Density fluctuation spectrum of superfluid ^{4}He at finite temperatures

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(Received 31 October 1978)

Very recently, Woods and Svensson have reported neutron scattering experiments which indicate that the temperature-dependent weight of the elementary excitation resonance in superfluid ^{4}He is directly proportional to the superfluid density $\rho_s(T)$. We argue that this result can be easily understood within the general framework of the standard field-theoretic formulation of a Bose-condensed system and that it gives strong evidence in favor of the regularity hypothesis introduced by Hohenberg and Martin.

Recently Woods and Svensson have reported[1] a careful set of measurements of $S(Q,\omega)$ for ^{4}He as a function of the temperature. They found that their results (from 1.00 to 2.27 K) could be well described by[2]

$$S(Q,\omega)=\frac{\rho_s(T)}{\rho}S_s(Q,\omega)+\frac{\rho_n(T)}{\rho}S_n(Q,\omega) . \quad (1)$$

Here $\rho_s(T)$ is the macroscopic superfluid density which enters in low-frequency long-wavelength phenomena described by two-fluid hydrodynamics [while $\rho_n(T)=\rho-\rho_s(T)$ is the macroscopic normal-fluid density]. Perhaps the simplest way of describing the decomposition in Eq. (1) is that $S_s(Q,\omega)$ is what $S(Q,\omega)$ reduces to for $T \ll T_\lambda$, while $S_n(Q,\omega)$ is what one obtains for $T > T_\lambda$. The "superfluid" part $S_s(Q,\omega)$ is strongly peaked at the Landau quasiparticle energy $\omega(Q)$ and thus the Woods-Svensson results mean that the weight of phonon-roton excitation is directly proportional to the superfluid density. For the values of ω and Q studied ($0.80\ \text{Å}^{-1} \leq Q \leq 1.93\ \text{Å}^{-1}$), it is well known[3] that such experiments are probing the collisionless region and not the hydrodynamic region. As a result, it might seem surprising[1] that the strength of the quasiparticle resonance in $S(Q,\omega)$ is proportional to $\rho_s(T)$.

In the present article, we show that the main results of Ref. 1 can be naturally explained as a direct consequence of the assumption that superfluid ^{4}He has a Bose condensate. More particularly, we point out that the general field-theoretic expression[4–6] for $S(Q,\omega)$ for a Bose-condensed system naturally separates into a "singular" and a "regular" part (to be defined shortly). If we then make the regularity hypothesis of Hohenberg and Martin[6] (namely, that the vertex functions and self-energies are nonsingular functions of Q and ω in the collisionless regime), then it immediately follows that: (a) The regular part of $S(Q,\omega)$ is a broad structure and can be identified with what Woods and Svensson call the "normal-fluid" part. (b) The singular part of $S(Q,\omega)$ exhibits a single sharp resonance with a weight proportional to $\rho_s(T)$ and can be identified with the "superfluid" part.

In fact, point (b) is already implicit in the paper by Hohenberg and Martin[6] as well as in some unpublished work by Pines and Nozières,[7] although no particular attention was called to it by these authors. In the present article, we critically review the key points of the Hohenberg-Martin analysis (largely unknown) and apply it specifically to understanding the Woods-Svensson results.

We also recall that one has a completely different structure for $S(Q,\omega)$ in the hydrodynamic region at *finite* temperatures since it is known[8] that Dyson-Beliaev self-energies have singularities related to the existence of second sound (and thus the regularity hypothesis is not valid). As a result, the first-sound resonance in $S(Q,\omega)$ has a weight which is essentially temperature independent [i.e., proportional to ρ, instead of $\rho_s(T)$]. Thus, a consequence of our analysis is that an expression with a form such as Eq. (1) will only be valid in the collisionless regime (say $Q \gtrsim 0.2\ \text{Å}^{-1}$).

A diagrammatic analysis[5,6] of the density-density correlation function χ_{nn} shows that all contributions can be divided into two categories, depending on whether they contain the single-particle Green's function $\tilde{G}_{\alpha\beta}(Q,\omega)$ as a separate factor or not. The contributions to $\chi_{nn}(Q,\omega)$ which do (proper parts) give the so-called "singular" or condensate part. The improper parts constitute the "regular" or background part. Thus we have

$$\chi_{nn}(Q,\omega)=\chi_{nn}^C(Q,\omega)+\chi_{nn}^B(Q,\omega) , \quad (2)$$

with

$$\chi_{nn}^C(Q,\omega)=\sum_{\alpha,\beta}\Lambda_\alpha(Q,\omega)\,\tilde{G}_{\alpha\beta}(Q,\omega)\,\Lambda^\beta(Q,\omega) , \quad (3)$$

and

$$\chi_{nn}^{B}(Q,\omega)=\frac{\chi_0^B(Q,\omega)}{1-V(Q)\chi_0^B(Q,\omega)}\ , \tag{4}$$

where $\chi_0^B(Q,\omega)$ is the irreducible density-density correlation function and $\Lambda_\alpha^{(\beta)}(Q,\omega)$ are certain vertex functions which arise from the presence of a condensate ($n_0 \neq 0$). The formal structure of these results is made more transparent by systematically decomposing all functions into irreducible (labeled by 0) and reducible parts, depending on whether diagrams can be split into two by cutting a single interaction line. In particular, one finds that[9–11] the single-particle self-energy is given by

$$\Sigma_{\alpha\beta}(Q,\omega)=\Sigma^0_{\alpha\beta}(Q,\omega)+V(Q)\frac{\Lambda^{0\alpha}(Q,\omega)\Lambda^0_\beta(Q,\omega)}{\epsilon_R(Q,\omega)}\ , \tag{5}$$

where

$$\epsilon_R(Q,\omega)\equiv 1-V(Q)\chi_0^B(Q,\omega)$$

and $V(Q)$ is the Fourier transform of the ^{4}He interatomic potential.

Physically, the distinction between χ_{nn} and χ_{nn}^B is quite simple. χ_{nn} involves density fluctuations of the whole system,

$$\hat{n}_Q=\sum_p \hat{a}_p^\dagger \hat{a}_{p+Q}\ , \tag{6}$$

while χ_{nn}^B gives the density fluctuation spectrum of the noncondensed atoms (i.e., atoms with finite momentum)

$$\tilde{n}_Q=\sum_{p\neq 0,-Q} \hat{a}_p^\dagger \hat{a}_{p+Q}\ . \tag{7}$$

Thus we have $\chi_{nn}^B=\chi_{\tilde{n}\tilde{n}}$. Clearly, if there is no condensate ($T>T_\lambda$), χ_{nn}^C vanishes and the full density response function is given by Eq. (4). The additional presence of a term in χ_{nn}, which is directly proportional to the single-particle Green's function, is entirely due to the existence of a condensate and makes the dynamics of superfluid ^{4}He quite different from normal ^{4}He. χ_{nn}^C describes density fluctuations which involve adding or removing atoms to and from the condensate.

We emphasize that the results summarized by Eqs. (2)–(5) are exact and apply to both the hydrodynamic and collisionless regions at all temperatures. At the present time, we know $\tilde{G}_{\alpha\beta}$ and χ_{nn} in the hydrodynamic region from the calculations of Hohenberg and Martin[6,8] but microscopic calculations for the collisionless region have only been carried out at $T=0$ °K in the long-wavelength limit.[5,11] Our procedure is to accept the regularity hypothesis of Hohenberg and Martin, namely, that the self-energies and vertex functions such as $\Lambda_\alpha^{(\beta)}$ are smooth functions of Q and ω in the collisionless domain. The reason for this assumption[6] is that if the self-energies were singular, then $\tilde{G}_{\alpha\beta}(Q,\omega)$ and hence $\chi_{nn}^C(Q,\omega)$ would exhibit *two* resonances (as in the hydrodynamic region), rather than *one* as observed in inelastic neutron scattering.[1,3] We note that any sharp resonance exhibited by $\chi_{nn}^B(Q,\omega)$ in Eq. (4) as a result of the vanishing of the denominator $\epsilon_R(Q,\omega)$ would also give rise to a singular self-energy [see Eq. (5)]. From the Chalk River results,[1] however, we see that $\chi_{nn}^B(Q,\omega)$ only exhibits a very broad structure. This is what we expect, of course, if the regularity hypothesis is valid. In the hydrodynamic domain, the two terms in Eq. (2) do not give rise to distinguishable parts of χ_{nn} and hence (2) is not useful. In this region, χ_{nn}^C and χ_{nn}^B *both* exhibit first and second sound poles.[8]

Momentum current-current correlation functions also may be split into singular and regular parts in a manner completely analogous to Eq. (2). We refer to Refs. 5 and 11 for a systematic diagrammatic analysis for all three correlation functions χ_{nn}, χ_{n,g_i}, and χ_{g_i,g_j}. All these functions have the same poles and moreover the regularity hypothesis must be valid for all or none. Our procedure is to first find the strength of the resonance in the longitudinal part of the momentum current-current correlation function, making use of the fact that certain exact limiting values of $\chi_{gg}^l(Q,\omega)$ are known. Then we use the equation of continuity given by

$$\chi_{nn}(Q,\omega)=\frac{Q^2}{m^2\omega^2}[\chi_{gg}^l(Q,\omega)+\rho] \tag{8}$$

to find the resonance in χ_{nn}^C. As discussed in Ref. 6, the condensate part of χ_{g_i,g_j} can be related to the superfluid velocity-velocity correlation function, which in turn is directly proportional to $\tilde{G}_{\alpha\beta}$. One finds[12] the longitudinal part of the condensate part of χ_{g_i,g_j} is [compare with Eq. (3)]

$$\chi_{gg}^{Cl}(Q,\omega)=F_C^l(Q,\omega)\left[\frac{Q^2}{n_0 m^2}\mathcal{G}(Q,\omega)\right]F_C^l(Q,\omega)\ , \tag{9}$$

where

$$\mathcal{G}(Q,\omega)=\tfrac{1}{2}[\tilde{G}_{11}(Q,\omega)-\tilde{G}_{12}(Q,\omega)]$$

and $F_C^l(Q,\omega)$ is a certain vertex function which we must determine. We proceed as follows.

As discussed at length in the literature[6,7,13] one of the most fundamental characterizations of the superfluid state of ^{4}He is that the condensate does not contribute to the transverse current response function, i.e.,

$$\lim_{Q\to 0}\chi_{gg}^t(Q,\omega=0)=-\rho_n\ . \tag{10}$$

In contrast, Eq. (8) gives

$$\chi_{gg}^l(Q,\omega=0)=-\rho=-(\rho_s+\rho_n)\ . \tag{11}$$

In terms of the field-theoretic decomposition into singular and regular parts discussed above, we have[6]

$$\lim_{Q\to 0} \chi_{gg}^{Bl}(Q,\omega=0) = -\rho_n \ , \qquad (12)$$

$$\lim_{Q\to 0} \chi_{gg}^{Cl}(Q,\omega=0) = 0 \ . \qquad (13)$$

Moreover, since the nonsuperfluid should not distinguish between a transverse and longitudinal response to a static, uniform vector potential, Eq. (12) implies that

$$\lim_{Q\to 0} \chi_{gg}^{Bl}(Q,\omega=0) = -\rho_n \ . \qquad (14)$$

Finally, combining Eq. (14) with Eq. (11), we have

$$\lim_{Q\to 0} \chi_{gg}^{Cl}(Q,\omega=0) = -\rho_s \ . \qquad (15)$$

Combining Eq. (15) with another exact result,[6,13] the so-called "Josephson sum rule"

$$\lim_{Q\to 0} \mathcal{G}(Q,\omega=0) = -\frac{n_0 m^2}{\rho_s Q^2} \ , \qquad (16)$$

it follows from Eq. (9) that

$$\lim_{Q\to 0} F_C^l(Q,\omega=0) = \rho_s(T) \ . \qquad (17)$$

We defer discussion of how small Q must be for Eq. (17) to be valid.

Using the regularity hypothesis, we next argue[5,6] that in the collisionless long-wavelength region, the Dyson-Beliaev self-energies in $\mathcal{G}(Q,\omega)$ may be expanded around their $Q=0, \omega=0$ values. Making use of Eq. (16), we thus conclude that the singular part of $\mathcal{G}(Q,\omega)$ *must* have the form

$$\mathcal{G}(Q,\omega) = \frac{n_0 m^2}{\rho_s} \frac{v^2}{\omega^2 - v^2 Q^2} \qquad (18)$$

in the phonon region.[14] Combining this with the assumption that, in the collisionless region, the vertex functions $F_C^l(Q,\omega)$ are smooth functions and hence can be approximated by Eq. (17), we see that the *resonant* part of Eq. (9) is given by

$$\chi_{gg}^{Cl}(Q,\omega) = \frac{\rho_s v^2 Q^2}{\omega^2 - v^2 Q^2} \ . \qquad (19)$$

Taking the imaginary part of the expression in Eq. (19), it immediately follows from Eq. (8) that the *resonant* part of $S_c(Q,\omega)$ is given by

$$S_C(Q,\omega) = \frac{\rho_s}{m} \frac{Q}{2mv} [\delta(\omega - vQ) - \delta(\omega + vQ)] \ . \qquad (20)$$

Thus we have derived an expression in agreement with the experimental results of Ref. 1. This agreement is yet another piece of evidence for the correctness of the regularity hypothesis in the collisionless region.

We see that the quasiparticle resonance has a weight proportional to ρ_s, even though we are not in the hydrodynamic two-fluid regime. Deviations from this simple result can be expected at values of Q such that Eq. (17) is no longer a good approximation. From general considerations discussed at length by Pines and Nozières[7,15] the zero-frequency response functions in Eqs. (10)–(15) are expected to become Q dependent when $Q\xi \gtrsim 1$, where ξ is the Bose coherence distance. Since $\xi \simeq$ interatomic spacing, Pines and Nozières have argued that Eq. (10) [and hence Eq. (17)] should be valid for Q up to about the roton wave-vector $Q_C \sim 2$ Å^{-1}, which in fact was the maximum wave vector studied by Woods and Svensson.[1] This predicted breakdown at larger Q values should be looked for experimentally.

It is now well known[1,3] that at low temperatures ($\lesssim 1\,°$K), superfluid ^{4}He exhibits both a quasiparticle resonance at $\omega(Q)$ plus a much broader "multiphonon" structure at higher frequencies, or order $2\omega(Q)$. We interpret this multiphonon structure as part of $\chi_{nn}^C(Q,\omega)$, arising from the nonsingular part of $\tilde{G}_{\alpha\beta}$ in Eq. (3). Thus we have

$$\chi_{nn}^C(Q,\omega) = \chi_{nn}^{\mathrm{I}}(Q,\omega) + \chi_{nn}^{\mathrm{II}}(Q,\omega) \ , \qquad (21)$$

where χ_{nn}^{I} is the resonant part which we have calculated above in Eq. (20). It should be kept in mind that the multiphonon part χ_{nn}^{II} is quite different from χ_{nn}^B, the latter being associated with a structure[1] which is concentrated at lower frequencies. χ_{nn}^B describes the density fluctuations in the excited atoms ($\vec{p} \neq 0$) and is present both above and below T_λ. In contrast, χ_{nn}^{II} is intimately tied in with the existence of a condensate. It ceases to exist (as does χ_{nn}^{I}) for $T > T_\lambda$. At the present time, we have no good microscopic calculations of either χ_{nn}^{I} or χ_{nn}^B at finite temperatures. Whether χ_{nn}^B can indeed be approximately by the second term given in Eq. (1) can only be answered by microscopic calculations based on Eq. (4).

As we mentioned in the Introduction, Hohenberg and Martin[6] as well as Pines and Nozières[7,15] were the first to emphasize that, in the collisionless region, the condensate and background parts in Eq. (2) might be expected to give rise to distinguishable parts of the inelastic neutron scattering cross section. In fact, Pines and Nozières did not give a clear microscopic definition of the two response functions but their physical description clearly corresponds to Eqs. (3) and (4) [or Eq. (6.38) of Ref. 6]. These early papers concentrated on the temperature dependence of the quasiparticle energy,[14] rather than on its strength as we have done.[16] Moreover, little discussion of $\chi^B_{nn}(Q, \omega)$ was given. On the other hand, we note that Pines[15,17] did conjecture that superfluid ^{4}He at finite temperatures might exhibit a quasiparticle resonance distinct from a broad zero-sound mode, in essential agreement with the experimental results of Woods and Svensson.[1]

ACKNOWLEDGMENTS

I would like to thank Dr. A. D. B. Woods for a copy of Ref. 1 prior to publication, and for encouraging me to think about the results. I have benefitted from discussions with Professor A. E. Jacobs. This work was supported by a grant from NSERC of Canada.

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[13] A nice summary is given by G. Baym, in *Mathematical Methods in Solid State and Superfluid Theory*, edited by R. C. Clark and G. H. Derrick (Oliver and Boyd, Edinburgh, 1969), p. 121.

[14] At $T=0$, one may show (Refs. 5–7) rigorously that $v^2 = dP/d\rho$. At the present time, one does not have a good theory of the phonon speed v as a function of the temperature. The compressibility sum rule argument given in Appendix C of Ref. 6 does not appear to be correct. Experimentally, $v(T)$ is remarkably temperature independent up to T_λ (Ref. 3).

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[16] However, a result equivalent to Eq. (20) is given in Eq. (C.6) of Ref. 6 and Eq. (6.248) of Ref. 7.

[17] D. Pines, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland, Amsterdam, 1966), p. 257 and p. 338.