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Phase transitions in treelike percolation

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The phase transition in treelike percolation occurs for a bond density slightly less than the transition in the usual unrestricted-bond-percolation problem.

I. INTRODUCTION

Treelike percolation is a problem concerning the statistics of bonds on a lattice which are constrained to form only clusters which are trees (i.e., no closed loops). This problem was originally proposed by Stephen,[1] who suggested the occurence of a phase transition in such systems, namely, that there is a concentration of bonds such that the infinite lattice with concentration of bonds exceeding this critical value will contain an infinite cluster (with probability unity). Wu[2] recently discussed this problem and argued that there can be no phase transition in two dimensions, using an argument based on the dual transformation. This paper will show that this result is incorrect.

Stephen and Wu agree that the interesting properties of treelike percolation lie in the function

$$Z = \sum_T x^{e(T)} \quad , \tag{1}$$

where the sum extends over the tree graphs, $e(T)$ is the number of bonds in graph T, x = p/(1-p), and p is the fraction of lattice bonds present. This may be regarded as the partition function for a special case of the Potts model[1] or as the y = 0 limit of the Whitney polynomial[2]

$$W = \sum_G x^{e(G)-c(G)} y^{c(G)} \quad , \tag{2}$$

where {G} is the unrestricted set of graphs, and c(G) is the number of independent cycles in a graph G.

These latter models have a dual transformation, which gives an equivalence between graphs on a lattice and its dual. The dual to a square lattice is the lattice made by the centers of the squares; adjacent dual sites are dual bonded if and only if the bond that runs along the shared square edge is absent. Since the dual lattice is also a square lattice, this is a mapping of the problem into itself, whereby a concentration p of bonds is related to a concentration 1-p of (dual) bonds.

Wu's argument depends on this (rigorous) dual transformation and the assumption (which he carefully identifies as such) that the criticality of a lattice statistical model depends continuously upon its parameters. This assumption fails in the present case, since the treelike percolation problem does not have the dual transformation properties of the general Potts model: it cannot be self-dual, since trees close pack at $p = \frac{1}{2}$. The dual to a tree is a graph which has no finite separated component; the dual transformation relates the function given in Eq. (1) to a similar function in which the sum is over this latter set of graphs. Since the two problems are not otherwise related, this does not locate (or forbid) a phase transition.

The treelike percolation problem (y = 0) can have a phase transition which is not shared by the general Potts model (finite y), just as the Ising model has a phase transition only in zero field.

The question of the existence of a phase transition in the treelike percolation problem is thus reopened. I offer below nonrigorous arguments and computer simulations which indicate that trees do percolate, at a bond density only slightly less than for the usual unrestricted bond percolation problem.

II. NONRIGOROUS ARGUMENTS

Given any tree graph, we may classify the bonds of the lattice into three groups: (i) bonds that belong to the tree, (ii) bonds that are committed by the tree, and (iii) the remaining ("uncommitted") bonds of the lattice. Here committed bonds are those bonds whose addition to the graph would form a closed loop, and which are therefore committed to be absent. Since a lattice with no bonds present is a trivial tree graph, we might attempt to construct a "typical" tree graph from it by an inductive process: given a tree graph of R bonds, choose at random a bond from the uncommitted set and add it to the graph, thus forming a tree graph of R+1 bonds.

If an isolated lattice site is regarded as a one-site tree, then the effect of adding a bond to a tree graph as just described always joins two trees. All bonds that were committed remain so, and some of the formerly uncommited bonds become committed (if there was more than one way to join the two trees); thus the fraction of committed bonds increases with the fraction of bonds present. The concentration of bonds required to close-pack a tree is the same for all configurations; it is generally 2/z, where z is the number of neighbors of a lattice point.

We will say that a graph percolates if it contains an infinitely large tree. We may estimate the fraction of bonds which must be present for this to occur by comparing treelike percolation to ordinary (unrestricted graph) percolation. This latter problem percolates on the square lattice when half of the bonds are present. Let us imagine constructing a cluster by sequentially adding bonds to the lattice; but we will draw in the bonds with two colors of ink, using black if this is an uncommitted bond of the black tree, and red if it is a committed bond of the black tree. When we have colored in half the bonds of the lattice, it will contain an infinite cluster, with probability one. This cluster contains a black tree which is also infinite; indeed, it contains all the sites of the cluster.

The density of black bonds is less than the density of colored bonds, because the density of red bonds is finite. In estimating the latter it must be noted that the set of red bonds of the cluster is not the entire set of committed bonds of the black tree— it is only the subset that has been chosen in the process of building the tree. But since the density of committed bonds has been finite throughout the building process, there has been a finite chance of adding a red bond at every step. The conclusion is that it is always possible to remove a finite density of bonds from a percolating cluster and retrieve a percolating tree. Trees percolate at a density below close packing on the square quadratic lattice.

The problem is much simpler on the honeycomb and higher-dimensional hypercubic lattices, since for all of these the percolation density for the unrestricted problem is below the close-packing density of the tree problem. It is possible that trees do not percolate on the triangular lattice, except for the trivial transition at tree close packing (p = 0.3333), since the unrestricted bond problem percolates only at p = 0.3473.

III. NUMERICAL SIMULATION

This problem was studied numerically by computer implementation of the red-and-black coloring algorithm described above. Bonds were sequentially added to a finite lattice, and were classified black if they joined two clusters together, or red if they joined sites already belonging to the same cluster. Of course, the lattice should percolate when the total density of bonds added (of either color) is 50% (two dimensions) or 24.7% (three dimensions); and indeed it was observed that the size of the largest cluster on the lattice was growing very rapidly near those concentrations. Several runs were made for 90 x 90 and 30 x 30 x 30 lattices using different random sequences. The following examples will typify how the total bond density partitioned itself into black (b) and red (r) bonds: two dimensions,

$$30\% = 29.5\%\, b + 0.5\%\, r \ ,$$

$$40\% = 38.3\%\, b + 1.7\%\, r \ ,$$

$$48.9\% = 44.5\%\, b + 4.4\%\, r \ ,$$

three dimensions,

$$10\% = 9.98\%\, b + 0.02\%\, r \ ,$$

$$20\% = 19.58\%\, b + 0.42\%\, r \ ,$$

$$25\% = 23.9\%\, b + 1.1\%\, r \ .$$

These numbers were quite reproducible from run to run. The fact that the fraction of red bonds is finite (albeit small) and reproducible strongly suggests that the treelike percolation problem has a transition at $p = 45\%$ (2D) or 24% (3D).

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[1]M. J. Stephen, Phys. Lett. A. 56, 149 (1976).

[2]F. Y. Wu, Phys. Rev. B 18, 516 (1978).