Zero-temperature renormalization-group method for quantum systems. II. Isotropic X-Y model in a transverse field in one dimension

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A zero-temperature real-space renormalization-group method is applied to the one-dimensional quantum spin-$\frac{1}{2}$ isotropic X-Y model in a transverse field. A line of unstable fixed points is found and is attributed to the oscillatory behavior of the correlation functions. Magnetization components and critical indices are in good agreement with the exact results. The behavior of the xx correlation function is studied near the critical field.

I. INTRODUCTION

In this paper we continue the study of a new renormalization-group method[1-4] applied to the Hamiltonian

$$H=-\sum_i\left[\tfrac{1}{2}J(1+\gamma)S_i^xS_{i+1}^x+\tfrac{1}{2}J(1-\gamma)S_i^yS_{i+1}^y+hS_i^z\right], \tag{1}$$

where

$$S_i^x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad S_i^y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad S_i^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

are a set of Pauli matrices on each site of an infinite chain with free ends. In the first paper of the series,[4] we studied the case $\gamma=1$, i.e., the Ising model in a transverse field. Here we consider the other limiting case $\gamma=0$, which corresponds to the isotropic X-Y model in a transverse field. The Hamiltonian (1) has been solved for $\gamma=1$ by Pfeuty[5] and for $h=0$ by several authors.[6,7] The general case which has interesting time-dependent properties[8] has been studied by Barouch and McCoy[7] but they could not calculate the correlations functions $\rho_{xx}=\rho_{yy}=\langle 0|S_i^xS_{i+1}^x|0\rangle$ for the isotropic case $\gamma=0$. This calculation has been done only very recently.[9] The isotropic case $\gamma=0$ is equivalent under the Wigner-Jordan transformation to a spinless one-dimensional (1-D) fermion gas without interaction, J being the half bandwidth and h the Fermi-level position. The quantum model (1) is known to be equivalent[10] to the classical 2-D Ising model only if $(h/J)^2+\gamma^2>1$. For $\gamma=0$, in the whole range $|h/J|<1$, this equivalence does not hold and the behavior of the model is specific to its quantum nature.

We present here the first renormalization-group approach to this interesting problem. In Sec. II we present the application of the two-level method already used in Paper I (Ref. 4). In Sec. III, we extend briefly the study by considering a four-level method. Then in Sec. IV we discuss the results. As an interesting result we find a line of unstable fixed points for $|h/J|<1$. This line is certainly related to the oscillatory behavior of the correlation functions already obtained in the case $\gamma\neq 0$.[7] From these results confirmed by a scaling analysis we conclude that the critical exponents are constant for $|h/J|<1$ and change suddenly when $|h/J|=1$. For instance η_x is equal to $\frac{1}{2}$ and jumps to 1 when $|h/J|=|h/J|_c=1$, ν jumps from 1 to $\frac{1}{2}$, and z jumps from 1 to 2. Our approximate renormalization-group method gives η_x roughly constants as well as z for $0<|h/J|\lesssim 0.8$, and a rapid rise towards $\eta_x\sim 1$ and $z\sim 2$ when $|h/J|$ goes to $(h/J)_c$. The known results for μ_z and ρ_{zz} are recovered with reasonable agreement.

II. TWO-LEVEL METHOD

A. Description of the method

Applying the method described in Ref. 4 to Hamiltonian (1) in the isotropic case $\gamma=0$, we consider the following form of the Hamiltonian at the iteration (n):

$$H^{(n)}=-\sum_i\left[\frac{J^{(n)}}{2}(S_i^{x(n)}S_{i+1}^{x(n)}+S_i^{y(n)}S_{i+1}^{y(n)})+h^{(n)}S_i^{z(n)}\right]+C^{(n)}\sum_i I_i^{(n)}. \tag{2}$$

The initial values of $J^{(n)}$, $h^{(n)}$, and $C^{(n)}$ are

$$J^{(0)}=J,\quad h^{(0)}=h,\quad C^{(0)}=0. \tag{3}$$

As in Ref. 4, we divide the chain into adjacent blocks of n_s sites and we write $H^{(n)}$ under the form

$$H^{(n)}=\sum_j\left[H_j^{(n)}+H_{j,j+1}^{(n)}+C^{(n)}\sum_{p=1,\ldots,n_s}I_{j,p}^{(n)}\right], \tag{4}$$

where $H_j^{(n)}$ is the block Hamiltonian

$$H_j^{(n)} = -\frac{J^{(n)}}{2}\sum_{p=1}^{n_s-1}(S_{j,p}^{x(n)}S_{j,p+1}^{x(n)} + S_{j,p}^{y(n)}S_{j,p+1}^{y(n)}) - h^{(n)}\sum_{p=1}^{n_s} S_{j,p}^{z(n)} \tag{5}$$

and $H_{j,j+1}^{(n)}$ is the interblock interaction

$$H_{j,j+1}^{(n)} = -\frac{J^{(n)}}{2}(S_{j,n_s}^{x(n)}S_{j+1,1}^{x(n)} + S_{j,n_s}^{y(n)}S_{j+1,1}^{y(n)}) \quad . \tag{6}$$

The diagonalization of $H_j^{(n)}$ is simplified by observing that $H_j^{(n)}$ commutes with the z projection of the total spin of the block

$$\Sigma_j^z = \sum_{p=1}^{n_s} S_{j,p}^{z(n)} \quad . \tag{7}$$

This special feature of the isotropic case $\gamma = 0$ simplifies considerably the calculations. We diagonalize then $H_j^{(n)}$ separately in each subspace ϵ_j^q corresponding to the eigenvalue $-n_s + 2q - 2$ of Σ_j^z where q varies from 1 to $n_s + 1$. Let us call $|q\rangle$ the ground state of $H_j^{(n)}$ in ϵ_j^q, i.e.,

$$|q\rangle = \sum_{\epsilon_1 \cdots \epsilon_{n_s}}^{q} \lambda_{\epsilon_1,\ldots,\epsilon_{n_s}}^{q} |\epsilon_1,\ldots,\epsilon_{n_s}\rangle \tag{8}$$

as in Ref. 4, $|\epsilon_1 \cdots \epsilon_{n_s}\rangle$ is the basis vector where $\epsilon_p = \pm 1$ represents the eigenstate of $S_{j,p}^{z(n)}$ in the block and $\sum^q$ designates a restricted sum on $\epsilon_1,\ldots,\epsilon_{n_s}$ in the subspace ϵ_j^q. Observing that all the diagonal elements of the matrix representing $H_j^{(n)}$ are equal, the energy E_q of $|q\rangle$ is given by

$$E_q = (n_s - 2q + 2)h^{(n)} + e_q J^{(n)} \quad , \tag{9}$$

where e_q is the lowest eigenvalue of the dimensionless nondiagonal operator

$$-\frac{1}{2}\sum_{p=1}^{n_s-1}(S_p^x S_{p+1}^x + S_p^y S_{p+1}^y) \quad . \tag{10}$$

The cases $q = 1$ and $q = n_s + 1$ are trivial

$$e_1 = e_{n_s+1} = 0; \quad |1\rangle = |---\cdots-\rangle; \quad |n_s+1\rangle = |+++\cdots+\rangle \quad . \tag{11}$$

The cases $q = 2$ and $q = n_s$ can be obtained analytically

$$e_2 = e_{n_s} = -2\cos\frac{\pi}{n_s+1}, \quad |2\rangle = \left(\frac{2}{n_s+1}\right)^{1/2}\sum_{p=1}^{n_s}\sin\frac{p\pi}{n_s+1}\phi_p \quad , \tag{12}$$

where ϕ_p has all the ϵ_i equal to -1 except $\epsilon_p = +1$. $|n_s\rangle$ is obtained from $|2\rangle$ by spin symmetry. In other cases $(2 \leq q \leq n_s - 1)$ e_q and $\lambda_{\epsilon_1\cdots\epsilon_{n_s}}^{q}$ have been computed by machine.[11]

We define a q-renormalization-group transformation by retaining only the adjacent states $|q\rangle$ and $|q+1\rangle$ and considering them as $|-\rangle^{(n+1)}$ and $|+\rangle^{(n+1)}$ states, respectively, of the z projection of a new spin $\frac{1}{2}$, $\vec{S}_j^{(n+1)}$, attached to the block j. The block Hamiltonian $H_j^{(n)}$ can be rewritten under the form

$$H_j^{(n)} = -h^{(n+1)}S_j^{z(n+1)} + \frac{E_q + E_{q+1}}{2} I_j^{(n+1)} \quad , \tag{13}$$

where the new field $h^{(n+1)}$ is given by

$$h^{(n+1)} = h^{(n)} + \tfrac{1}{2}(e_q - e_{q+1})J^{(n)} \quad . \tag{14}$$

Taking the matrix element of the old spin operator $S_{j,p}^{x(n)}$ between the block states $|\pm\rangle^{(n+1)}$ we obtain the following spin recursion relation:

$$S_{j,p}^{x(n)} = \xi_p^q S_j^{x(n+1)} \tag{15}$$

with

$$\xi_p^q = \sum_{\epsilon_1 \cdots \epsilon_{n_s}}^{q} \lambda_{\epsilon_1\cdots\epsilon_p\cdots\epsilon_{n_s}}^{q}\lambda_{\epsilon_1,\ldots,-\epsilon_p\cdots\epsilon_{n_s}}^{q+1} \quad . \tag{16}$$

The same relation holds for $S_{j,p}^{y(n)}$. As in Ref. 4, for symmetry reasons we have $\xi_p^q = \xi_{n_s-p+1}^q$.[11] Relation (15) allows us to rewrite the interblock interaction (6) under the form

$$H_{j,j+1}^{(n)} = -\frac{J^{(n+1)}}{2}(S_j^{x(n+1)}S_{j+1}^{x(n+1)} + S_j^{y(n+1)}S_{j+1}^{y(n+1)}) \tag{17}$$

with

$$J^{(n+1)} = (\xi_1^q)^2 J^{(n)} \quad . \tag{18}$$

Then, if we consider the new constant to be

$$C^{(n+1)} = n_s C^{(n)} + (n_s - 2q + 1)h^{(n)} + \tfrac{1}{2}(e_q + e_{q+1})J^{(n)} \quad , \tag{19}$$

the Hamiltonian takes the same form at each iteration.

The constant $C^{(n)}$ is useful to find the ground state energy per site which is given by

$$\frac{E_0}{N} = \lim_{n\to\infty}\frac{C^{(n)}}{n_s^n} = \frac{1}{n_s}\sum_{n=0}^{\infty}\left[(n_s - 2q + 1)\frac{h^{(n)}}{n_s^n} + \tfrac{1}{2}(e_q + e_{q+1})\frac{J^{(n)}}{n_s^n}\right] \quad . \tag{20}$$

An important difference with Paper I (Ref. 4) is that the numbers e_q and $\lambda_{\epsilon_1\cdots\epsilon_{n_s}}^{q}$ (and thus ξ_1^q) do not depend on the iteration number. As a consequence the recursion relations can be integrated easily.

B. Fixed points and discussion of the recursion relations

From Eqs. (14) and (18), we get the recursion relation for the dimensionless parameter h/J,

$$\left[\frac{h}{J}\right]^{(n+1)} = \frac{1}{(\xi_1^q)^2}\left[\left[\frac{h}{J}\right]^{(n)} + \frac{1}{2}(e_q - e_{q+1})\right] . \quad (21)$$

This linear relation is represented by a straight line in the plane $(h/J)^{(n)}$ - $(h/J)^{(n+1)}$. We find a fixed point at the intersection of this straight line and the bissectrice of the axis

$$\left[\frac{h}{J}\right]_q^* = \frac{1}{2}\frac{e_{q+1} - e_q}{1 - (\xi_1^q)^2} . \quad (22)$$

Since $(\xi_1^q)^2$ is smaller than 1, this fixed point is unstable. Starting with $h/J \neq (h/J)_q^*$ we always converge towards the stable fixed points $(h/J)^* = \pm\infty$. Considering all possible renormalization-group transformations, we find n_s unstable fixed points of this sort. An example for $n_s = 6$ is given in Fig. 1 where six fixed points are obtained from the different straightlines (21) for $q = 1,\ldots,n_s$. For $q = 1$ and $q = n_s$ we get the symmetrical values

$$\left[\frac{h}{J}\right]_{n_s}^* = -\left[\frac{h}{J}\right]_1^* = \frac{\cos\frac{\pi}{n_s+1}}{1 - \frac{2}{n_s+1}\sin^2\frac{2\pi}{n_s+1}} < 1 , \quad (23)$$

between which all the other fixed points are located. When $n_s \to \infty$, expression (23) tends to 1 and the n_s fixed points tend to form a semicontinuum covering the segment -1, $+1$.

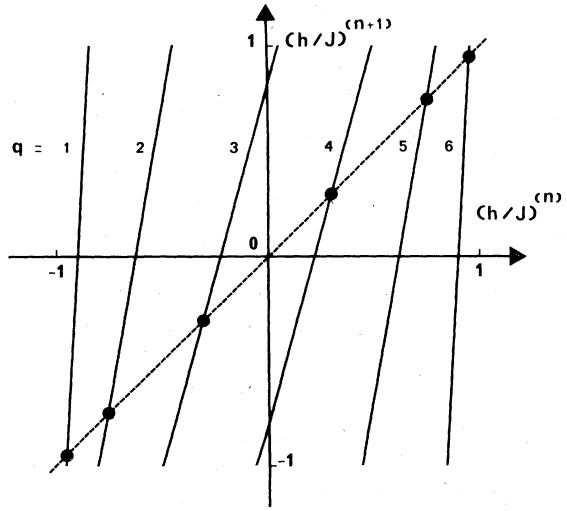

FIG. 1. Diagram in the plane $(h/J)^{(n)}$-$(h/J)^{(n+1)}$ showing the linear recursion relations for the dimensionless parameter h/J for each value of q. The fixed points are obtained by cutting by the bissectrice of the axis.

Knowing the fixed point the recursion relation can be integrated to find the limiting values h^∞ and J^∞ of $h^{(n)}$ and $J^{(n)}$ and to find E_0/N for the q-renormalization-group transformation

$$J^{(n)} = (\xi_1^q)^{2n} J \to J^\infty = 0 , \quad (24)$$

$$h^{(n)} = h - \left[\frac{h}{J}\right]_q^* J[1 - (\xi_1^q)^{2n}] \to h^\infty = h - \left[\frac{h}{J}\right]_q^* J , \quad (25)$$

$$\frac{E_0}{N} = \frac{n_s - 2q + 1}{n_s - 1} h^\infty + \frac{(n_s - 2q + 1)\left[\frac{h}{J}\right]_q^* + \frac{1}{2}(e_q + e_{q+1})}{n_s - (\xi_1^q)^2} J . \quad (26)$$

Thus, remaining within the q-renormalization-group transformation we end up with a finite gap for the whole chain, except in the special case where h/J is initially chosen to be just equal to the corresponding fixed point $(h/J)_q^*$. In this special case both $h^{(n)}$ and $J^{(n)}$ tend to zero with a constant ratio, we get $h^\infty = 0$, and the system has a degenerated ground state. Moreover if $h/J = (h/J)_q$ we observe that $|q\rangle$ and $|q+1\rangle$ always remain the two lowest energy states at any iteration step. This justifies physically to use the corresponding q-renormalization-group transformations in this case.

When $h/J > (h/J)_{n_s}^*$ [respectively $h/J < (h/J)_1^*$] $|n_s\rangle$ and $|n_s+1\rangle$ (respectively $|1\rangle$ and $|2\rangle$) remain also the two lowest energy states. Then, using the n_s-(respectively 1)-renormalization-group transformation we find a finite gap $\Delta = 2h^\infty$ and formula (25) gives a linear behavior for this gap, with an exponent s equal to one, as in the exact solution.[6] The gap is opened at the critical ratio $(h/J)_c = (h/J)_{n_s}^*$ which is near the exact value $(h/J)_c = 1$, this critical ratio tending to one when $n_s \to \infty$.

Let us now consider the case where we start with an initial ratio h/J located between two unstable fixed points, one of them being $(h/J)_q^*$. In this case we observe that $|q\rangle$ and $|q+1\rangle$ do not stay always the lowest energy states of the system. The q-renormalization-group transformation becomes more or less rapidly physically irrelevant. If we impose to keep the two lowest energy states, we must change the q value at each iteration. Doing so, we can find, for special values of h/J, some cyclic fixed points, also unstable.[12] But, in the general case, after jumping from one q value to another, we finally end up in the range $|h/J| > (h/J)_{n_s}^*$.

Eliminating this unphysical last case, we have chosen to calculate the quantities of interest only just at a fixed point or in the range $|h/J| > (h/J)_{n_s}^*$.

C. Magnetization components and critical indices

We have calculated the magnetization components in the ground state in a slight different way than in Ref. 4. Instead of considering the recursion relations for a spin at the center (or near the center of the block) we have considered the recursion relations of the following block averages:

$$\sigma^x=\frac{1}{n_s}\sum_{p=1}^{n_s} S_p^x \text{ and } \sigma^z=\frac{1}{n_s}\sum_{p=1}^{n_s} S_p^z=\frac{1}{n_s}\Sigma^z \ . \tag{27}$$

Here also, by averaging, we avoid edge effects. This choice is justified by the fact that σ^z is a good quantum number. The recursion relations are

$$\sigma^{x(n+1)}=\left[\frac{1}{n_s}\sum_{p=1}^{n_s}\xi_p^q\right]\sigma^{x(n)} \ ,$$
$$\sigma^{z(n+1)}=\frac{1}{n_s}\sigma^{z(n)}-\frac{n_s-2q+1}{n_s}I^{(n)} \ . \tag{28}$$

We find that the x magnetization μ_x is always zero (as well as μ_y) as in the exact solution and the z magnetization is given by

$$\mu_z=\langle\sigma^{z(0)}\rangle=\frac{-n_s+2q-1}{n_s-1} \ . \tag{29}$$

Thus, we find $\mu_z=\pm 1$ for $|h/J|\geq(h/J)^*_{n_s}$ as it is for $|h/J|\geq(h/J)_c$ in the exact solution. Moreover we observe that $-\mu_z$ corresponds to the derivative of the ground-state energy, given by Eq. (26), with respect to h. (This was not the case in Ref. 4).

The results for μ_z, as well as for E_0/N, are given in Fig. 2 (left-hand part) for each fixed point in the range $0<h/J<1$ for $n_s=6$ and $n_s=8$ and compared with the exact results[7] (dashed curves)

$$\mu_z=\frac{2}{\pi}\arcsin\left(\frac{h}{J}\right) \ ,$$
$$-\frac{E_0}{NJ}=\frac{2}{\pi}\left\{\left[1-\left(\frac{h}{J}\right)^2\right]^{1/2}+\frac{h}{J}\arcsin\left(\frac{h}{J}\right)\right\} \ . \tag{30}$$

From this renormalization-group method we obtain directly (as in Ref. 4) the critical behavior associated with the "line" of unstable fixed points. At each fixed point we get the critical exponents z, ν, $\eta_x=\eta_y$, and η_z where ν is a "correlation length" exponent, z is an exponent related to the energy dilatation, and η_x and η_z give the power-law behavior of the x-x and z-z correlation function in the ground state. These exponents have been more extensively defined in Ref. 4. We find that for any fixed point $z\nu=s=1$ and z is given by

$$z=1/\nu=-2\log(\xi_1^q)/\log n_s \ . \tag{31}$$

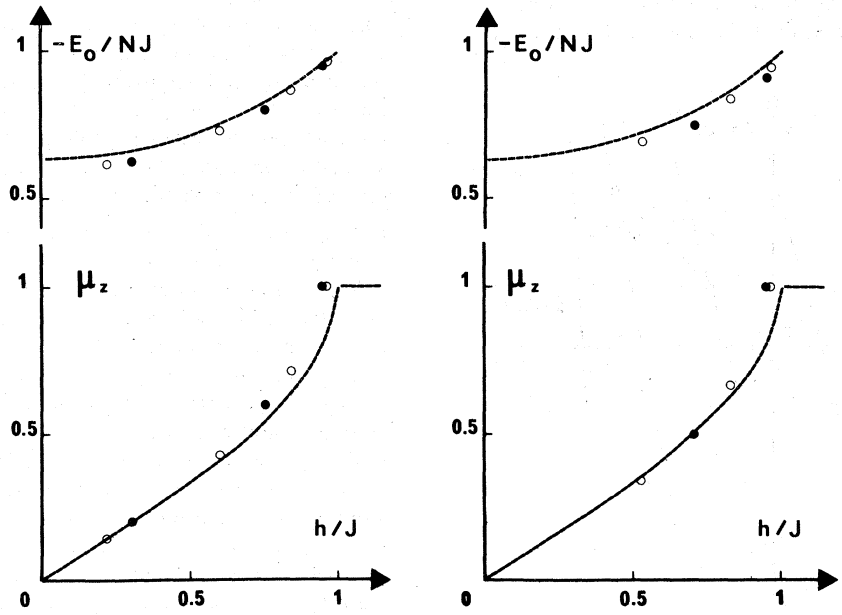

FIG. 2. Numerical results for μ_z and $-E_0/NJ$ at each fixed point for positive h/J. The left-hand side corresponds to $n_L=2$ with $n_s=6$ (full circles) and $n_s=8$ (open circles), and the right-hand side to $n_L=4$ with $n_s=3$ (full circles) and $n_s=4$ (open circles). Exact results[5–7] are represented by dashed curves.

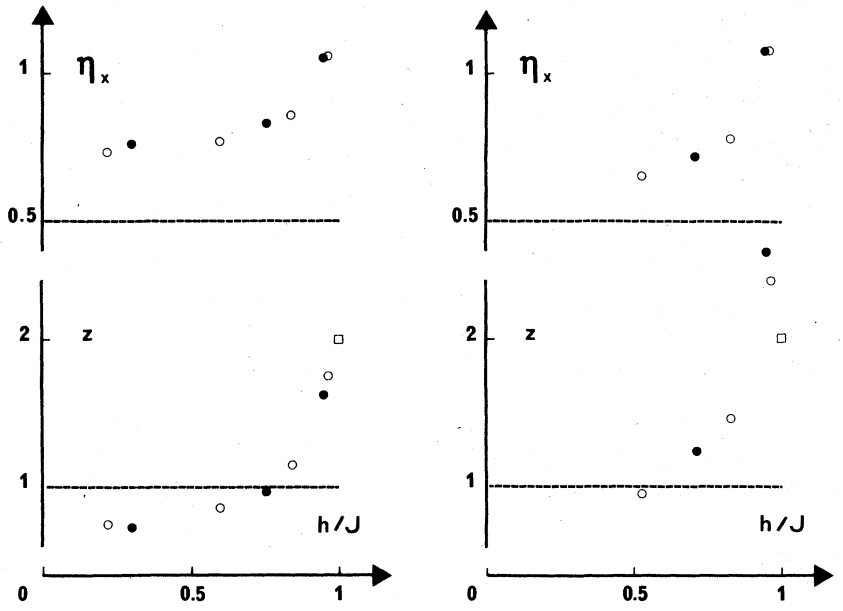

FIG. 3. Numerical results for the critical exponents z and η_x at each fixed point for positive h/J. The left-hand side corresponds to $n_L=2$ with $n_s=6$ (full circles) and $n_s=8$ (open circles), and the right-hand side to $n_L=4$ with $n_s=3$ (full circles) and $n_s=4$ (open circles). Exact results (Refs. 5–9) are represented by dashed lines and open squares.

Moreover $\eta_x=\eta_y$ and η_z are given by

$$\eta_x=-2\log\left|\frac{1}{n_s}\sum_{p=1}^{n_s}\xi_p^q\right|/\log n_s \ , \qquad \eta_z=-2\log(1/n_s)/\log n_s=2 \ . \tag{32}$$

$\eta_z=2$ corresponds to the exact result and this is consistent with the scaling law $\eta_z=2[1+(1/\nu)(s-1)]$. The results for z and η_x for each fixed point in the range $0<h/J<1$ are given in Fig. 3 (left-hand part) for $n_s=6$ and $n_s=8$. For the last fixed point $(h/J)_{n_s}{}^*$ we find analytically that $\eta_x\to 1$ and $z\to 3$ (and thus $\nu\to\frac{1}{3}$) when $n_s\to\infty$.

III. FOUR-LEVEL METHOD

The preceeding study has been checked and completed by taking four levels at each iteration ($n_L=4$). There, we consider initially sites with two spins $\frac{1}{2}$ as shown in Fig. 4. Then at the step (n) of the iteration we must consider the following Hamiltonian:

$$H^{(n)}=-\sum_i (T_i^{+(n)}S_{i+1}^{-(n)}+T_i^{-(n)}S_{i+1}^{+(n)}) - \sum_i (2h^{(n)}S_i^z+J'^{(n)}D_i+C^{(n)}\sum_i I_i^{(n)}) \ , \tag{33}$$

where in the two-spin-site representation we have

$$t^-=|--\rangle, \quad t^0=\frac{|+-\rangle+|-+\rangle}{2^{1/2}}, \quad s=\frac{|+-\rangle-|-+\rangle}{2^{1/2}}, \quad t^+=|++\rangle \ ; \tag{34}$$

$T^{+(n)}$, $S^{+(n)}$, S^z, and D are 4×4 matrices given by

$$T^{+(n)}=\begin{pmatrix}0&0&0&0\\ a^{(n)}&0&0&0\\ -b^{(n)}&0&0&0\\ 0&b^{(n)}&a^{(n)}&0\end{pmatrix}, \quad S^{+(n)}=\begin{pmatrix}0&0&0&0\\ a^{(n)}&0&0&0\\ b^{(n)}&0&0&0\\ 0&-b^{(n)}&a^{(n)}&0\end{pmatrix}, \quad S^z=\begin{pmatrix}-1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&+1\end{pmatrix}, \quad D=\begin{pmatrix}0&0&0&0\\ 0&+1&0&0\\ 0&0&-1&0\\ 0&0&0&0\end{pmatrix}, \tag{35}$$

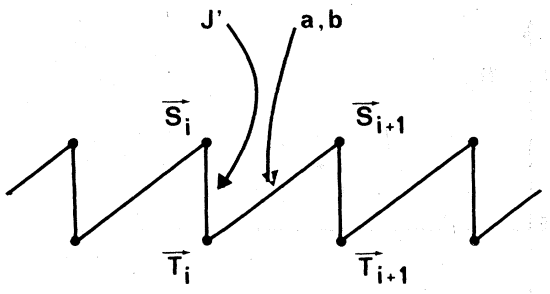

FIG. 4. Sketch of the $n_L = 4$ method. Each site contains two spins $\frac{1}{2}\vec{S}i$ and $\vec{T}i$. The initial interspin interaction J is replaced at step (n) by an intrasite parameter J' and two intersite parameters a and b.

with $T^{-(n)}$ and $S^{-(n)}$ the transposed of $T^{+(n)}$ and $S^{+(n)}$. The initial values of the parameters $a^{(n)}$, $b^{(n)}$, $h^{(n)}$, and $J'^{(n)}$ are

$$a^{(0)} = b^{(0)} = (J/2)^{1/2} ,$$
$$h^{(0)} = h , \tag{36}$$
$$J'^{(0)} = J .$$

After iterating, the parameter space becomes larger with $a^{(n)} \neq b^{(n)}$. A systematic study in the whole parameter space is not in the scope of this paper; it would correspond to the study of two coupled X-Y chains in a field (another example but without a field has been studied in Ref. 3). Here we restrict our discussion by varying the unique initial parameter h/J; in this one-dimensional parameter space we get "critical points" instead of "fixed points" as in the preceding study.

We proceed as before by considering

$$\Sigma^z = \sum_{p=1}^{n_s} S_p^z , \tag{37}$$

which commutes with the block Hamiltonian. We consider again q subspaces corresponding now to the eigenvalues $-n_s + q - 1$ of Σ^z, where q varies from 1 to $2n_s + 1$. We define a q-renormalization-group transformation by considering the two lowest energy states of the q subspace as $t^{0(n+1)}$ and $s^{(n+1)}$ states for the block, $t^{0(n+1)}$ being the ground state and $s^{(n+1)}$ being the first excited state, and by considering the ground states of the $q-1$ and $q+1$ subspaces as, respectively, $t^{-(n+1)}$ and $t^{+(n+1)}$ states for the block. The new values $h^{(n+1)}$ and $J'^{(n+1)}$ are obtained from the relative differences of energy of these four states and the new values $a^{(n+1)}$, $b^{(n+1)}$ are obtained by expressing the old matrices $T^{+(n)}$, $S^{+(n)}$ for the site at the edge of a block in term of the new states.

This has been done by machine. Let us exclude the case $q = n_s + 1$, which corresponds to $h = 0$ and for which we do not observe a critical behavior if we start with $J' = J$.[13] For all the other values [$2(n_s - 1)$ values] we find a critical ratio $(h/J)_q^c$ having all the features of the fixed points found in Sec. II. For the critical exponents we find $\eta_z = 2$ and $s = \nu z = 1$. The results for μ_z, E_0/N, z, and η_x obtained numerically for each critical point in the range $0 < h/J < 1$, are reported in the right-hand side of Figs. 2 and 3 for $n_s = 3$ and $n_s = 4$.

To conclude, let us note that this method can be extended to higher values of n_L. The number of critical points will depend on both n_s, n_L and the way the states are chosen but the main features of the results will be conserved.

IV. DISCUSSION OF THE RESULTS AND CONCLUSIONS

One of the interesting features of these results is the existence of a line of unstable fixed points in the low-field region. This line of fixed points is certainly due to the oscillatory behavior of the correlation functions.[6(b),7] This oscillatory behavior has been established for $\gamma = 0$ $h = 0$ and when $\gamma \neq 0$ for all h below the critical value.[9] The correlation functions contain terms of the form real $e^{2i\psi R}$ where ψ is a function of h/J [the exact result is $\psi = \arccos(h/J)$].[7] $i\psi^{-1}$ is like an imaginary correlation length which diverges when $\psi = 0$ (respectively $\psi = \pi$) at a critical point $h/J = (h/J)_c$. In a renormalization-group transformation where the length scale is changed by a factor n_s, the phase factor stays unchanged for n_s values of $\psi = \psi_p = p\pi/(n_s - 1)$ $(p = 0, \ldots, n_s - 1)$ leading to n_s fixed points $(h/J)_p^* = F(\psi_p)$, as shown previously for the case $n_L = 2$ [for $n_L = 2^m$ we get by the same argument $n_s m - (m - 1)$ fixed points].

Among the exponents (ν, z, η_x, η_z) associated to each fixed point ν tells how ψ depends on h close to ψ_p, $\psi - \psi_p \sim [h/J - (h/J)_p^*]^{\nu}$. From the exact result[7] ν is expected to be one except for the last fixed point when $\psi = 0$ where $\nu = \frac{1}{2}$. z is known to be equal to one except for the last fixed point where it is equal to two.[7] From a continuum generalization[14] of the spin Hamiltonian considered here, we can show that when $|h/J| < (h/J)_c$ the effect of h/J is just to change the Fermi velocity of the equivalent fermion gas[15] but the exponent η_x remains unchanged equal to $\frac{1}{2}$ as shown exactly recently[9]; however for $h/J = (h/J)_c$ when the gap opens the continuum approach fails. The main new result of this work is to show that when $h/J = (h/J)_c$ η_x jumps suddenly to a new value $\eta_x = 1$. This result is consistent with what is known for $\rho_{xx}(R)$ when $\gamma \neq 0$ but small $\to 0$ and $h/J \to (h/J)_c = 1$,[6] if we assume that scaling is obeyed. We propose then that when we have $h/J \to (h/J)_c = 1$, $\rho_{xx}(R)$ behaves like

$$\rho_{xx}(R) \sim \frac{1}{R} G\left(\frac{R}{R_c}\right) \mathrm{Re}\left\{e^{2i\psi R + i\phi(h/J)}\right\} , \tag{38}$$

where $R_c \sim (1-h/J)^{-1/2}$ and $G(x \to 0) =$ constant and $G(x \to \infty) \sim x^{1/2}$

$$\phi(1) = \pi/2 \ ,$$

$$\psi = \arccos(h/J) \ .$$

Our results shown in Fig. 3 agree with the preceding picture. The exponents z and η_x are fairly constant up to h/J close to $(h/J)_c$. We do not find the predicted discontinuity of the exponents but only some kind of fast "crossover" due to the approximations. The two levels and the four levels schemes give similarly results when $h/J \neq (h/J)_c$ (the four-level scheme is slightly better). At $(h/J)_c$ the results become much better when we go from $n_L = 2$ to $n_L = 4$: for $n_L = 2$ z tends to 3 and ν tends to $\frac{1}{3}$ when n_s increases; for $n_L = 4$ no analytical results are available but it seems that z converges towards the exact result 2 and ν converges towards $\frac{1}{2}$.

These results give much confidence in the method which could be applied to other quantum systems and extended to higher dimensionality. The study of the transverse Ising model in two dimensions is achieved.[16] Other quantum systems will be studied in the near future.

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[11]We have used a further simplification which considers only the block-symmetric states (the block-antisymmetric states being much more excited).

[12]For example

$$(h/J)^* = (h/J)_q^* \frac{1-(\xi_1^q)^2}{1+(\xi_1^q)^2}$$

is a cyclic fixed point where h changes into $-h$ at each iteration. It has the same critical indices as the "regular" fixed point.

[13]Case $h = 0$, with $J \neq J'$, corresponds to the dimerized X-Y chain recently studied with similar methods by J. N. Fields (unpublished). In this case we have always $a^{(n)} = b^{(n)}$ and Eq. (1) contains only one dimensionless parameter J/J'. In our calculation the critical value $(J/J')_c$ is not exactly 1 due to our two-spins initial blocking, which does not respect the $J \leftrightarrow J'$ symmetry. See also, R. Jullien and J. N. Fields, Phys. Lett. A 69, 214 (1978).

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