). \quad (5.2)$$

If we apply the Sommerfeld method to Eq. (5.1), we find a finite-temperature correction given by

$$\Delta\lambda^{(0)}_j(q) = (\pi/6\beta^2 p_F^3 q)(1/g_+^3 + 1/g_-^3), \quad (5.3)$$

where $g_\pm$ have been defined by Eq. (4.11). Unfortunately, this result is still singular for zero temperature. This singularity is caused by the zeros of the denominator of Eq. (5.1) at $p=p_\pm$.

In order to avoid this difficulty, we introduce a linear approximation to $f(p)$:

$$f(p) = \begin{cases} 0, & p_F+\epsilon < p \\ -(1/2\epsilon)(p-p_F)+\tfrac{1}{2}, & p_F-\epsilon < p < p_F+\epsilon, \\ 1, & p < p_F-\epsilon \end{cases} \quad (5.4)$$

where

$$\epsilon = 1/\beta p_F. \quad (5.5)$$

We then find

$$\lambda^{(0)}_j(q) = (1/2\pi)F_j(q), \quad (5.6)$$

where

$$F_j(q) = 1 - (1/4q\epsilon)[T(p_F+\epsilon) - T(p_F-\epsilon)], \quad (5.7)$$

and where in terms of $p_\pm$ of Eq. (5.2),

$$T_\pm(k) = k(p_\pm^2 - k^2)^{1/2} + p_\pm^2 \sin^{-1}\frac{k}{p_\pm} . \quad (5.8)$$

In the zero-temperature limit, we find

$$\lim_{\beta\to\infty} F_j(q) = 1 - \frac{1}{2q}\left(\frac{\partial}{\partial k} T(k)\right)_{p_F} = 1 - \frac{1}{2s}(g_+ + g_-) = F(s,u), \quad (5.9)$$

in agreement with Eq. (4.9). We also find

$$F_2(s,u) = -(\epsilon_F/8\epsilon)\{k_1^2 s^2[V(k_1) + (p_F-\epsilon)W(k_1)] - k_2^2 s^2[V(k_2) + (p_F+\epsilon)W(k_2)]\}, \quad (5.10)$$

where we have used

$$k_1 = p_F + \epsilon, \quad k_2 = p_F - \epsilon, \quad (5.11)$$

$$V(k) = V_+(k) + V_-(k), \quad (5.12)$$

$$V_\pm(k)=\frac{k^2}{16q^5}\left((-15u^2\pm 6isu-s^2)\sin^{-1}\frac{2}{\omega_\pm}-2\frac{u^2\mp 10isu+12-s^2}{(\omega_\pm^2-4)^{1/2}}\right), \tag{5.13}$$

$$W(k)=W_+(k)+W_-(k), \tag{5.14}$$

$$W_\pm(k)=\frac{2}{k^4s^5}\left(\frac{u^4\mp 3isu^3+3(2-s^2)u^2\pm isu(s^2-6)+2(3-s^2)}{(\omega_\pm^2-4)^{3/2}}\right), \tag{5.15}$$

$$W(k)=\partial V(k)/\partial k, \tag{5.16}$$

$$\omega_\pm=s\pm iu. \tag{5.17}$$

In terms of these quantities, we evaluate the combined contribution of the ring and exchange graphs which has been denoted by $\ln\Xi_{r1x}^{(1b)}$. Since Eq. (4.32) includes the two terms, i.e., $n=1,2$ of Eq. (4.27), and for clarity, let us evaluate these two terms separately.

For the case $n=1$, we can write

$$\ln\Xi_{r1x}^{(1b),n=1}=\ln\Xi_{1x}^{(1b)} =-(A\beta e^2a^4/12\pi^2p_F) \times\{J_{1b}^{(1)}+\tfrac{1}{4}[\ln(e^2/p_Fs_c)-\tfrac{7}{16}\pi-\tfrac{11}{15}]\}. \tag{5.18}$$

Here, the first equality indicates that the term $n=1$ coincides with the first-order-exchange contribution. The term J_{1b} is

$$J_{1b}^{(1)}=-\frac{1}{8p_F\epsilon}\left(k_1^5\int_{s_c}^\infty s^3\,ds\int_0^\infty du\,[V(k_1)+(p_F-\epsilon)W(k_1)]-k_2^5\int_{s_c}^\infty s^3\,ds\int_0^\infty du\,[V(k_2)+(p_F+\epsilon)W(k_2)]\right)$$
$$=-\frac{1}{8p_F\epsilon}\{k_1^5[v(k_1)+(p_F-\epsilon)w(k_1)]-k_2^5[v(k_2)+(p_F+\epsilon)w(k_2)]\}, \tag{5.19}$$

where

$$v(k)=-\frac{\ln s_c}{2k^3}+\frac{1}{k^3}(\ln 2-\tfrac{1}{4})+O(s_c^3\ln s_c),$$
$$w(k)=-\frac{\ln s_c}{k^4}+\frac{2\ln 2}{k^4}+O(s_c^2). \tag{5.20}$$

In the limit $\beta\to\infty$, we get

$$\lim_{\beta\to\infty}J_{1b}^{(1)}=\tfrac{1}{4}\ln s_c-\tfrac{1}{4}-\tfrac{1}{2}\ln 2. \tag{5.21}$$

Note that in this limit the divergent term ϕ of Eq. (4.29) does not appear. Except for this divergent term, Eq. (5.21) agrees with Eq. (4.29). We arrive at

$$\lim_{\beta\to\infty}\ln\Xi_{1x}^{(1b)} =(A\beta e^2a^4/12\pi^2p_F)\{-\tfrac{1}{4}\ln s_c+\tfrac{1}{4}+\tfrac{1}{2}\ln 2 +\tfrac{1}{4}[-\ln(e^2/p_Fs_c)+\tfrac{7}{16}\pi+\tfrac{11}{15}]\}$$
$$=(A\beta e^2a^4/12\pi^2p_F)\{-\tfrac{1}{4}\ln(e^2/2p_F)+\tfrac{1}{4}(\ln 2+\tfrac{7}{16}\pi+\tfrac{26}{15}\}. \tag{5.22}$$

The term for $n=2$ consists of

$$\ln\Xi_{r1x}^{(1b),n=2}=(A\beta e^4a^4/12\pi^2p_F^2)\{J_{1b}^{(2)A}+J_{1b}^{(2)B} -(\tfrac{9}{128}\pi-\tfrac{1}{4})(1/s_c)\}, \tag{5.23}$$

where the first term in the curly bracket is

$$J_{1b}^{(2)A}=-\frac{1}{8\epsilon}\left(k_1^4\int_{s_c}^\infty ds\int_0^\infty du\,[V(k_1)+(p_F-\epsilon)W(k_1)] -k_2^4\int_{s_c}^\infty ds\int_0^\infty du\,[V(k_2)+(p_F+\epsilon)W(k_2)]\right). \tag{5.24}$$

In the zero-temperature limit, we find

$$\lim_{\beta\to\infty}J_{1b}^{(2)A}=-1/4s_c+O(s_c). \tag{5.25}$$

The second term in the curly bracket of Eq. (5.23) is given by [cf. Appendix A and Eq. (5.7)]

$$J_{1b}^{(2)B}=-\frac{1}{4\epsilon}\int_{s_c}^\infty\frac{ds}{s}\int_0^\infty du\,F_2(s,u)[T(k_1)-T(k_2)] =\frac{9\pi}{128s_c}+\frac{11\pi}{192}+0(s_i). \tag{5.26}$$

Introducing Eqs. (5.25) and (5.26) into Eq. (5.23), we find

$$\ln\Xi_{r1x}^{(1b)n=2}=\frac{A\beta e^4a^4}{12\pi^2p_F^2}\left[-\frac{1}{4s_c}+\frac{9\pi}{128s_c} +\frac{11\pi}{192}-\left(\frac{9\pi}{128}-\frac{1}{4}\right)\frac{1}{s_c}\right] =\frac{A\beta e^4a^4}{48\pi^2p_F^2}\left(\frac{11\pi}{48}\right). \tag{5.27}$$

Combination of Eqs. (5.22) and (5.27) yield the divergent-free result of Eq. (4.32).

VI. TOTAL FIELD-DEPENDENT GRAND-PARTITION FUNCTION

For convenience, we assemble all the field-dependent terms of the grand-partition function which we have evaluated. Our result to order a^4 in magnetic field is

$$\ln\Xi^{(a)} = A\beta a^4\{(1/12\pi)(\tfrac{3}{4}g^2-1) + c[(1/12\pi^2)(\tfrac{3}{4}g^2-1)+(1/48\pi^2)(\tfrac{26}{15}+\tfrac{7}{16}\pi)] - (1/48\pi^2)c\ln\tfrac{1}{4}c + [-(1/96\pi)(\tfrac{3}{4}g^2-1) + (11/2304\pi)]c^2+\cdots\}, \tag{6.1}$$

where

$$c = e^2/p_F.$$

VII. FERMI MOMENTUM

From the grand-partition function given by Eq. (6.1) we shall evaluate relevant physical quantities. The first quantity to be evaluated is the Fermi momentum p_F as a function of density. For this purpose, we make use of the grand-ensemble relation:

$$n = \frac{1}{A}\left(\frac{\partial \ln\Xi}{\partial \ln z}\right)_{\beta,A}. \tag{7.1}$$

For a density parameter let us use

$$r_s = e^2/2(\pi n)^{1/2}. \tag{7.2}$$

Since we are interested in weak fields, it is appropriate to write

$$p_F = p_F^{(0)} + p_F^{(a)}, \tag{7.3}$$

where the first term is field independent and the second term represents the first field effect. The former has been evaluated in our previous paper[11] as follows:

$$p_F^{(0)} = p_0[1-0.4501r_s - (0.1427\pm 0.04\times 10^{-2})r_s^2] \tag{7.4}$$

The second term $p_F^{(a)}$, which is new, is found to be

$$p_F^{(a)} = p_0\alpha r_s^4[(1.407\times 10^{-2}g^2 - 3.997\times 10^{-3})r_s - 4.689\times 10^{-3}r_s\ln r_s + (3.372\times 10^{-3}g^2+2.287\times 10^{-2})r_s^2 - 6.333\times 10^{-3}r_s\ln r_s], \tag{7.5}$$

where α is a dimensionless combination of the field energy and Fermi energy:

$$\alpha = a^4/r_s^4 p_0^4 = (\omega_0\hbar/4\,\text{Ry})^2. \tag{7.6}$$

Note that $p_F^{(a)}$ disappears when $r_s = 0$. That is, only in the presence of Coulomb interaction, the first field-dependent term $p_F^{(a)}$ appears. This is a 2-D peculiarity.

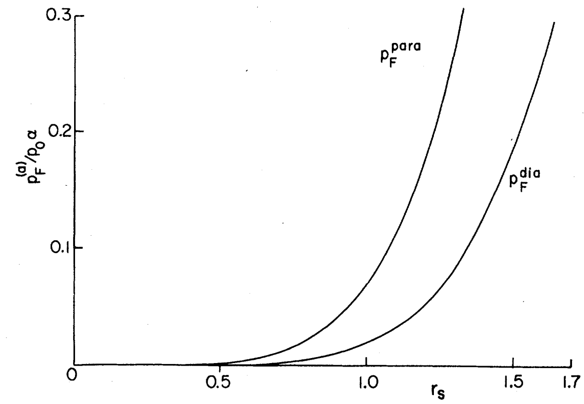

FIG. 1. Field-dependent parts of the Fermi momentum in the units of $p_F^{(a)}/p_0$ plotted against r_s. The paramagnetic case corresponds to $g=2$.

The field-dependent term $p_F^{(a)}$ consists of the paramagnetic and diamagnetic parts. These parts are illustrated in Fig. 1. The spin-dependent paramagnetic part varies more strongly than the orbital diamagnetic part on the density parameter r_s. The field effect on the Fermi momentum appears differently in 2-D than in 3-D. In 3-D, the first term which appears in the square bracket of Eq. (7.5) is independent of r_s. That is, there is an ideal-gas contribution. In 2-D, on the other hand, the electrons are all in Landau levels unless perturbed out by Coulomb interaction. Therefore, the first term depends on r_s.

VIII. MAGNETIC SUSCEPTIBILITY

The magnetic susceptibility is evaluated from

$$\chi = \lim_{H\to 0}\frac{1}{A\beta H}\left(\frac{\partial \ln\Xi}{\partial H}\right)_{A,\beta,z}. \tag{8.1}$$

The susceptibility thus obtained is the total response of the system to a magnetic field and includes both the paramagnetic and diamagnetic parts. These two parts can be easily separated from each other because the former depends on the spin. Note that the susceptiblity has a dimension. This is a fundamental difference from the 3-D case where the susceptibility is dimensionless. The difference arises because the magnetic field energy is defined in exactly the same way. Equation (8.1) yields

$$\chi = (2e^2/c^2)[(1/12\pi)(\tfrac{3}{4}g^2-1) + (8.956\times 10^{-3}g^2r_s) + 4.409\times 10^{-4}r_s - 2.985\times 10^{-3}r_s\ln r_s + (-9.425\times 10^{-4}g^2 + 8.525\times 10^{-3})r_s^2 - 1.3438\times 10^{-3}r_s^2\ln r_s]. \tag{8.2}$$

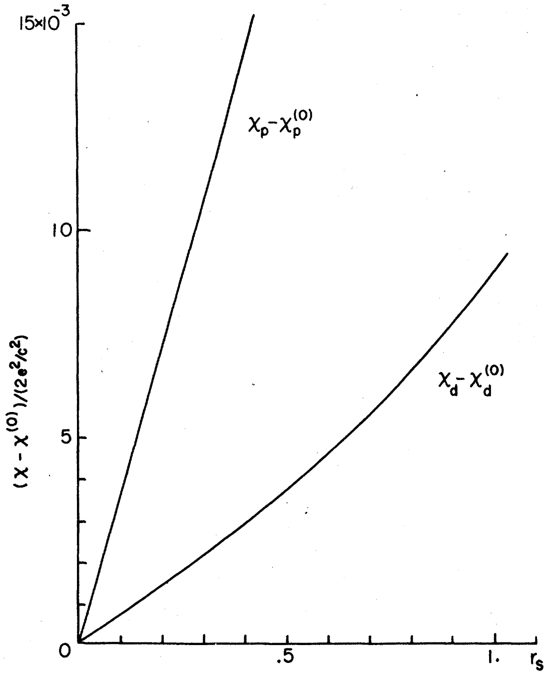

FIG. 2. The density dependences of the paramagnetic and diamagnetic susceptibilities. In both cases, the ideal susceptibilities are subtracted off so that the curves show the effects of Coulomb interaction. The susceptibility differences are in the units of $2e^2/c^2$.

In deriving this result, Eqs. (7.4) and (7.5) have been used.

We have plotted the total susceptibility in Fig. 2. The units of the ordinates are $2e^2/c^2=2\mu_B^2$, where μ_B is the Bohr magneton. For the paramagnetic susceptibility we have chosen $g=2$. It is interesting to observe in these curves that the paramagnetic susceptibility and the absolute magnitude of the diamagnetic susceptibility increase with r_s, the former more strongly. Their limiting values at $r_s=0$ maintain the 3 to 1 ratio as in the 3-D ideal case. The interaction effects in 3-D can be seen by plotting $\chi-\chi_0$ against r_s, which starts from a constant. The corresponding plot for 2-D should be for $(\chi-\chi_0)/2\mu_B^2$, which starts from the origin. This is a 2-D peculiarity.

IX. FIELD ENERGY

The ground-state energy is obtained from

$$\epsilon=\lim_{\beta\to\infty}\left[-\frac{1}{N}\left(\frac{\partial\ln\Xi}{\partial\beta}\right)_{z,A}\right]. \tag{9.1}$$

The field-dependent part of the ground-state energy is defined as

$$\epsilon=\epsilon^{(0)}+\epsilon^{(a)}, \tag{9.2}$$

where the field-independent part $\epsilon^{(0)}$ has been evaluated recently by us. The field-dependent part $\epsilon^{(a)}$ is as follows:

$$\begin{aligned}\epsilon^{(a)}=\alpha r_s^2\{&(\tfrac{1}{3}-\tfrac{1}{4}g^2)+(-5.540\times10^{-3}-0.1125g^2)r_s\\&+0.03751\,r_s\ln r_s\\&+(1.184\times10^{-2}g^2-0.1071)r_s^2\\&+1.689\times10^{-2}r_s^2\ln r_s\}.\end{aligned} \tag{9.3}$$

For $g=2$ both the first and second terms in Eq. (9.3) are negative, indicating that the magnetic field reduces the ground-state energy in the first approximation. However, the paramagnetic and diamagnetic parts counteract each other in this respect. In Fig. 3, we have plotted the paramagnetic part of the field energy. The upper curve represents the ideal-gas field energy. These energies can be separated from the total energy by choosing the g -dependent terms. In the actual evaluation, we have chosen $g=2$. Due to interaction, the paramagnetic energy is lowered. The difference between the two curves increases with r_s.

In Fig. 4, we have plotted the diamagnetic ground-state energy as a function of r_s. Again, the upper

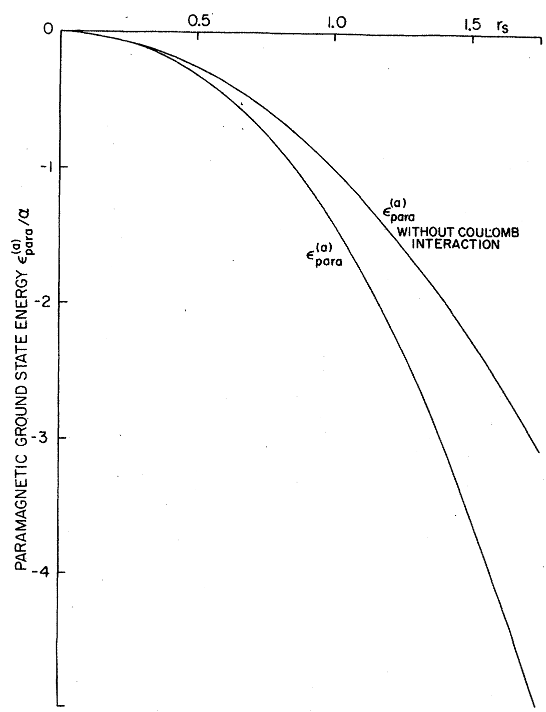

FIG. 3. Paramagnetic ground-state energy as a function of r_s. The upper curve represents the ideal case so that the lower curve shows the effects of Coulomb interaction. Both curves approach each other in the small r_s limit.

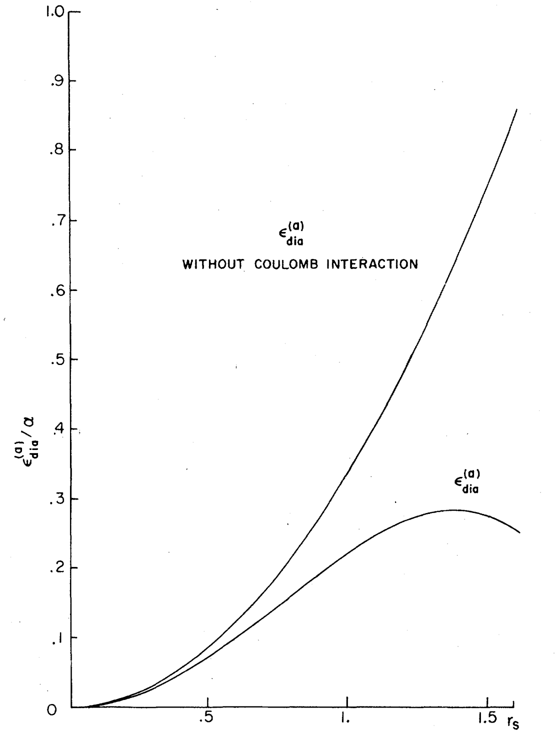

FIG. 4. Diamagnetic-ground-state energy. The upper curve represents the case without Coulomb interaction.

curve represents the ideal case. Although the two curves come closer and closer as r_s becomes smaller, as they should, the diamagnetic ground-state energy shows a maximum. The maximum value is found to be 0.2833α and the maximum point is $r_s = 1.384$. The maximum appears in two dimensions in distinction from the 3-D case in the same approximation. In all these ground-state energy curves, the ordinate is the energy divided by α which is defined by Eq. (7.6). This is a dimensionless combination of the field energy and the Fermi momentum. Expressing the Fermi energy in Ry, we have the second expression in Eq. (7.6) which shows that the constant is proportional to the field strength squared.

X. CONCLUDING REMARKS

We have evaluated the Fermi momentum, susceptibility, and magnetic correlation energy of a 2-D electron gas as functions of r_s. Since the results are obtained for an idealized 2-D system, certain assignments are needed in order to compare the results with experimental data on the electrons in a metal-oxide-semiconductor (MOS) inversion layer.

Let us assume that the average dielectric constant is 7.8 and the effective mass of the electrons is 0.2 m. The dielectric constant reduces the Coulomb interaction, and the effective mass increases the Bohr radius. A valley degeneracy factor 2 may also be introduced. With these corrections we find an effective r_s^* given by

$$r_s^* = (3.867/n^{1/2}) \times 10^6 , \qquad (10.1)$$

where n is the number density.

Let us now try to derive an effective g factor. The paramagnetic susceptibility can be used for this purpose. It is obtained from the total susceptibility by selecting the terms with Landé's g factor. As shown in Fig. 2, the paramagnetic susceptibility is very nearly proportional to r_s. Therefore, with good accuracy we find

$$g^{*2}/g^2 = 1 + (1.74/n^{1/2}) \times 10^6 - (0.708/n) \times 10^{12} . \qquad (10.2)$$

In Fig. 5 the theoretical effective g factor given by Eq. (10.2) is plotted against n. Our theoretical curve is close to the experimental curve of Fang and Stiles at densities around 3×10^{12} cm^{-2}. The deviation at smaller densities may be due to the neglect of higher-order graphs such as the third-order-exchange graphs. Since these higher-order terms contribute relatively more at high r_s, it is safer to take only the first-order term. Assuming that this term is small, we can use a linearized form:

$$g^*/g = 1 + (0.87/n^{1/2}) \times 10^6 . \qquad (10.3)$$

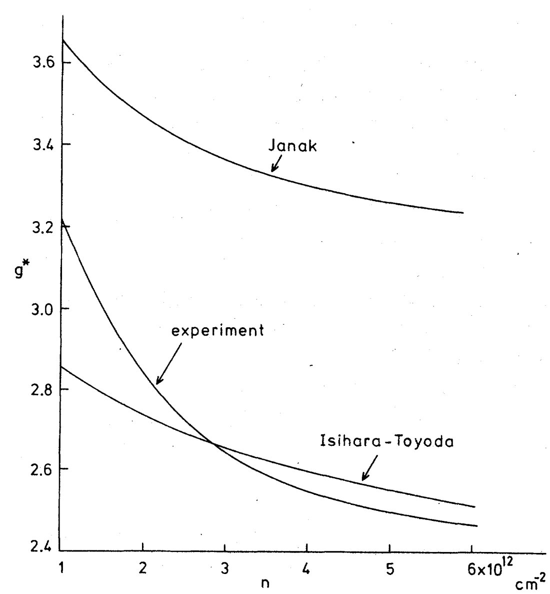

FIG. 5. Theoretical effective g factor due to Eq. (10.2) is compared with the experimental curve of Fang and Stiles. For comparison, Janak's original theoretical curve is also given.

TABLE I. Effective g factor and mass for electrons in a MOS inversion layer.

n (10^{12} cm^{-2})	2	4	6	8	Formula
g^*	2.74	2.60	2.52	2.47	Eq. (10.2)
	3.23	2.87	2.71	2.62	Eq. (10.3)
$\frac{m^*-m}{m}$ (%)	2.27	1.61	1.31	1.13	Eq. (10.4)

At higher densities, the theoretical and experimental curves are almost parallel with each other, indicating that a small adjustment of the parameters such as the average dielectric constant will yield a mcuh better agreement.

The diamagnetic susceptibility is independent of the Landé's factor and is determined by electron orbital motion. Therefore, its r_s dependence may be represented by an effective mass. Note in this respect that

$$\mu_B = e\hbar/4\pi mc .$$

Different from the paramagnetic susceptibility, the diamagnetic susceptibility is nonlinear when plotted against r_s. For simplicity and avoiding a possible difficulty arising from the higher-order terms which we have not considered, we use the first linear term that represents the exchange contribution to the diamagnetic susceptibility. After linearization, we find

$$m^*/m = 1 + (3.21/n^{1/2}) \times 10^4 . \tag{10.4}$$

Note that the coefficient to the second density-dependent term is much smaller than that for the effective g factor. This smallness is due to the small coefficient 4.4×10^{-4} in contrast to the paramagnetic coefficient 8.9×10^{-3} in Eq. (8.2). Because of the smallness of the coefficient, the above approximate formula is better than a formula which we would obtain by retaining other terms. In any case, the above formula indicates that the effective mass will increase by a few percent if the density is decreased.

The theoretical values of the effective g factor and mass are listed in Table I. The theoretical effective g factor thus evaluated is valid for a wider range of density than that for the effective mass, simply because the paramagnetic susceptibility is very nearly linear when plotted against r_s while the diamagnetic susceptibility is not. For this reason, a much stronger variation of m^* with density is expected for low densities. An experimental determination of m^* for a wide density range will be important for our theoretical studies. Finally we remark that we get $g = 2.87$ for $n = 4 \times 10^{12}$ cm^{-2} according to Eq. (10.3) which we proposed for low densities.

ACKNOWLEDGMENTS

This work was supported in part by the Dutch Science Foundation (FOM). It is submitted by one of the authors (T.T.) to the State University of New York at Buffalo in partial fulfillment of the Ph.D. degree.

APPENDIX A

Combination of Eq. (4.2) with Eqs. (4.9) and (4.12) gives

$$\ln \Xi_r^{(1a), n=2} = \frac{A\beta e^4 a^4}{12\pi^2 p_F^2}\left(\frac{3g^2}{4} - 1\right)(J_{1a}^{(1)} + J_{1a}^{(2)} + J_{1a}^{(3)}), \tag{A1}$$

where

$$J_{1a}^{(1)} = \int_0^\infty \frac{ds}{s}\int_0^\infty du\left(\frac{1}{g_+^3} + \frac{1}{g_-^3}\right), \tag{A2}$$

$$J_{1a}^{(2)} = -\frac{1}{2}\int_0^\infty \frac{ds}{s^2}\int_0^\infty du\left(\frac{1}{g_+^2} + \frac{1}{g_-^2}\right), \tag{A3}$$

$$J_{1a}^{(3)} = -\frac{1}{2}\int_0^\infty \frac{ds}{s^2}\int_0^\infty du\left(\frac{g_-}{g_+^3} + \frac{g_+}{g_-^3}\right), \tag{A4}$$

where $g_\pm$ is defined in Eq. (4.11).

Using the method given in Appendix D we obtain

$$J_{1a}^{(1)} = \int_0^2 \frac{ds}{s} i\int_{\Gamma_3} \frac{dz}{(z^2-4)^{3/2}} = -\int_0^2 \frac{ds}{2(4-s^2)^{1/2}} = -\frac{\pi}{4}, \tag{A5}$$

$$J_{1a}^{(2)} = -\frac{1}{2}\int_0^2 \frac{ds}{s^2} i\int_{\Gamma_3}\frac{dz}{z^2-4} = \frac{\pi}{4}\int_0^2 \frac{ds}{s^2} = -\frac{\pi}{8} + \lim_{s\to 0}\frac{\pi}{4s}, \tag{A6}$$

$$J_{1a}^{(3)} = -\frac{1}{2}\int_0^\infty \frac{ds}{s^2} \times 2\int_0^\infty du \frac{u^4 + 2(4-3s^2)u^2 + (s^2-4)^2}{\{[u^2+(s+2)^2][u^2+s-2)^2]\}^{3/2}}$$
$$= -\frac{1}{4}\int_0^1 \frac{dk}{k^2}[2E(k) - K(k)] - \frac{1}{4}\int_0^1 dk\left(\frac{2[E(k)-K(k)]}{k} + kK(k)\right)$$
$$= \lim_{s\to 0} -\frac{\pi}{4s} + \frac{\pi}{4}. \tag{A7}$$

Introducing Eqs. (A5), (A6) and (A7) into (A1), we obtain Eq. (4.22).

APPENDIX B

From Eqs. (4.14) and (4.25) we find

$$J_{1b}^{(1)}=2\int_{s_c}^{\infty}\frac{ds}{s^2}\int_0^{\infty}du\left(\frac{-s^2u^2+i2s(s^2-3)u+(s^4-4s^2+6)}{g_+^5}+\text{c.c.}\right), \tag{B1}$$

where g_+ is defined in Eq. (4.11). The last term is the complex conjugate of the first term. Using the method shown in Appendix D we obtain

$$\begin{aligned}J_{1b}^{(1)}&=2\int_{s_c}^{2}\frac{ds}{s^2}\,i\int_{\Gamma_3}dz\,\frac{s^2z^2-6sz+2s^2+6}{(z^2-4)^{5/2}}\\&=\int_{s_c}^{2}ds\left(\frac{-1}{s(4-s^2)^{1/2}}+\frac{2}{s(4-s^2)^{3/2}}\right)\\&=-\tfrac{1}{4}-\tfrac{1}{2}\ln 2+\tfrac{1}{4}\ln s_c+\lim_{s\to 2}1/2(4-s^2)^{1/2}.\end{aligned} \tag{B2}$$

The second term in Eq. (4.25) is given by

$$\begin{aligned}J_{1b}^{(2)}&=\int_{s_c}^{\infty}ds\int_0^{\infty}du\,F_2(s,u)\\&\quad-\frac{1}{2}\int_{s_c}^{\infty}\frac{ds}{s}\int_0^{\infty}du\,F_2(s,u)(g_++g_-)\\&\equiv I_A+I_B,\end{aligned} \tag{B3}$$

where I_A represents the first integral which is

$$I_A=-1/4s_c+\phi,$$

where ϕ is the divergent term defined by Eq. (4.31). It can be expressed in the form of the last term of Eq. (B2).

The second integral is

$$\begin{aligned}I_B&=-\frac{1}{2}\int_{s_c}^{\infty}\frac{ds}{s}\int_0^{\infty}du\,F_2(s,u)(g_++g_-)\\&=I_{B1}+I_{B2}.\end{aligned} \tag{B4}$$

The first term I_{B1} can be easily evaluated by using the method shown in Appendix D:

$$\begin{aligned}I_{B1}&=-\frac{1}{2}\int_{s_c}^{\infty}\frac{ds}{s}\frac{2}{s^3}\int_0^{\infty}du\left(\frac{-s^2u^2+2s(s^2-3)u+(s^4-4s^2+6)}{g_+^4}+\text{c.c.}\right)\\&=-\frac{1}{2}\int_{s_c}^{2}\frac{ds}{s}\frac{2i}{s^3}\int_{\Gamma_3}\frac{s^2z^2-6sz+2(s^2+3)}{(z^2-4)^2}=-\int_{s_c}^{2}\frac{ds}{s^4}\frac{\pi(3-s^2)}{8}=\frac{\pi}{8}\left(-\frac{11}{24}+\frac{1}{s_c}-\frac{1}{s_c^{\,3}}\right).\end{aligned} \tag{B5}$$

The second integral I_{B2} is more involved. We have calculated some new integrals appeared in I_{B2}. They are shown in Appendix E. The final results are

$$\begin{aligned}I_{B2}&=\tfrac{1}{8}\pi(-7/16s_c+1/s_c^3)\\J_{1b}^{(2)}&=\tfrac{1}{4}(9\pi/32-1)(1/s_c)-\tfrac{11}{192}\pi+\phi.\end{aligned} \tag{B6a}$$

I_{B2} can be written as follows:

$$\begin{aligned}I_{B2}&=-\frac{1}{2}\int_{s_c}^{\infty}\frac{ds}{s}\frac{2}{s^3}\int_0^{\infty}du\left(\frac{[-s^2u^2+i2s(s^2-3)u+(s^4-4s^2+6)]g_-^6}{g_+^5g_-^5}+\text{c.c.}\right)\\&=-\int_{s_c}^{\infty}\frac{ds}{s^4}\sum_{\nu=0}^{4}D_{2\nu}\int_0^{\infty}\frac{u^{2\nu}\,du}{\{[u^2+(s+2)^2][u^2+(s-2)^2]\}^{5/2}},\end{aligned} \tag{B6b}$$

where the coefficients $D_{2\nu}$ are:

$$\begin{aligned}D_0&=2(s^2-4)^3(s^4-4s^2+6),\\D_2&=4(s^2-4)(-2s^6-2s^4-5s^2+36),\\D_4&=-4(5s^6-36s^4+51s^2+36),\\D_6&=-4(2s^4+10s^2+3),\\D_8&=2s^2.\end{aligned} \tag{B7}$$

Using Table II, given in Appendix E, and in terms of the complete integrals of the first and second kinds, we obtain

$$I_{B2}=-\int_{s_c}^{\infty}\frac{ds}{s^4}\left(\frac{3-s^2}{4}\right)\left[\frac{1}{s-2}E\left(\frac{2\sqrt{k}}{1+k}\right)-\frac{1}{s+2}K\left(\frac{2\sqrt{k}}{1+k}\right)\right], \tag{B8}$$

where

$$k=s/2 \text{ or } 2/s\,. \tag{B9}$$

The argument of the elliptic integrals is invariant in the choice in Eq. (B9). Using k as an integral variable we obtain

$$I_{B2}=\frac{1}{8}\left[\int_{k_c}^{1}\frac{dk}{k^4}\left(\frac{3}{4}-k^2\right)\frac{E(k)}{1-k^2}-\int_0^1 dk\,k\left(\frac{3}{4}k^2-1\right)\left(\frac{E(k)}{1-k^2}-K(k)\right)\right]$$
$$=\frac{\pi}{8}\left(\frac{-7}{16s_c}-\frac{1}{s_c^3}\right). \tag{B10}$$

In the first integral of Eq. (B10) the first expression in Eq. (B9) was used and in the second integral the second expression, i.e., $2/s$, was adopted.

APPENDIX C

$$J_{1b}^{c}=\int_0^\infty du\,R_B(u)\int_0^{s_c}\frac{ds}{s+(e^2/p_F)R(u)}$$
$$=\int_0^\infty du\,R_B(u)\left[\ln\left(\frac{s_c p_F}{e^2}\right)+\ln\left(1+\frac{e^2}{s_c p_F}R(u)\right)-\ln R(u)\right]$$
$$=\ln\left(\frac{s_c p_F}{e^2}\right)\int_0^\infty du\,R_B(u)-\int_0^\infty du\,R_B(u)\ln R(u)+\frac{e^2}{s_c p_F}\int_0^\infty du\,R_B(u)R(u)+O\left(\left(\frac{e^2}{s_c p_F}\right)^2\right). \tag{C1}$$

These integrals can be easily evaluated in terms of a new variable θ defined by

$$u=2\tan\theta \tag{C2}$$

as follows:

$$J_{1b}^{c}=\tfrac{1}{96}\ln\left(\frac{e^2}{s_c p_F}\right)-\tfrac{7}{1536}\pi-\tfrac{11}{1440}$$
$$+\left(\tfrac{3}{1024}\pi-\tfrac{1}{96}\right)e^2/s_c p_F+O(e^4)+O(s_c)\,. \tag{C3}$$

APPENDIX D

The integrals (A2), (A3), (B1), and (B4) have the following form:

$$\int_0^\infty du\left(\frac{\mathfrak{F}(s+iu)}{g_+^{m/2}}+\frac{\mathfrak{F}(s-iu)}{g_-^{m/2}}\right)\equiv\mathfrak{I}\,. \tag{D1}$$

This integral can be simplified by introducing a new complex variable z:

$$s\pm iu=z\,, \tag{D2}$$

$$\mathfrak{I}=-i\int_{s+i\epsilon}^{s+i\infty}dz\,\frac{\mathfrak{F}(z)}{(z^2-4)^{m/2}}$$
$$+i\int_{s-i\epsilon}^{s-i\infty}dz\,\frac{\mathfrak{F}(z)}{(z^2-4)^{m/2}}\qquad(\epsilon\to 0^+)$$
$$=-i\int_{\Gamma_1+\Gamma_2}dz\,\frac{\mathfrak{F}(z)}{(z^2-4)^{m/2}}\,.$$

The contours Γ_1, Γ_2, Γ_3, and Γ_4 are shown in Fig. 6. Applying Cauchy's theorem we obtain

$$\mathfrak{I}=i\int_{\Gamma_3}dz\,\frac{\mathfrak{F}(z)}{(z^2-4)^{m/2}}\,,\quad 0<s<2 \tag{D3}$$

$$\mathfrak{I}=0\,,\quad 2<s\,.$$

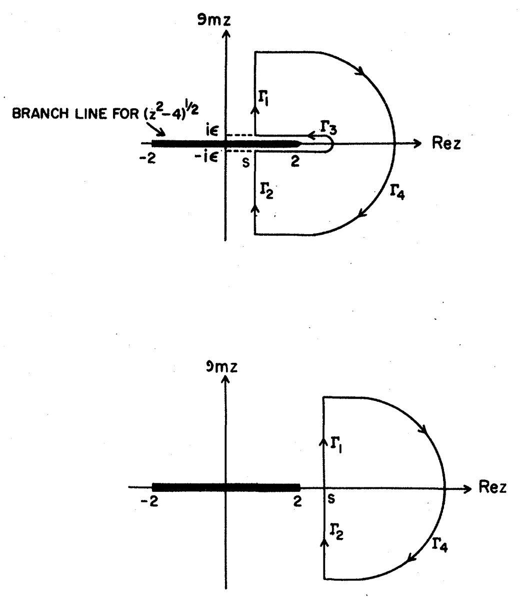

FIG. 6. Contour diagrams for the integrals in Appendix D.

TABLE II. The coefficients Π_{2n} and Λ_{2n} of the integral (*E*1).

n	Π_{2n}	Λ_{2n}
0	$\frac{2(\xi^6-5\xi^4\eta^2-5\xi^2\eta^4+\eta^6)}{3\xi^3\eta^4\Delta}$	$\frac{-(\xi^4-18\xi^2\eta^2+\eta^4)}{3\xi^3\eta^2\Delta}$
1	$\frac{\xi^4+14\xi^2\eta^2+\eta^4}{3\xi\eta^2\Delta}$	$\frac{-8(\xi^2+\eta^2)}{3\xi\Delta}$
2	$\frac{-8\xi(\xi^2+\eta^2)}{3\Delta}$	$\frac{3\xi^4+10\xi^2\eta^2+3\eta^4}{3\xi\Delta}$
3	$\frac{\xi(\xi^4+14\xi^2\eta^2+\eta^4)}{3\Delta}$	$\frac{-8\xi\eta^2(\xi^2+\eta^2)}{3\Delta}$
4	$\frac{2\xi(\xi^6+5\xi^4\eta^2-5\xi^2\eta^4+\eta^6)}{3\Delta}$	$\frac{-\xi\eta^2(\xi^4-18\xi^2\eta^2+\eta^4)}{3\Delta}$
5	$\frac{-\xi(8\xi^8-25\xi^6\eta^2+18\xi^4\eta^4-25\xi^2\eta^6+8\eta^8)}{3\Delta}$	$\frac{4\xi\eta^2(\xi^2+\eta^2)(\xi^4-4\xi^2\eta^2+\eta^4)}{3\Delta}$

APPENDIX E

We have evaluated the following integrals which appeared in I_{B2}.

$\xi>\eta>0$,

$$\int_0^\infty \frac{u^{2n}du}{[(u^2+\xi^2)(u^2+\eta^2)]^{5/2}}=\Pi_{2n}E(\theta)+\Lambda_{2n}K(\theta)+\delta\,, \tag{E1}$$

where K and E are complete elliptic integrals of the first and second kinds, respectively. θ and δ are defined as

$$\theta=(\xi^2-\eta^2)^{1/2}/\xi\,, \tag{E2}$$

$$\delta=\begin{cases}0, & n=0,1,2,3,4\\ \lim\limits_{u\to\infty} u\left(\dfrac{u^2+\xi^2}{u^2+\eta^2}\right)^{1/2}, & n=5\,.\end{cases} \tag{E3}$$

The coefficients Π_{2n} and Λ_{2n} are given in Table II in which we used

$$\Delta=(\xi^2-\eta^2)^4\,. \tag{E4}$$

APPENDIX F

In this Appendix, we give exact analytical expressions for the relevant physical quantities. These analytical expressions will help to confirm and extend our calculations.

$$\ln\Xi^{(a)}=A\beta a^4\left(B_1+B_2\frac{e^2}{p_F}+B_3\frac{e^2}{p_F}\ln\frac{e^2}{4p_F}+B_4\frac{e^4}{p_F^2}\right),$$

$$B_1=\frac{1}{12\pi}\left(\frac{3}{4}g^2-1\right),$$

$$B_2=\frac{1}{12\pi^2}\left(\frac{3}{4}g^2-1\right)+\frac{1}{48\pi^2}\left(\frac{26}{15}+\frac{7\pi}{16}\right)$$

$$=6.333\times10^{-3}g^2-1.883\times10^{-3}\,,$$

$$B_3=-1/48\pi^2=-2.1109\times10^{-3}\,,$$

$$B_4=-\frac{1}{96\pi}\left(\frac{3}{4}g^2-1\right)+\frac{11}{2304\pi}$$

$$=-2.487\times10^{-3}g^2+4.835\times10^{-3}\,,$$

$$p_F^{(a)}=p_0(a^4/p_0^4)(D_1r_s+D_2r_s\ln r_s+D_3r_s^2+D_4r_s^2\ln r_s)\,,$$

$$D_1=(\pi/2^{1/2})[B_2+(1-\tfrac{3}{2}\ln 2)B_3]$$

$$=1.407\times10^{-2}g^2-3.997\times10^{-3}\,,$$

$$D_2=(\pi/2^{1/2})B_3=-4.689\times10^{-3}\,,$$

$$D_3=(4-\tfrac{9}{2}\ln 2)B_3+3B_2+2\pi B_4$$

$$=3.372\times10^{-3}g^2+2.287\times10^{-2}\,,$$

$$D_4=3B_3=-6.333\times10^{-3}\,,$$

$$\chi=(2e^2/c^2)(M_0+M_1r_s+M_2r_s\ln r_s+M_3r_s^2+M_4r_s^2\ln r_s)\,,$$

$$M_0=B_1=(1/12\pi)(\tfrac{3}{4}g^2-1)\,,$$

$$M_1=2^{1/2}(B_2-(3\ln 2/2)B_3)$$

$$=8.956\times10^{-3}g^2+4.409\times10^{-4}\,,$$

$$M_2=2^{1/2}B_3=-2.985\times10^{-3}\,,$$

$$M_3 = (2/\pi)[B_2 + (1 - \tfrac{3}{2}\ln 2)B_3 + \pi B_4]$$
$$= -9.425\times 10^{-4} g^2 + 8.525\times 10^{-3},$$
$$M_4 = (2/\pi)B_3 = -1.3438\times 10^{-3},$$
$$\epsilon^{(a)} = (a^4/p_0^4)(W_0 + W_1 r_s + W_2 r_s \ln r_s + W_3 r_s^2 + W_4 r_s^2 \ln r_s),$$
$$W_0 = -4\pi B_1 = -\tfrac{4}{3}(\tfrac{3}{4}g^2 - 1),$$
$$W_1 = 4\sqrt{2}\,\pi[-B_2 + (3\ln 2/2)B_3]$$
$$= -0.1125 g^2 - 5.540\times 10^{-3},$$
$$W_2 = -4\sqrt{2}\,\pi B_3 = 0.03751,$$
$$W_3 = -8[B_2 + (1 - \tfrac{3}{2}\ln 2)B_3 + \pi B_4]$$
$$= 1.184\times 10^{-2} g^2 - 0.1071,$$
$$W_4 = -8B_3 = 1.689\times 10^{-2}.$$

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