Alternative approach to the dynamic renormalization group

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The dynamics (time-dependent) renormalization group is reformulated. Probability distributions for configurations at a discrete time sequence are parametrized in a manner similar to the parametrization of static probability distributions. The spirit of this approach is that of phenomenological fitting without making reference to any differential equations for microscopic dynamics. Simple illustrative calculations using the new formulation are presented.

I. INTRODUCTION

The purpose of this article is to provide an alternative formulation of the dynamic renormalization group (RG).[1] Let us briefly review a few concepts. Under a change of length scale by a factor s and time scale by a factor s^z,

$$x \to xs\ , \qquad t \to ts^z\ , \tag{1.1}$$

the values of parameters specifying a physical system are transformed. Such a transformation is a "scale transformation" or a "dynamic RG transformation." When the parameters assume certain particular values, and the value of z is appropriately chosen, such that they do not change under the transformation, then the system is "scale invariant" or "at a fixed point of the dynamic RG." The exponent z is called the "dynamic exponent." It measures the asymmetry between space and time. If $z=1$, then space and time are symmetric. Electrodynamics in the absence of matter is scale invariant with $z=1$, for example. The scale invariance of interest here appears in material systems at their critical points. For a material medium, there is in general no symmetry between space and time. The value of z is therefore generally not 1, and depends on some gross features (such as conservation laws and various symmetry properties) of the material. The dynamic exponent will be of major concern in this article. Of course, it is not the only thing of interest in the dynamics of systems at their critical points. The dynamic RG has applications other than the determination of z.

Let us mention briefly the various approaches to carrying out the dynamic RG. The perturbation theory approach has been successful for models in special space dimensions $d=6-\epsilon$, $4-\epsilon$, $2+\epsilon$, where ϵ is regarded as a small expansion parameter.[1,2] The calculations have been done analytically. The major drawback of this approach is its limitation on d. The values of interest for d are 3 and 2.

For $d=3$ or 2, one has to carry out the dynamic RG numerically. There is the Monte Carlo approach,[3] which carries out the dynamic RG by direct numerical simulation and measurement. Recently there has been more extensive numerical work, without Monte Carlo simulation, by cluster-expansion techniques.[4,5] All the numerical approaches so far have been done on the kinetic Ising model on a discrete lattice, and limited mostly to $d=2$.

The formulation presented here aims at numerical calculations. So we shall say no more about the perturbation-theory approach, but mention some technical points in the numerical approaches. The technical difficulties in formulating and carrying out the dynamic RG include those in the static RG plus many more. The static RG deals with the equilibrium probability distribution expressed through an effective Hamiltonian. The parameters specifying the effective Hamiltonian are transformed. The main difficulties are the following. First, to define the RG transformation completely, one has to write the effective Hamiltonian in the most general form with an infinite number of parameters. In practice, parametrization with only a few parameters is desired, i.e., one needs to truncate the effective Hamiltonian, hopefully in such a way that important results are not affected seriously. Second, to carry out the RG transformation, further truncations are needed so that the calculation would involve only a finite number of spins, preferably not too many spins. These truncations are difficult fitting programs—fitting complicated numerical data with a small number of parameters. These difficulties have not been resolved in a satisfactory way, but there has been considerable progress and experience gained. For example, it seems evident through experience that the nearest-neighbor interaction (imagine an Ising

model) is more important than the next-next-neighbor interaction, and a small number of parameters specifying the shortest-range interactions seem to provide at least a good qualitative representation of the scale-invariant effective Hamiltonian.

Statics is only a part of dynamics. Parametrization in dynamics must specify the law of time evolution in addition to statics. Most often in the literature, differential equations of motion are used as the basis for defining the dynamic RG. Differential equations are the most natural starting point, because almost all dynamic problems in physics have been formulated in terms of differential equations. However, calculations so far have indicated that relatively simple differential equations cannot provide even a qualitatively accurate representation of the scale-invariant dynamics. In other words, the fixed point of the dynamic RG does not appear to be a simple differential equation of motion.

Therefore, at this stage of development, it seems that an alternative approach to the dynamic RG would be valuable. The approach formulated in this article does not involve differential equations of motion. It is designed to make the parametrization and fitting program straightforward and mechanical. It is a purely phenomenological approach. The numerical calculation involved would be similar to that in statics. It might make the dynamics in d dimension look like the statics in $d+1$ dimension with time as the additional dimension. However, there are important details which make time very different from space.

The applications of the new formulation presented here are to very simple and crude calculations. They are only to illustrate the basic mechanics of how the new formulation works. More elaborate calculations aiming at precise results have not been completed.

The outline of this article is as follows. Section II provides the basic formalism defining quantities through which various probability distributions of spin variables at discrete time t are specified. Section III discusses the parametrization of these quantities and the important normalization conditions they must satisfy. Connections to well-known kinetic Ising models and the time-dependent Ginzburg-Landau models are pointed out. Transformations of the parameters under the change of time scale are discussed in Sec. IV, and under the space scale in Sec. V. They are illustrated with trivial models. The combined time-space scale transformations give the dynamic RG. In Sec. VI, applications are made to the dynamics of a one-dimensional Ising model and a $(1+\epsilon)$-dimensional model. The latter is of a speculative nature. Section VII gives a crude but slightly nontrivial calculation of the dynamic exponent for a dynamic two-dimensional Ising model. This calculation is to serve as a prototype of a more elaborate computing program.

The discussion in this paper is limited to purely dissipative models. Mode-mode couplings, conserved energy densities, and other complications are not discussed. The formalism developed here is sufficiently general to accommodate these complications, however. It may also be used as a basis for the determination of dynamic models over intermediate space-time scales.

II. BASIC FORMALISM

Consider a system of spins $[\sigma]=\{\sigma(x),\ x$ are lattice sites$\}$. The configuration at time t will be denoted by σ_t. Let the probability distribution of spin configurations be

$$P[\sigma]=e^{H[\sigma]}, \tag{2.1}$$

which is assumed to be the same for all times, i.e., we have s system in equilibrium. H is the Hamiltonian divided by $-kT$. An additive constant is included in H so that $P[\sigma]$ is normalized.

Consider the spin configurations σ_t at times $t=1,2,3,\ldots$, measured in units of Δt, chosen arbitrarily. Let the joint distribution of σ_1 and σ_2 be

$$P[\sigma_1,\sigma_2]=\exp(H[\sigma_1]+H[\sigma_2]+A[\sigma_1,\sigma_2]), \tag{2.2}$$

which defines $A[\sigma_1,\sigma_2]$. The joint distribution for σ_1, σ_2, and σ_3 is

$$\begin{aligned}P[\sigma_1,\sigma_2,\sigma_3]=\exp(&H[\sigma_1]+H[\sigma_2]+H[\sigma_3]\\&+A[\sigma_1,\sigma_2]+A[\sigma_2,\sigma_3]\\&+A_3[\sigma_1,\sigma_2,\sigma_3]),\end{aligned} \tag{2.3}$$

which defines $A_3[\sigma_1,\sigma_2,\sigma_3]$. Similarly one can write down joint distributions for spins at more times and define $A_4[\sigma_1,\sigma_2,\sigma_3,\sigma_4],\ldots,$ $A_m[\sigma_1,\ldots,\sigma_m],\ldots$.

We define the transition probability from σ_2 to σ_1 as the conditional probability of finding σ_1 given σ_2:

$$\begin{aligned}W[\sigma_1|\sigma_2]&=P[\sigma_1,\sigma_2]/P[\sigma_2]\\&=\exp(H[\sigma_1]+A[\sigma_1,\sigma_2]).\end{aligned} \tag{2.4}$$

$W[\sigma_1|\sigma_2]$ must be normalized, i.e.,

$$\begin{aligned}\sum_{\sigma_1} W[\sigma_1|\sigma_2]&=\sum_{\sigma_1}\exp(H[\sigma_1]+A[\sigma_1,\sigma_2])\\&=1.\end{aligned} \tag{2.5}$$

This poses a restriction on A. Assuming time reversal symmetry, we must have $P[\sigma_1,\sigma_2]=P[\sigma_2,\sigma_1]$, or

$$A[\sigma_1, \sigma_2] = A[\sigma_2, \sigma_1], \tag{2.6}$$

which assures the detailed balance condition

$$W[\sigma_1 | \sigma_2] / W[\sigma_2 | \sigma_1] = \exp(H[\sigma_1] - H[\sigma_2]) \tag{2.7}$$

for the transition probabilities.

If $A[\sigma_1, \sigma_2] = 0$, then spins at different times are independent. The joint distribution of $\sigma_1, \ldots, \sigma_N$ becomes simply

$$P[\sigma_1, \sigma_2, \ldots, \sigma_N] = \prod_{t=1}^{N} e^{H[\sigma_t]}. \tag{2.8}$$

We expect that when $\Delta t \to \infty$, (2.8) becomes valid.

If $A_m[\sigma_1, \ldots, \sigma_m] = 0$ for $m \geq 3$, then the time evolution of the system is called "Markovian." The joint distribution is

$$\begin{aligned} P[\sigma_1, \ldots, \sigma_N] &= W[\sigma_1 | \sigma_2] W[\sigma_2 | \sigma_3] \cdots W[\sigma_{N-1} | \sigma_N] e^{H[\sigma_N]} \\ &= \exp(H[\sigma_1] + \cdots + H[\sigma_N] + A[\sigma_1, \sigma_2] + \cdots + A[\sigma_{N-1}, \sigma_N]). \end{aligned} \tag{2.9}$$

H and A specify completely the statics and dynamics.

The generalization to cases where some of the A_m, $m \geq 3$ do not vanish is straightforward. Define

$$\begin{aligned} W[\sigma_1 | \sigma_2, \ldots, \sigma_m] = \exp(&H[\sigma_1] + A[\sigma_1, \sigma_2] + A_3[\sigma_1, \sigma_2, \sigma_3] \\ &+ \cdots + A_m[\sigma_1, \sigma_2, \ldots, \sigma_m]), \end{aligned} \tag{2.10}$$

which is the conditional probability of finding σ_1 given $\sigma_2, \ldots, \sigma_m$. The normalization conditions are

$$\sum_{\sigma_1} W[\sigma_1 | \sigma_2, \ldots, \sigma_m] = 1. \tag{2.11}$$

The time reversal conditions are

$$A_m[\sigma_1, \sigma_2, \ldots, \sigma_m] = A_m[\sigma_m, \sigma_{m-1}, \ldots, \sigma_1]. \tag{2.12}$$

Equations (2.4), (2.5), and (2.6) are just Eqs. (2.10), (2.11), and (2.12), respectively, for $m = 2$. Note that so far all the W's are just definitions. There has been no condition imposed on them (apart for the time reversal symmetry and equilibrium). Equations (2.10)–(2.12) must be satisfied for all m simultaneously. The joint distribution $P[\sigma_1, \ldots, \sigma_N]$ can be expressed in terms of the W's as

$$\begin{aligned} P[\sigma_1, \ldots, \sigma_N] &= W[\sigma_1 | \sigma_2, \ldots, \sigma_N] P[\sigma_2, \ldots, \sigma_N] \\ &= W[\sigma_1 | \sigma_2, \ldots, \sigma_N] W[\sigma_2 | \sigma_3, \ldots, \sigma_N] \cdots W[\sigma_{N-1} | \sigma_N] e^{H[\sigma_N]} \\ &= \exp\left(\sum_{t=1}^{N} H[\sigma_t] + \sum_{t=1}^{N-1} A[\sigma_t, \sigma_{t+1}] + \sum_{t=1}^{N-2} A_3[\sigma_t, \sigma_{t+1}, \sigma_{t+2}] + \cdots \right). \end{aligned} \tag{2.13}$$

Of course, we hope that in cases of interest, A_m would be negligible for m larger than some small integer such as 2 or 3. Then H, A, and A_3 can be parametrized in some simple fashion to afford a simple fitting of the static and dynamic properties.

III. PARAMETERS AND NORMALIZATION

We now turn to questions regarding the forms of the A's in (2.13) and the choice of parameters. In principle, one can start with microscopic equations of motion for continuous time, usually dif-

ferential equations, and then integrate the equations over a time period Δt to find the transition probability $W[\sigma_1|\sigma_2]$, and hence $A[\sigma_1,\sigma_2]$. If the dynamic processes at time scales shorter than Δt are Markovian, then they are also Markovian at the scale Δt, i.e., $A_m=0$, for $m\geq 3$. However, our formulation here is intended as a basis for phenomenological analysis. We want to choose some simple forms of A and perhaps A_3, A_4, with a few parameters. Then we adjust the parameters to describe phenomena over the time scale of Δt and longer. Our main interest in the following sections will be how these parameters and forms change as Δt is changed and under the change of space scale (the renormalization group).

Let us consider a simple model to illustrate some basic aspects of parametrization. Take $H[\sigma]$ to be that of a conventional Ising model:

$$H[\sigma]=J\sum_{\langle x,y\rangle}\sigma(x)\sigma(y)\,, \tag{3.1}$$

where x, y are restricted to nearest neighbors on the lattice, and $\sigma(x)=\pm 1$. Now we want to choose a form for A. Let us try

$$A_0[\sigma,\sigma']=-\tfrac{1}{2}\lambda\sum_x[\sigma(x)-\sigma'(x)]^2$$

$$=\lambda\sum_x\sigma(x)\sigma'(x)+\text{const}\,. \tag{3.2}$$

There is only one parameter, λ. Since the transition probability is proportional to e^A, Eq. (3.2) restricts σ' to near σ. If σ is obtained from σ' by flipping n spins, then $e^{A_0}=e^{-2n\lambda}$. The larger the n, the smaller the transition probability.

The form (3.2) is not quite right yet for A because it does not satisfy the normalization condition (2.5). To fix it, we introduce $V[\sigma]$ so that

$$A[\sigma,\sigma']=-V[\sigma]-V[\sigma']+A_0[\sigma,\sigma'] \tag{3.3}$$

will satisfy the normalization condition, which is now

$$\sum_{\sigma'}\exp(H[\sigma']-V[\sigma']+A_0[\sigma,\sigma'])=e^{V[\sigma]}\,, \tag{3.4}$$

which must be solved for V. Thus, even though one can easily choose an $A_0[\sigma,\sigma']$, to get an $A[\sigma,\sigma']$ which satisfies the normalization condition is a nontrivial task. Let us look at a special case, that with very large λ, just to get some rough idea about what A might look like.

For very large λ, $e^{-2\lambda}$ is small. Let us keep only the zeroth and first order in $e^{-2\lambda}$ in the sum of (3.4). This means keeping $\sigma'=\sigma$ and those σ' which differ from σ by flipping one spin only. Then (3.4) becomes

$$\exp(H[\sigma]-V[\sigma])+e^{-2\lambda}\sum_x\exp\{Q_x(H[\sigma]-V[\sigma])\}$$

$$=e^{V[\sigma]}\,, \tag{3.5}$$

where Q_x changes the sign of the spin at x, but leaves all other spins unchanged:

$$Q_x f[\sigma]=f[\sigma]\,|_{\sigma(x)\to-\sigma(x)}\,. \tag{3.6}$$

We can now solve (3.5) easily for $V[\sigma]$ to first order in $e^{-2\lambda}$:

$$V[\sigma]=\tfrac{1}{2}H[\sigma]+\tfrac{1}{2}e^{-2\lambda}\sum_x e^{\sigma(x)h(x)}\,, \tag{3.7}$$

where $h(x)$ is the local field seen by $\sigma(x)$:

$$\sigma(x)h(x)\equiv\tfrac{1}{2}(1-Q_x)\,H[\sigma]\,. \tag{3.8}$$

It is something like the derivative of $H[\sigma]$ with respect to $\sigma(x)$. Using the $H[\sigma]$ of (3.1), $h(x)$ is just J times the sum of neighboring spins of $\sigma(x)$. The transition probability $W[\sigma|\sigma']$ is, by Eqs. (2.4), (3.3), and (3.7),

$$W[\sigma|\sigma']=\exp(H[\sigma]-V[\sigma]-V[\sigma']+A_0[\sigma,\sigma'])$$

$$=1-e^{-2\lambda}\sum_x\,,\quad\text{no flip}\,,$$

$$=\exp[-2\lambda+\sigma(x)h(x)]\,,\quad\sigma(x)\text{ flips}\,. \tag{3.9}$$

$\sum_x$ of course gives the total number of spins. Equation (3.9) is a version of the kinetic Ising model often seen in the literature if we identify

$$e^{-2\lambda}=\Gamma\Delta t\,. \tag{3.10}$$

Here Γ is the spin flip rate. This model describes an Ising model whose spins are flipped at random (to simulate the action of the thermal reservoir) at a rate Γ. In view of (3.10), the large λ limit is the short time limit, i.e., small Δt.

Now let us consider the model with a small λ. Since A_0 is now small, we can solve for V from (3.4) by expanding in powers of λ. Let us ignore the additive constant in (3.2) and expand (3.4). Assuming $H[\sigma]=H[-\sigma]$, there is no $O(\lambda)$ contribution to V. The leading term of V is of $O(\lambda^2)$. One finds

$$V[\sigma]=\tfrac{1}{2}\lambda^2\sum_{x,y}\sigma(x)\sigma(y)C(x-y)+O(\lambda^4), \tag{3.11}$$

where $C(x-y)$ is the static correlation function

$$C(x-y)\equiv\langle\sigma(x)\sigma(y)\rangle$$

$$=\sum_\sigma e^{H[\sigma]}\sigma(x)\sigma(y)\,. \tag{3.12}$$

$C(x-y)$ has a range, which is the correlation length ξ by definition. Thus $V[\sigma]$ is an interaction of range ξ. If $H[\sigma]$ describes the system at its critical point, ξ becomes infinite. However, this

does not mean $V[\sigma]$ will have an infinite range. In fact, one must solve (3.4) self-consistently for small λ and $V[\sigma]$ under this circumstance. One finds that $V[\sigma]$ will still have a finite range proportional to some inverse power of λ. We shall see this more explicitly when we discuss the Gaussian model.

The above discussion gives a rough idea about the connection between the magnitude of $A[\sigma, \sigma']$ and the length of the time interval Δt. Small Δt means little change from σ' to σ, and requires a large A for tight restriction. Large Δt means the system has relaxed to a large extent over this period, and a small A is all that is needed for the memory.

One can certainly introduce more terms to A_0 of (3.2), for example,

$$A_0[\sigma, \sigma'] = -\tfrac{1}{2}\lambda \sum_x [\sigma(x) - \sigma'(x)]^2 + \kappa \sum_{(x,y)} \sigma(x)\sigma'(y), \tag{3.13}$$

where x, y are restricted to nearest neighbors. The additional parameter κ can be adjusted to afford more flexibility in fitting this model to whatever case is of interest.

To gain further insight into the above discussion, let us consider a Gaussian model, which is exactly solvable. The model is defined by

$$H[\sigma] = -\frac{1}{2}\int d^d x[(\nabla\sigma)^2 + r_0\sigma^2],$$

$$A_0[\sigma, \sigma'] = -\frac{\lambda}{2}\int d^d x[\sigma(x) - \sigma'(x)]^2. \tag{3.14}$$

We shall not write out explicitly the additive constant in H which keeps e^H normalized. The spin variable $\sigma(x)$ is now regarded as a real valued function of the continuous d-dimensional space coordinates x. In terms of the Fourier component σ_k of $\sigma(x)$, we have

$$H[\sigma] = -\frac{1}{2}\sum_k \epsilon_k |\sigma_k|^2,$$

$$A_0[\sigma, \sigma'] = -\frac{1}{2}\sum_k \lambda |\sigma_k - \sigma_k'|^2, \tag{3.15}$$

$$\epsilon_k \equiv r_0 + k^2.$$

To be precise, the independent variables are $\mathrm{Re}\sigma_k$ and $\mathrm{Im}\sigma_k$ with the restriction $\mathrm{Re}\sigma_k = \mathrm{Re}\sigma_{-k}$, $\mathrm{Im}\sigma_k = -\mathrm{Im}\sigma_{-k}$. For our purpose here, we can simply pretend that each σ_k is an independent real variable. Now we have to fix a V to A_0 to satisfy the normalization condition [see Eqs. (3.3) and (3.4)]. Write

$$V[\sigma] = \frac{1}{2}\sum_k v_k |\sigma_k|^2,$$

$$A[\sigma, \sigma'] = -\frac{1}{2}\sum_k [v_k(|\sigma_k|^2 + |\sigma_k'|^2) + \lambda |\sigma_k - \sigma_k'|^2]. \tag{3.16}$$

We can calculate each v_k separately. This is the simplicity of the Gaussian model. Equation (3.4) now reads, for each k,

$$e^{v\sigma^2/2} = \int d\sigma' \exp[-\tfrac{1}{2}(\epsilon + v)\sigma'^2 - \tfrac{1}{2}\lambda(\sigma - \sigma')^2]. \tag{3.17}$$

We have dropped all subscripts k for simplicity of notation. It is also understood that $d\sigma'$ carries a factor $(2\pi/\lambda)^{-1/2}$ so that there is no need to carry along a constant with $V[\sigma]$. The integral of (3.17) is elementary. One obtains

$$v = \frac{\lambda^2}{\epsilon + v + \lambda} - \lambda = -\lambda - \tfrac{1}{2}\epsilon + (\lambda^2 + \tfrac{1}{4}\epsilon^2)^{1/2}, \tag{3.18}$$

which, for special cases gives

$$\lambda \gg \epsilon, \quad v \approx -\tfrac{1}{2}\epsilon$$
$$\lambda \ll \epsilon, \quad v \approx -\lambda. \tag{3.19}$$

Let us find the transition probability $W[\sigma|\sigma']$ for the case of large λ. Consider a single σ_k and again drop the subscript k.

$$W[\sigma|\sigma'] = e^{H+A} \approx \exp[\tfrac{1}{4}(\epsilon\sigma'^2 - \epsilon\sigma^2) - \tfrac{1}{2}\lambda(\sigma - \sigma')^2]. \tag{3.20}$$

Since λ is large, the transition probability is very small except for σ very close to σ', and

$$\frac{1}{2}\left(\frac{\epsilon}{2}\sigma'^2 - \frac{\epsilon}{2}\sigma^2\right) \approx (\sigma - \sigma')\left(-\frac{1}{2}\frac{\partial H}{\partial \sigma}\right). \tag{3.21}$$

This transition probability gives an average displacement away from σ'

$$\langle \Delta\sigma \rangle \equiv \langle (\sigma - \sigma') \rangle_{\sigma'} = -\frac{1}{2\lambda}\frac{\partial H}{\partial \sigma}, \tag{3.22}$$

and fluctuations in the displacement characterized by

$$\langle (\Delta\sigma)^2 \rangle = 1/\lambda. \tag{3.23}$$

We can identify the Gaussian distribution of the displacement with the results of diffusion with a diffusion constant D:

$$\exp[-\tfrac{1}{2}\lambda(\sigma - \sigma')^2] = \exp[-(\sigma - \sigma')^2/4D\Delta t],$$

$$\lambda = \frac{1}{2D\Delta t}. \tag{3.24}$$

Thus (3.22) becomes the average displacement due to dissipation with a friction coefficient D:

$$\frac{\langle \Delta\sigma \rangle}{\Delta t} = -D\frac{\partial H}{\partial \sigma}. \tag{3.25}$$

These conclusions concerning $W[\sigma|\sigma']$ for large λ, in fact, is not limited to a Gaussian model. One can use any H (but the same A_0). The derivation will be a little different. When one uses the Ginzburg-Landau form of $H[\sigma]$, i.e., (3.14) with an additional $u\sigma^4$ term in the integrand, the large-λ limit of the model has the same transition probability as the time-dependent Ginzburg-Landau model extensively studied in the literature. If the λ in Eq. (3.15) is proportional to $1/k^2$, instead of being independent of k, then we get a model with conserved total spin, and $D \propto k^2$.

Note that (3.24) is quite different from (3.10) of the discrete Ising model in spite of the formal similarity between the way λ appears in A_0. [See Eqs. (3.2) and (3.14).]

Now we turn to the question as to whether $V[\sigma]$ has a long range when the system is near its critical point. In this model, the critical point is given by $r_0 = 0$. Let us write $V[\sigma]$ explicitly, using (3.18) and putting back the subscripts and summing over k:

$$V[\sigma] = \frac{1}{2}\sum_k \left(\frac{\lambda^2}{k^2 + r_0 + v_k + \lambda} - \lambda \right) |\sigma_k|^2 . \tag{3.26}$$

The last term in the sum, namely λ, contains no k dependence, so it has no range. For $r_0 \to 0$ and very small k, the first term in the sum is

$$\frac{\lambda^2}{\frac{1}{2}(k^2 + r_0) + \lambda}, \tag{3.27}$$

since $v_k \to -\frac{1}{2}\epsilon_k = -\frac{1}{2}(r_0 + k^2)$ by (3.19). Thus the range of $V[\sigma]$ is finite near and at the critical point, namely $(2\lambda)^{-1/2}$.

On the other hand, if $r_0 \gg \lambda$, the first term in the sum of (3.26) becomes simply

$$\lambda^2/(k^2 + r_0) \tag{3.28}$$

since $v_k \approx -\lambda$. This is just λ^2 times the Fourier transform of the spin-correlation function $C(x-y)$ and Eq. (3.26) becomes the same as Eq. (3.11) [apart from the last term of (3.26)]. The range of C is just $\xi = (r_0)^{-1/2}$.

The above simple examples show roughly how one constructs models within this formalism, and how the normalization condition can be satisfied. They also demonstrated qualitatively that no long range interactions need to appear even when the system is near its critical point. In order to apply the formalism to the analysis of dynamics near a critical point, it is of great importance that no long range interaction appears.

IV. TRANSFORMATION UNDER A CHANGE OF TIME SCALE

The values of the parameters of a dynamic model formulated above depend on the time interval Δt. For a given model, if we change the size of Δt, the values of parameters change so that the physical content of the model remains the same. This is a change of the scale unit of time. Like a rotation of coordinate axes or other changes of references, a change of time scale is merely a change of description, not of content. We proceed to investigate how the parameters transform under a change of time scale.

Consider the joint distribution $P[\sigma_1, \sigma_2, \ldots, \sigma_N]$ of spin configurations at times $t = 1, 2, 3, \ldots, N$. Let us sum over every other configuration, $\sigma_2, \sigma_4, \sigma_6, \ldots$ to obtain the transformed distribution

$$P''[\sigma_1, \sigma_3, \ldots, \sigma_N] = \sum_{\sigma_2, \sigma_4, \ldots, \sigma_{N-1}} P[\sigma_1, \sigma_2, \ldots, \sigma_N], \tag{4.1}$$

if N is odd. If N is even, just sum over σ_N, too. P'' gives the joint distribution of spin configurations separated by two units of time. Thus P'' is the transformation of P under the time scale change $\Delta t \to 2\Delta t$. As far as properties of the system over time scales larger than Δt are concerned, P'' has the same content as P. We can now easily express (4.1) in terms of transformations of the A's. Since

$$\sum_{\sigma_2} P[\sigma_1, \sigma_2, \sigma_3] = P''[\sigma_1, \sigma_3] \equiv \exp(H[\sigma_1] + H[\sigma_3] + A''[\sigma_1, \sigma_3]), \tag{4.2}$$

we have A'', the transformed A, given by

$$\exp(A''[\sigma_1, \sigma_3]) = \sum_{\sigma_2} \exp(H[\sigma_2] + A[\sigma_1, \sigma_2] + A[\sigma_2, \sigma_3] + A_3[\sigma_1, \sigma_2, \sigma_3]). \tag{4.3}$$

See Eq. (1.3) for the expression of $P[\sigma_1, \sigma_2, \sigma_3]$ in terms of H, A, and A_3. Similarly $A''_m[\sigma_1, \sigma_3, \ldots, \sigma_{2m+1}]$ can be defined via $P''_m[\sigma_1, \sigma_3, \ldots, \sigma_{2m+1}]$. For example,

$$\exp(A''_3[\sigma_1, \sigma_3, \sigma_5] = P''[\sigma_1, \sigma_3, \sigma_5]\{\exp(H[\sigma_1] + H[\sigma_3] + H[\sigma_5]) \times \exp(A''[\sigma_1, \sigma_3] + A''[\sigma_3, \sigma_5])\}^{-1}. \tag{4.4}$$

Note that $H[\sigma]$ does not change under the time scale change. If $A_m = 0$ for $m \geq 3$, then $A''_m = 0$ for

$m \geq 3$. Otherwise, one cannot say in general which A_m'' must be zero.

Let us use the simple models of the previous section to illustrate the calculation of A''. Consider first the kinetic Ising model defined by Eqs. (3.1) and (3.2) in the limit of large λ, and keep terms up to $O(e^{-2\lambda})$. Under this approximation, we consider only the cases where $\sigma_1 = \sigma_3$, and $\sigma_1 = Q_x\sigma_3$ [i.e., σ_1 differs from σ_3 by the flip of one spin σ_x only, see (3.6) for the definition of Q_x]. For $\sigma_1 = \sigma_3 = \sigma$, we have, using (4.3) with $A_3 = 0$, and keeping in the sum over σ_2 only the intermediate configurations $\sigma_2 = \sigma$,

$$\exp(A''[\sigma, \sigma]) = \exp(H[\sigma] + 2A[\sigma, \sigma]) . \tag{4.5}$$

We call σ_2 the intermediate configurations since they happen between the initial configuration σ_3 and final σ_1. Summing over intermediate configurations here means averaging over processes with fixed initial configuration σ_3 and final σ_1. In (4.5) we included only the process

$$\sigma \longleftrightarrow \sigma \longleftrightarrow \sigma . \tag{4.6}$$

Any other process would involve two or more flips. By Eqs. (3.3) and (3.7),

$$\begin{aligned} A[\sigma, \sigma'] &= A_0[\sigma, \sigma'] - V[\sigma] - V[\sigma'] \\ &= -\frac{\lambda}{2} \sum_x [\sigma(x) - \sigma'(x)]^2 \\ &\quad -\frac{1}{2}\Big(H[\sigma] + H[\sigma'] + e^{-2\lambda} \sum_x (e^{\sigma(x)\, h(x)} \\ &\qquad + e^{\sigma''(x)\, h(x)})\Big) . \end{aligned} \tag{4.7}$$

Substituting this in (4.5), we obtain

$$A''[\sigma, \sigma] = -H[\sigma] - 2e^{-2\lambda} \sum_x e^{\sigma(x)\, h(x)} . \tag{4.8}$$

Similarly, for $\sigma_1 = Q_x\sigma$, we take into account the intermediate configuration $\sigma_2 = \sigma$ and $\sigma_2 = Q_x\sigma$, i.e., the processes

$$\begin{aligned} &Q_x\sigma \longleftrightarrow Q_x\sigma \longleftrightarrow \sigma , \\ &Q_x\sigma \longleftrightarrow \sigma \longleftrightarrow \sigma . \end{aligned} \tag{4.9}$$

We obtain

$$\exp(A''[Q_x\sigma, \sigma]) = 2 \exp\{-2\lambda - \tfrac{1}{2}(H[\sigma] + H[Q_x\sigma])\} . \tag{4.10}$$

To $O(e^{-2\lambda})$, Eqs. (4.8) and (4.10) are consistent with the identification

$$e^{-2\lambda''} = 2e^{-2\lambda} , \tag{4.11}$$

which is the transformation formula for the parameter λ under the change of time scale $\Delta t \to 2\Delta t$. The identification of $e^{-2\lambda}$ as proportional to Δt [see (3.10)] in Sec. III is thus justified for large λ.

As a more detailed illustration, let us look at the Gaussian model defined by (3.14) or (3.15), and see how the parameter λ changes under the time scale transformation. The formula (4.3) becomes

$$\begin{aligned} \exp(A''[\sigma, \sigma']) &= \int \delta\sigma'' \exp(H[\sigma''] + A[\sigma, \sigma''] + A[\sigma'', \sigma']) \\ &= \prod_k \int d\sigma_k'' \exp[-\tfrac{1}{2}(\epsilon_k + v_k)\,|\sigma_k''|^2 - \tfrac{1}{2}\lambda\,|\sigma_k - \sigma_k''|^2 - \tfrac{1}{2}\lambda\,|\sigma_k' - \sigma_k''|^2] \exp[-\tfrac{1}{2}v_k(|\sigma_k|^2 + |\sigma_k'|^2)] . \end{aligned} \tag{4.12}$$

We have used (3.15) and (3.16) for the expressions for H and A. The integrals can be performed separately for each k. After a little algebra, one obtains

$$\begin{aligned} A''[\sigma, \sigma'] &= -\frac{1}{2} \sum_k [\lambda_k''\,|\sigma_k - \sigma_k'|^2 \\ &\qquad + v_k''(|\sigma_k|^2 + |\sigma_k'|^2)] , \\ \lambda_k'' &= \lambda^2(4\lambda^2 + \epsilon_k^2)^{-1/2} . \end{aligned} \tag{4.13}$$

One immediately notices that A'' has a more complicated form than A as given by (3.15). Now λ_k'' depends on k, whereas in (3.15) λ is one constant. One may thus conclude that, in general, A will be transformed into a different form. A'' may be more complicated than A.

For the special case $\lambda \gg \epsilon_k$, we can ignore ϵ_k in (4.13) and obtain the simple formula

$$\lambda'' \approx \lambda/2 . \tag{4.14}$$

This indicates that λ reduces to $\lambda/2$ as Δt doubles, and thus justifies the identification (3.24) of λ as proportional to $1/\Delta t$ for large λ. The transformation can be repeated l times. Now let us denote the transformed time scale as

$$\Delta t = (\Delta t)_0 2^l \tag{4.15}$$

and $\lambda_{kl} \equiv \lambda_k(\Delta t)$ as the parameters at this time scale. Equation (4.13) provides a recursion relation

$$\lambda_{kl+1} = \lambda_{kl}^2 (4\lambda_{kl}^2 + \epsilon_k^2)^{-1/2}, \tag{4.16}$$

which allows us to work out $\lambda_k(\Delta t)$ as a function of Δt. For large λ_k, i.e., short time scale, we have already seen that

$$\lambda_k(\Delta t) \propto 1/\Delta t. \tag{4.17}$$

When Δt is increased, $\lambda_k(\Delta t)$ eventually becomes small compared to ϵ_k (assuming $\epsilon_k \neq 0$). Then it will decrease exponentially:

$$\lambda_k(\Delta t) \sim e^{-\Delta t/\tau_k} \tag{4.18}$$

where τ_k is some characteristic time scale such that $\lambda_k(\tau_k)$ is already small compared to ϵ_k.

The above illustrations show roughly how parameters might change when the time scale is increased. They also show that the form of A in general changes too. To define a time scale transformation completely one would need in general to introduce infinite numbers of parameters (e.g., one λ_k for each k in the Gaussian model). In practice, one needs some truncations to dispose of parameters which are of no interest. For example, if we happen to be interested in a range of Δt where $\lambda_k \gg \epsilon_k$, in the Gaussian model, then we might take λ as just one parameter.

V. TRANSFORMATION UNDER THE COMBINED TIME-SPACE SCALE CHANGE

The transformation of parameters under a change of length scale in space, with the modern name "renormalization group (RG)," has been extensively studied. When combined with a time scale change, it is called "the dynamic RG." In this section, we illustrate how the dynamic RG works in the present formulation. We start with the Gaussian model.

The Gaussian model is defined by (3.14) or (3.15). We now introduce a cutoff Λ so that the wavevectors are limited to $k < \Lambda$. Denote the Fourier components of σ_t by σ_{tk}. Given the joint distribution $P[\sigma_1, \ldots, \sigma_N]$, the transformed distribution P' under RG (space only) is defined by

$$P'[\sigma_1, \ldots, \sigma_N] = \left(\int P[\sigma_1, \ldots, \sigma_N] \prod_{t=1}^{N} \prod_{k_t' > \Lambda/s} d\sigma_{tk_t'} \right)_{\sigma_{tk} \to s^{1-\eta/2} \sigma_{tks}}. \tag{5.1}$$

This transformation involves two operations. We first eliminate by integration short wave ($k' > \Lambda/s$) variations of the spins. The labels k of the remaining spin variables are then replaced by ks and the spin variables are multiplied by $s^{1-\eta/2}$. Here $1 < s < \infty$ is the factor of scale change. Thus the transformation effectively is $x \to x/s$ with the cutoff Λ unchanged. For the Gaussian model, all the σ_{tk}'s are independent. The integrations of (5.1) simply mean dropping $\sigma_{tk'}$ with $k' > \Lambda/s$. The second step is also trivial. The transformed H is

$$H'[\sigma] = -\frac{1}{2} \sum_{k < \Lambda} (r_0' + k^2 s^{-\eta}) |\sigma_k|^2,$$
$$r_0' = s^{2-\eta} r_0. \tag{5.2}$$

For the special value $r_0 = r_0^* = 0$, and by choosing $\eta = 0$, we get a "fixed point" $H^*[\sigma]$, which does not change under RG.

The transformation of λ is easily read off from Eq. (3.15) since the transformation of A is just the change of variables σ, σ' to $s\sigma, s\sigma'$. We have

$$\lambda' = s^2 \lambda. \tag{5.3}$$

Now we combine this transformation with the time scale transformation (4.13), i.e., given λ, we first apply (4.13) and then apply (5.3) to the result. We get, for small k, the transformation under the dynamic RG:

$$\lambda' = \tfrac{1}{2} s^2 \lambda. \tag{5.4}$$

The terms dropped are of $O(k^4/\lambda^2)$, assuming $r_0 = 0$. Thus, if we choose $s = \sqrt{2}$, we get $\lambda' = \lambda$, i.e., invariance under the dynamic RG. By definition, the time scale is changed by s^z. Thus, we have

$$z = 2 \tag{5.5}$$

for the dynamic exponent for the Gaussian model.

What we have learned is that, while a time scale change decreases λ, and a space scale change increases it, a proper combination can keep λ the same, provided that λ is fairly large. If λ is not large, this cannot be done in this model. We expect the following generalization of this conclusion to other models: Over a certain range of values of parameters, invariance under the dynamic RG can be possible with a proper choice of z.

Having discussed the most basic aspects of the dynamic RG within the Gaussian model, we proceed to remark on more general models. The definition of RG transformation (5.1) is not limited to the Gaussian model. To facilitate discussion, let us rewrite (5.1) as follows. Let μ_t denote the remaining and rescaled spin variables after the transformation, and write

$$P'[\mu_1, \ldots, \mu_N] = \prod_{t=1}^{N} \sum_{\sigma_t} R[\mu_t | \sigma_t] P[\sigma_1, \ldots, \sigma_N], \tag{5.6}$$

where the sum over σ_t with the restriction R replaces the integrals in (5.1). For (5.1), we have

$$R[\mu_t | \sigma_t] = \prod_{k<\Lambda/s} \delta(\mu_{tk} - \sigma_{tk}),$$

$$\sum_{\sigma_t} = \prod_k \int d\sigma_{tk}. \tag{5.7}$$

Equation (5.6) also allows other forms of R. In general we can regard $R[\mu|\sigma]$ as the conditional probability of finding μ given σ. It must satisfy the normalization condition

$$\sum_{\mu} R[\mu|\sigma] = 1. \tag{5.8}$$

From the definitions of H, A, $A_3, \ldots$, one can obtain the transformed quantities H', A', $A_3', \ldots$ from (5.6). (For the moment, let us forget about the time scale change.) To obtain H', we need only H. To obtain A', we need H and A, but not A_3:

$$e^{H'[\mu]} = \sum_{\sigma} R[\mu|\sigma] e^{H[\sigma]},$$

$$\exp(H'[\mu_1] + H'[\mu_2] + A'[\mu_1, \mu_2])$$

$$= \sum_{\sigma_1 \sigma_2} R[\mu_1|\sigma_1] R[\mu_2|\sigma_2] \times \exp(H[\sigma_1] + H[\sigma_2] + A[\sigma_1, \sigma_2]), \tag{5.9}$$

etc.

The calculation of A_n' is not affected by A_m with $m > n$. This is an important advantage of the present formulation. It allows the calculation of H', A', A_3', $A_4', \ldots$ sequentially. There is no need to deal with H, A, $A_3, \ldots$ all at once.

It should be pointed out, however, that even if $A_m = 0$ for all $m \geq 3$, A_m' for $m \geq 3$ in general do not vanish (Gaussian model excepted). This situation is different from the case of the time scale transformation, where A_m'' for all $m \geq 3$ would vanish if $A_m = 0$ for all $m \geq 3$. In other words, the RG transformation would generate a non-Markovian form of the joint probability distribution, even though the time scale transformation would not. The extent to which A_m' with $m \geq 3$ are important depends on the choice of $R[\mu|\sigma]$. For example, if we choose an R different from Eq. (5.7) for the Gaussian model, A_m' with $m \geq 3$ would no longer vanish. One can design a program to optimize the choice of $R[\mu|\sigma]$ in a calculation, so as to minimize A_m' with $m \geq 3$.

There are also other considerations in the choice of $R[\mu|\sigma]$. For the Gaussian model it is clear from a physical viewpoint that the choice (5.7) is a sound one, because, in this choice, the slowly varying variables (σ_{tk} with small k) and the fast varying ones (with large k) are neatly separated. For a non-Gaussian model, such separation is difficult. Even though a good choice of R is not obvious, one can easily tell that some choices are bad. For example, one should avoid those which generate fast variations which should not be there. An example is the choice

$$R[\mu|\sigma] \propto \prod_y e^{p\mu(y) S(y)},$$

$$S(y) = \sum_{x \in y} \sigma(x), \tag{5.10}$$

where y labels a block containing several lattice cells, and p is an adjustable parameter. This choice is often used in static RG calculations for the Ising model, where $\mu = \pm 1$, $\sigma = \pm 1$. This choice is expected to be bad for dynamic calculations because it generates fast variations in μ which are ficticious in the following sense. Even if all $\sigma(x)$ in a block $y = +1$, $R[\mu|\sigma]$ for $\mu(y) = -1$ does not vanish. Namely, the block spin can fluctuate even if no spin ever flips. Obviously, this choice of R masks the dynamic content of the original model, not to mention that it violates the spin conservation law in certain models.

We now proceed to illustrative calculations.

VI. DYNAMICS IN ONE DIMENSION AND IN $1+\epsilon$ DIMENSION

We now apply the dynamic RG to a one-dimensional Ising model and look for a fixed point, and its associated dynamic exponent z.

First, we note that the critical point for the one-

dimensional Ising model is at zero temperature, or $J=\infty$ [J as defined by $H[\sigma]$ of (3.1)]. Therefore, let us consider only the case of very large J. In this case the system is characterized by large domains of parallel spins. The border point of two domains is often referred to as a "kink" or a "domain wall." The size of a domain is on the average $\xi = O(e^{2J})$. The system is thus essentially an ideal gas of kinks for very large J. The dynamics is fully characterized by the motion of a single kink.

Let us look at a portion of the system small compared to ξ but much larger than the lattice spacing. If there is no kink in this portion, there is no change under the dynamic RG. Now consider the case where there is one kink in this portion, located at the position x_t at time t. Let us try the dynamic model defined by

$$A(x,x') = -\tfrac{1}{2}\lambda(x-x')^2 . \tag{6.1}$$

We need not consider the normalization condition, since H and hence V are all independent of x, and Eq. (6.1) is correct up to an additive constant. The time scale transformation of A is easily worked out:

$$\exp[A''(x,x')] = \int dx'' \exp[A(x,x'') + A(x'',x')] ,$$

$$A''(x,x') = -\tfrac{1}{2}\lambda''(x-x')^2 ,$$

$$\lambda'' = \tfrac{1}{2}\lambda . \tag{6.2}$$

This is a trivial version of (4.12). We have assumed that the portion of the system under consideration is large compared to $1/\sqrt{\lambda}$, even though smaller than $\xi \sim e^{2J}$, so that the limits of the x'' integral in (6.2) can be regarded as $\pm\infty$. If we change the scale of x by s, then evidently λ is multiplied by s^2 in view of (6.1). Thus, together with the time scale change (6.2), the dynamic RG changes λ into

$$\lambda' = \tfrac{1}{2}s^2\lambda . \tag{6.3}$$

When we choose $s=\sqrt{2}$, $z=2$, λ would be invariant. We thus have a fixed point of the dynamic RG.

The reader may wonder why the special form (6.1) is chosen. The main reason is that this special form gives the transition probability $\exp A(x,x')$ of a diffusion process. Since we expect the kink motion to be that of a random walk, and for the time interval Δt large compared to the time for making one spin flip, and $|x-x'|$ large compared to one lattice spacing, Eq. (6.1) is a reasonable choice to start with. Note that the λ of (6.1) is not that of (3.2), but the connection is easy to determine.

Another question is, when terms of other forms are added to (6.1), how would A transform under the dynamic RG? Would the additional terms grow (then they are "relevant") or would they diminish ("irrelevant")? To answer this, let us define

$$\alpha(x-x') \equiv A(x,x') ,$$

$$e^{\alpha(\kappa)} \equiv \int dx \exp[-i\kappa x + \alpha(x)] . \tag{6.4}$$

Under the time scale transformation (6.2), which is just a convolution integral, we get simply

$$\tilde{\alpha}''(\kappa) = 2\tilde{\alpha}(\kappa) . \tag{6.5}$$

Since $\alpha(x)$ is an even function, so is $\tilde{\alpha}$, and we can introduce quite generally the parameters $a_2, a_3, \ldots$, so that

$$\tilde{\alpha}(\kappa) = -\kappa^2/2\lambda - a_2\kappa^4 - a_3\kappa^6 - \cdots . \tag{6.6}$$

In view of Eq. (6.5) and the fact that κ has the unit of an inverse length, the transformation of these parameters under the dynamic RG is thus

$$a_2' = 2s^{-4}a_2 ,$$

$$a_3' = 2s^{-6}a_3 , \text{ etc.} \tag{6.7}$$

Upon choosing $s=\sqrt{2}$, we see that $a_2', a_3', \ldots$ all diminish under the dynamic RG transformation, i.e., these parameters are all "irrelevent" and (6.1) is a "stable" fixed point.

There have been few precise results on the static and dynamic exponents in $1+\epsilon$ dimensions, with small ϵ. The following calculation of z in $1+\epsilon$ dimensions is only speculative. It is based on the simple picture in one dimension plus extrapolation.

First, let us review the simple argument which leads to the conclusion that J^*, the critical value of J, is of $O(1/\epsilon)$ [i.e., a critical temperature of $O(\epsilon)$]. Suppose that all spins are pointing up. The probability for later finding a domain of size L with spins down will be proportional to

$$\exp(-JL^{\epsilon} + \ln L) \tag{6.8}$$

where JL^{ϵ} is, up to a constant factor, the energy of the domain wall, and $\ln L$ is the entropy of the fluctuation of the domain wall, again up to a factor probably independent of L. We see that the size of the domain is limited if $JL^{\epsilon} > \ln L$, i.e., if

$$J > J^* \sim 1/\epsilon . \tag{6.9}$$

This completes the argument.

Near the critical point, we expect large domains of parallel spins of size $\sim e^{1/\epsilon}$. Since the domains are very large, the motion of one part of the domain wall is expected to be quite independent of other parts. Therefore, let us choose an arbitrary

direction as the x axis. Let y denote the coordinate perpendicular to the x axis. The space of y is an ϵ-dimensional space. Consider a domain wall and describe its shape by $x(y)$, which is a "surface" in the $(1+\epsilon)$-dimensional space. The simplest dynamic model in our formulation is given by a Gaussian model:

$$A_0(x,x') = -\frac{\lambda}{2}\int d^{\epsilon}y[x(y)-x'(y)]^2 . \tag{6.10}$$

This is, of course, a direct generalization of Eq. (6.1). Instead of a single kink, we have here a wall. Since this is a Gaussian model, the analysis of Sec. V applies, and, since x is a length, the transformation of λ under scale change is evident. Under the dynamic RG, λ goes to

$$\lambda' = \tfrac{1}{2}s^{2+\epsilon}\lambda \tag{6.11}$$

in view of (6.10). Thus

$$z = 2+\epsilon .$$

This result is an extrapolation from the one-dimensional analysis, and seems natural in the present formulation.[6] It still lacks firm mathematical or physical grounds, however.

VII. DYNAMIC RENORMALIZATION GROUP ON A DISCRETE LATTICE

The extension of the above analysis to models more general and realistic than Gaussian is straightforward in principle, although not quite so in practice. One can use perturbation theory in the present framework to obtain the ϵ expansions of the dynamic exponent. One can also work with approximate recursion formulas like that of Wilson. We shall not go into these approaches here. Instead, let us consider the dynamic RG on a discrete lattice, in a way similar to the static RG studied extensively in the literature.

The following is an oversimplified calculation, working out the dynamic RG transformation on a two-dimensional Ising model lattice. The purpose is to illustrate the basic mechanics of calculation and the necessary truncations. A more serious calculation would follow the same steps but would be more elaborate. It would involve programmed computing, which is not described in this paper.

The model we shall use is the Ising model with a simple form of A defined by Eqs. (3.1) and (3.2). There are two parameters J, the static nearest neighbor interaction, and λ, containing the dynamic information.

Before going into dynamics, we first determine J^*, the fixed point value of J under RG. This is a static problem. The RG transformation of J is obtained as follows. We use the RG defined by "decimation," which eliminates a half of the spins of a square lattice, as shown in Fig. 1. This corresponds to a scale change of $s=\sqrt{2}$. For our calculation, we consider only four spins, σ_1, σ_2, σ_3, σ_4 (note that the subscripts now denote positions, not times, see Fig. 1) and ignore the rest. This is a truncation. We write

$$\begin{aligned} H[\sigma] &= J(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1) \\ &= J(\sigma_1+\sigma_3)(\sigma_2+\sigma_4) . \end{aligned} \tag{7.1}$$

The decimation is then carried out by summing over σ_2 and σ_4:

$$\exp(J'\sigma_1\sigma_3+\text{const.}) = \sum_{\sigma_2,\,\sigma_4} e^{H[\sigma]} . \tag{7.2}$$

The transformed interaction J' is then given by

$$e^{2J'} = \tfrac{1}{2}(1+\cosh 4J) . \tag{7.3}$$

The fixed point value of J and dJ'/dJ are given by

$$\begin{aligned} e^{2J^*} &= 3.38 , \\ \left(\frac{dJ'}{dJ}\right)_{J^*} &= 1.67 . \end{aligned} \tag{7.4}$$

The resulting value of the exponent $\nu = \ln\sqrt{2}/\ln 1.67 = 0.68$ is quite far from the exact value 1. This indicates that this truncated RG is very crude.

To find the fixed point under the dynamic RG, we do the following. First, we determine λ as a function of Δt, i.e., the transformation of λ under time scale change. This function is plotted as the lower curve in Fig. 2. Second, we determine how λ transforms under RG in space. This transformation maps the lower curve of Fig. 2 into the upper curve. The change in space scale is $\sqrt{2}$. Over the range of λ where the two curves are approximately parallel, it is possible to return to the same value of λ after changing the time scale by $(\sqrt{2})^z$ followed by a change in space scale by $\sqrt{2}$. This is shown by the arrows in Fig. 2. The dynamic exponent z is determined by measuring Δ, the length of the horizontal line in Fig. 2. We now proceed to describe these steps in some detail.

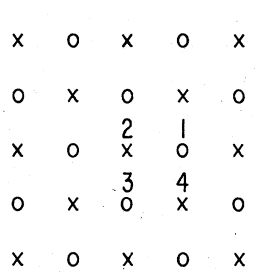

FIG. 1. Decimation eliminates spin variables located at × positions.

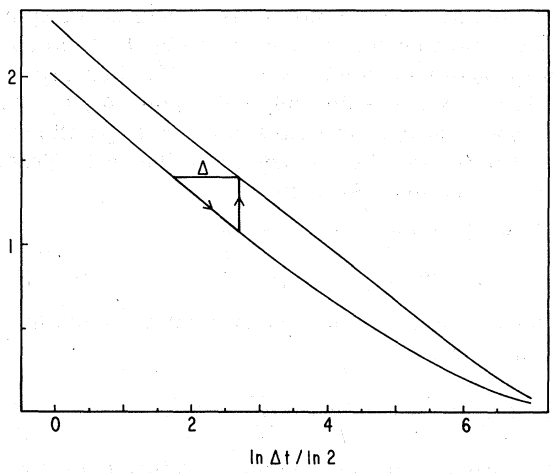

FIG. 2. Lower curve is λ vs Δt, and the upper curve is the mapping of the lower curve by the RG transformation (see Sec. VII).

Step (i): Determination of λ as a function of Δt

Let us now be extremely crude. We take only two spins (not even four) and write

$$A_0[\sigma,\sigma'] = \lambda(\sigma_1\sigma_1' + \sigma_2\sigma_2'),$$
$$A[\sigma,\sigma'] = A_0[\sigma,\sigma'] - V[\sigma] - V[\sigma']. \tag{7.5}$$

We need to find $V[\sigma]$ to satisfy the normalization condition (2.5). $V[\sigma]$ is defined by (3.3) and (3.4), and is obtained from the equation

$$e^{V[\sigma]} = \sum_{\sigma'} \exp(H[\sigma'] - V[\sigma'] + A_0[\sigma,\sigma']). \tag{7.6}$$

Since there are only two spins, we write

$$V[\sigma] = v\sigma_1\sigma_2,$$
$$H[\sigma] = J\sigma_1\sigma_2. \tag{7.7}$$

Additive constants are ignored. Substituting (7.7) and (7.5) in (7.6), we find, after a little algebra,

$$e^{2v} = e^{2J_0}\left(\frac{1+\operatorname{sech}2\lambda e^{-2J_0}}{1+\operatorname{sech}2\lambda e^{2J_0}}\right).$$
$$J_0 \equiv J - v. \tag{7.8}$$

This is a fairly complicated equation for v, but its numerical solution needs only a desk calculator.

Let us proceed to obtain an equation for λ'', the transformed λ when Δt is enlarged by 2. The transformation is defined by (4.3), i.e.,

$$\exp(A''[\sigma,\sigma']) = \sum_{\sigma''} \exp(H[\sigma''] + A[\sigma,\sigma''] + A[\sigma'',\sigma']). \tag{7.9}$$

Since we consider only two spins, we write

$$A''[\sigma,\sigma'] = \lambda''(\sigma_1\sigma_1' + \sigma_2\sigma_2') - V''[\sigma] - V''[\sigma'] \tag{7.10}$$

following the form (7.5). Now Eq. (7.10) implies a truncation. Namely, if we carry out (7.9), we shall have not only terms proportional to $\sigma_1\sigma_1' + \sigma_2\sigma_2'$ in A'' but also those proportional to $\sigma_1\sigma_2' + \sigma_2\sigma_1'$. Thus, (7.10) is really incomplete, and is therefore a truncated version. We now define λ'' by a fitting procedure as follows: We compute $e^{A''}$ for $\sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = 1$, and then compute it for $\sigma_1 = \sigma_2 = -\sigma_1' = -\sigma_2' = 1$. Then we take the ratio of the two results, which, according to (7.10), is $e^{4\lambda''}$. Note that $V[\sigma] = V[-\sigma]$. So the V's drop out when we take the ratio. We obtain

$$e^{4\lambda''} = \frac{\cosh 4\lambda + e^{-2J_-}}{1 + e^{-2J_-}},$$
$$J_- \equiv J - 2v. \tag{7.11}$$

This result, together with (7.8), gives λ''. Starting with any value of λ, we can perform the above transformation, and then repeat it. The sequence of values we get give us $\lambda(\Delta t)$ as a function of Δt for a discrete sequence of values of Δt. We can plot the results on a graph paper and join the points with a smooth curve to get $\lambda(\Delta t)$ as a function of continuous Δt. The lower curve of Fig. 2 follows. Δt is on a logarithmic scale in Fig. 2. For large λ, Eqs. (7.8) and (7.11) give approximately

$$v \approx \tfrac{1}{2}J,$$
$$e^{-2\lambda''} \approx 2e^{-2\lambda}. \tag{7.12}$$

These results are consistent with the short time limit discussed earlier [see (4.11)], and account for the straight portion of the curve in Fig. 2. The value of J used in Fig. 2 is J^* [see (7.4)].

Step (ii): Transformation of λ under RG (space only)

The generalization of (7.2) to include A is

$$P'[\sigma,\sigma'] = \sum_{\sigma_2\sigma_4\sigma_2'\sigma_4'} P[\sigma,\sigma'], \tag{7.13}$$

where σ, σ' on the left-hand side refer to only σ_1, σ_3 and σ_1', σ_3', i.e., only two spins (at two different times) are left after σ_2, σ_4, σ_2', and σ_4' are summed over on the right-hand side. Let us write

$$P'[\sigma,\sigma'] = \exp(H'[\sigma] + H'[\sigma'] + A'[\sigma,\sigma']), \quad \sigma = (\sigma_1,\sigma_3),$$
$$A'[\sigma,\sigma'] = \lambda'(\sigma_1\sigma_1' + \sigma_3\sigma_3') - V'[\sigma] - V'[\sigma'], \tag{7.14}$$
$$P[\sigma,\sigma'] = \exp(H[\sigma] + H[\sigma'] + A[\sigma,\sigma']), \quad \sigma = (\sigma_1,\sigma_2,\sigma_3,\sigma_4).$$

Again, when we carry out (7.13), we shall find A' having not only terms proportional to $\sigma_1\sigma_1' + \sigma_3\sigma_3'$, but also $\sigma_1\sigma_3' + \sigma_3\sigma_1'$. Using only one parameter λ' implies truncation. We shall define λ' by fitting as we did for λ''. We calculate P' for $\sigma_1 = \sigma_3 = \sigma_1' = \sigma_3' = 1$ and call it P'_{++}. Then we calculate P' for $\sigma_1 = \sigma_3 = -\sigma_1' = -\sigma_3' = 1$ and call it P'_{+-}. According to Eq. (7.14), the ratio gives

$$P'_{++}/P'_{+-} = e^{4\lambda'} . \tag{7.15}$$

All the V's drop out when we have taken the ratio. This formula defines λ'. This gives the transformation of λ under a space scale change of $s = \sqrt{2}$. After some algebra, one obtains from Eq. (7.15)

$$\begin{aligned} &\lambda' = \lambda + \tfrac{1}{4}\ln[f(\lambda)/f(-\lambda)] , \\ &f(\lambda) \equiv e^{2\lambda}\cosh 8J_0 + 4\cosh 4J_0 + 2\cosh 2\lambda + e^{-2\lambda} , \\ &J_0 \equiv J - v , \end{aligned} \tag{7.16}$$

where v has to be solved from (7.8). This transformation takes the lower curve $\lambda(\Delta t)$ of Fig. 2 to the upper curve $\lambda'(\Delta t)$.

Step (iii): Determination of the exponent z

The two curves in Fig. 2 tell us how λ changes under changes of time or space scales. The dynamic RG combines the change of space scale s and the change in time scale s^z. If z is chosen appropriately, λ could remain unchanged. Such a value of z can be obtained graphically using Fig. 2. The interval Δ is simply

$$\Delta = \ln(\sqrt{2})^z \tag{7.17}$$

from which we find $z \approx 1.85 \pm 0.15$ for the portion of curves where they are approximately straight. The error indicated is numerical.

Having described how this crude calculation is done, we shall now examine its implications and questions involved.

The above calculation shows clearly the role of truncations or fitting. We kept only two parameters. We used four spins to determine the RG and two spins to determine the time scale transformation. We still needed further truncations to avoid the generation of more parameters. In general, one needs infinitely many parameters to define the transformations completely. In practice, the calculation would always be a fitting procedure involving a small number of parameters.

The truncations and fitting procedures used in the above calculation are quite arbitrary. We have given some ideas on how they can be improved. In fact, the most important general questions are how to optimize the parametrization and fitting programs, and what are the criteria for optimization. Satisfactory answers to these questions are not available. We certainly will get more ideas when a more elaborate calculation is performed.

In the above calculation, we found that λ is invariant under the dynamic RG of a fixed z only over a certain range. (The approximately straight portion of the curves in Fig. 2.) This feature of limitation is expected to be general and to appear in various forms and contexts. In general, we expect that a given parametrization and fitting program can be adequate only in a certain range of values of the parameters. This also implies a limited range of time scale and space scale over which the parametrization and fitting are good. Such limitation is not surprising since any phenomenological model is limited in its validity to a certain range of values of its parameters. The parameters such as J and λ are really phenomenological in nature, not microscopic specifications in the exact solutions in the Onsager tradition.

For the sake of simplicity, the RG transformation used in the above calculation is via decimation. In terms of $R[\mu\,|\sigma]$ [see Eqs. (5.6), (5.9)], we have

$$R[\mu\,|\sigma] = \delta_{\mu_1\sigma_1}\delta_{\mu_2\sigma_3} \tag{7.18}$$

for the four spins σ_1, σ_2, σ_3, and σ_4. In view of the remarks at the end of Sec. V, decimation is not well suited for dynamic RG, although not as bad as the example (5.10). It chooses one of the two spins in each block to be the new spin variable μ. Therefore the variation of μ does not quite reflect that of the total spin in the block. If the model is one which requires the conservation of total spin, this $R[\mu\,|\sigma]$ would be a wrong one to use. Although the total spin is not conserved in the above model, the slowly varying variables are expected to be the long wavelength spin fluctuations, and an $R[\mu\,|\sigma]$ whose μ for a block represents the net spin more faithfully would be better. In principle, the results of the calculation should be independent of R provided that the choice is not too far off and the truncations are very few and mild. We expect, in practice, that a physically more reasonable choice of R would allow faster numerical convergence and the right answers to survive more truncations.

In a more elaborate calculation, there will be many features which the above simple calculation fails to illustrate, for example, the role of A_3 and irrelevant parameters. These will be discussed in future publications.

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[1]For reviews of work done on dynamic RG, see S. Ma, *Modern Theory of Critical Phenomena* (W. A. Benjamin, Inc., Reading, Mass. 1976), Chaps. XI–XIV, and P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977).

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[6]There have been different results, which unfortunately are not in published form and therefore are not quoted here.