Interaction energy of superconducting vortices

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By means of a constrained variational calculation we determine the interaction energy of two-vortex configurations in the Ginzburg-Landau theory or, equivalently, in the Abelian Higgs model. The energy is evaluated as a function of the separation between vortices and of the parameter λ, which measures the relative strength of the matter self-coupling and the electromagnetic coupling. Our results provide a precise determination of the inter-vortex potential, attractive for $\lambda < 1$ and repulsive for $\lambda > 1$. They also show that for $\lambda = 1$ the lower bound on the energy which can be then derived is actually reached at all separations and, therefore, that in this case vortices do not interact.

I. INTRODUCTION

It has been known for a long time that the Ginzburg-Landau equations[1] admit localized solutions of the vortex type.[2] More recently it has been observed that vortices of essentially the same structure appear as solutions to model field theories for strongly interacting particles.[3] A typical model, known as the Abelian Higgs model, describes the interaction of a matter (Higgs) field (analogous to an order parameter) with an Abelian gauge field, the electromagnetic potential in the Ginzburg-Landau theory, in a minimal gauge-invariant fashion. The energy functional, and the field equations which follow from its minimization, are mathematically identical to those in the Ginzburg-Landau theory.

The purpose of this paper is to present a highly accurate variational computation of the interaction of vortices for arbitrary separation. Our starting point is a trial configuration where the matter field ϕ vanishes at two points, the locations of the vortices, and which reduces, for large separation, to two single-vortex configurations. The trial field is then modified in the interaction region, constraining the zeros of ϕ, until the energy is minimized. The minimum can be interpreted as the energy of two vortices kept at a fixed separation. The main relevance of our results lies in a complete description of the behavior of the interaction energy as a function of the separation and of the coupling constant which measures the relative strength of the matter self-coupling and the electromagnetic coupling. The energy functional contains three coupling constants, but two inessential ones may be eliminated by straightforward rescaling of the fields. The remaining coupling constant λ (related to the Ginzburg-Landau parameter) is physically significant. With our normalization convention, in the sector where $\lambda < 1$ the range of the matter self-interaction exceeds that of the electromagnetic one, whereas for $\lambda > 1$ the opposite is true. Materials with $\lambda < 1 (\lambda > 1)$ exhibit Type I (Type II) superconductivity. The interaction between widely separated vortices has been studied and it has been found that, asymptotically, vortices attract (repel) each other for $\lambda < 1 (\lambda > 1)$.[4] Also, the stability of a rotationally symmetric configuration of many superimposed vortices has been investigated: it is found that for $\lambda > 1$ the system is unstable against decay into separated vortices.[5] In the case where $\lambda = 1$, it is possible to derive a lower bound on the energy.[6] The bound is saturated if the fields satisfy a set of first-order (nonlinear) partial differential equations and the energy is then proportional to the number of vortices. These first-order equations have been solved for two vortices at zero separation[7] and, of course, are solved by a configuration of infinitely separated vortices. Since the bound is additive in the number of vortices, the existence of solutions to the first-order equations for arbitrary separations implies that $\lambda = 1$ vortices do not interact. However, solutions with finite separation between vortices have not been constructed nor existence proofs been given, and therefore one can not infer the absence of interaction from the bound.

Our results show that two vortices attract each other for $\lambda < 1$ and repel for $\lambda > 1$ at all separations. Moreover, for $\lambda = 1$ we find that the energy of a two-vortex field configuration is constant as a function of the separation within an error of less than two parts in 10^4, attributable to the method of approximation. We thus obtain a very strong indication that $\lambda = 1$ vortices do not interact, implying a high degeneracy of the solutions of the Ginzburg-Landau equations for this value of the coupling constant.

The paper is organized as follows: In Sec. II we discuss the asymptotic behavior of the fields, derive the bound on the energy, and summarize our results. In Sec. III we construct the variational *Ansatz* and explain the procedure followed to minimize the energy functional. Section IV contains a discussion of the results.

II. THE ENERGY FUNCTIONAL AND ITS MINIMUM VALUE

The expression for the free energy in the Ginzburg-Landau theory (or equivalently the potential energy in the Abelian Higgs model) is

$$E=\int d^3x\,[\tfrac{1}{2}|(\partial_i-ieA_i)\phi|^2+\tfrac{1}{4}F_{ij}F^{ij}+c_4(|\phi|^2-c_0^2)^2]\ . \tag{2.1}$$

ϕ is a complex scalar field, A_i is the gauge potential, and $F_{ij}=\partial_iA_j-\partial_jA_i$ is the field strength.[8] The minimum of the energy density is reached when $|\phi|=c_0\neq 0$. Rescaling lengths and fields as follows:

$$x^i=\frac{1}{ec_0}\tilde{x}^i,\quad \phi=c_0\tilde{\phi},\quad A_i=c_0\tilde{A}_i\ , \tag{2.2}$$

the energy functional is written

$$E=\frac{c_0}{e}\int d^3\tilde{x}\,[\tfrac{1}{2}|(\tilde{\partial}_i-i\tilde{A}_i)\tilde{\phi}|^2+\tfrac{1}{4}\tilde{F}_{ij}\tilde{F}^{ij}+\tfrac{1}{8}\lambda^2(|\tilde{\phi}|^2-1)^2]\ , \tag{2.3}$$

where

$$\lambda^2=8c_4/e^2\ . \tag{2.4}$$

We shall be interested in field configurations invariant under translations along a definite axis. Taking this one to be the third axis, the fields depend only on the coordinates x_1, x_2, and $A_3=0$. It is then convenient to introduce complex coordinates

$$z=\tilde{x}_1+i\tilde{x}_2,\quad \bar{z}=\tilde{x}_1-i\tilde{x}_2 \tag{2.5}$$

and a complex potential

$$A=\tfrac{1}{2}(\tilde{A}_1-i\tilde{A}_2),\quad \bar{A}=\tfrac{1}{2}(\tilde{A}_1+i\tilde{A}_2)\ . \tag{2.6}$$

In terms of these variables the energy per unit length along the third axis is

$$E=\frac{c_0\pi}{e}\mathcal{E} \tag{2.7}$$

with

$$\mathcal{E}=\frac{1}{2\pi}\int dz\,d\bar{z}\,[\,|(\partial-iA)\tilde{\phi}|^2+|(\bar{\partial}-i\bar{A})\tilde{\phi}|^2+2|\bar{\partial}A-\partial\bar{A}|^2+\tfrac{1}{8}\lambda(|\tilde{\phi}|^2-1)^2]\ . \tag{2.8}$$

In this last equation ∂, $\bar{\partial}$ stand for

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial\tilde{x}_1}-i\frac{\partial}{\partial\tilde{x}_2}\right),\quad \frac{\partial}{\partial\bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial\tilde{x}_1}+i\frac{\partial}{\partial\tilde{x}_2}\right),$$

and the factor $1/\pi$ has been introduced for later convenience. The tilde over ϕ will be omitted henceforth.

For $\mathcal{E}$ to be finite, $|\phi|$ must tend to 1 as $|z|$ goes to infinity. Hence ϕ must approach the value $e^{i\chi(\theta)}$ as $z\rightarrow\infty$ with fixed argument θ. Continuity demands that we have $e^{i\chi(\theta+2\pi)}=e^{i\chi(\theta)}$ and therefore

$$\chi(\theta+2\pi)=\chi(\theta)+2\pi n\ , \tag{2.9}$$

with integer n. The local behavior of the phase χ as a function of θ is immaterial because it can be changed at will through a gauge transformation. But the integer n has physical significance. Indeed, finiteness of $\mathcal{E}$ also requires

$$\lim_{|z|\rightarrow\infty}(\partial-iA)\phi=0\ , \tag{2.10}$$

or

$$A=-i\,\partial\ln\phi+o\left(\frac{1}{|z|}\right)=\partial\chi+o\left(\frac{1}{|z|}\right)\ , \tag{2.11}$$

as $|z|\rightarrow\infty$. The total magnetic flux through the plane then follows from the Gauss theorem:

$$\Phi(B)=\frac{-i}{e}\int dz\,d\bar{z}\,(\partial\bar{A}-\bar{\partial}A)=\frac{1}{e}\lim_{|z|\rightarrow\infty}\oint(A\,dz+\bar{A}\,d\bar{z})=\frac{1}{e}\oint d\chi=\frac{2\pi n}{e}\ . \tag{2.12}$$

Thus, finite-energy field configurations are divided into classes, labelled by n. Each class contains all field configurations (with finite $\mathcal{E}$) which can be continuously distorted into each other (for this reason n is called a topological invariant) and within each class the total magnetic flux is $2\pi n/e$.

Continuity also imposes a relation between n and the number of zeros of ϕ. The integral

$$n_\gamma=\frac{-i}{2\pi}\oint_\gamma d\ln\phi\ , \tag{2.13}$$

along any closed contour γ, takes an integer value which can change under continuous deformations of the contour only when it goes through a zero of ϕ. Assuming that there are n_+ points $z_i^{(+)}$, where ϕ vanishes as $z-z_i^{(+)}$ and n_- points $z_i^{(-)}$ where it vanishes as $\bar{z}-\bar{z}_i^{(-)}$, taking for γ, first, a circle which encloses all the points z_i and letting then the radius of γ decrease to zero, one readily verifies that

$$n=n_+-n_-\ . \tag{2.14}$$

We shall say that the field configuration exhibits a vortex (an antivortex) located at z_i, if ϕ has there a zero of the type $z - z_i(\bar{z} - \bar{z}_i)$. The topological invariant n is then equal to the number of vortices minus the number of antivortices.

Fields which make $\mathcal{E}$ stationary obey the Euler-Lagrange equations,

$$(\partial - iA)(\bar{\partial} - i\bar{A})\phi + (\bar{\partial} - i\bar{A})(\partial - iA)\phi - \tfrac{1}{4}\lambda^2\phi(\phi\bar{\phi} - 1) = 0 \ , \tag{2.15}$$

$$4\partial\bar{\partial}A - 4\partial^2 A - i\bar{\phi}\partial\phi + i\phi\partial\bar{\phi} - 2A\phi\bar{\phi} = 0 \ . \tag{2.16}$$

One can insert into these equations a rotationally symmetric *Ansatz*

$$\phi = e^{in\theta}f(r), \quad A = -(ni/2z)a(r), \quad r \equiv |z| \ , \tag{2.17}$$

with $f(\infty) = a(\infty) = 1$. The symmetry is compatible with the Euler-Lagrange equations, which reduce to the following ordinary nonlinear differential equations for $f(r)$ and $a(r)$:

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2(a-1)^2}{r^2}f - \tfrac{1}{2}\lambda^2 f(f^2 - 1) = 0 \ ,$$
$$\frac{d^2a}{dr^2} - \frac{1}{r}\frac{da}{dr} - (a-1)f^2 = 0 \ . \tag{2.18}$$

From the asymptotic behavior of the fields one recognizes that the configuration has vorticity n. Correspondingly ϕ must have n zeros, which, because of the symmetry, must be degenerate at $z = 0$. Hence we have $f(r) = O(r^n)$ for $r \to 0$. Regularity of $A(z,\bar{z})$ requires $a(r) = O(r^2)$ for $r \to 0$.

Equations (2.18) are not amenable to an analytic solution, but one can easily see that the asymptotic values $f = a = 1$ are approached exponentially as

$$f(r) - 1 = O(e^{-\lambda r}) \ ,$$
$$a(r) - 1 = O(e^{-r}) \tag{2.19}$$

for $r \to \infty$. The coupling constant λ thus determines the spatial rate of decay of the perturbation of the matter field, relative to the rate of decay of the electromagnetic perturbation. For intermediate values of r the equations must be solved numerically. Alternatively, one can insert the *Ansatz* into the expression for $\mathcal{E}$ and search for stationary points. Using a variational procedure, which will be described in Sec. III, we have found the minimum of $\mathcal{E}$ for various values of λ and for $n = 1$ and 2. The results are given in Table I and displayed in Fig. 1, where $\mathcal{E}(\lambda, n=2)$ and $2\mathcal{E}(\lambda, n=1)$ are plotted. One notes a crossover at $\lambda = 1$. A configuration of two superimposed vortices has a lower (greater) energy than the energy of two widely separated vortices for $\lambda < 1 (\lambda > 1)$.

The value $\lambda = 1$ is of particular interest. One can derive a lower bound on the energy functional as follows: we use an integration by parts,

$$\int dz\, d\bar{z}\, [(\partial - iA)\phi(\bar{\partial} + i\bar{A})\bar{\phi}] = \int dz\, d\bar{z}\, [(\bar{\partial} - i\bar{A})\phi(\partial + iA)\phi - i(\partial\bar{A} - \bar{\partial}A)\phi\bar{\phi}] \ , \tag{2.20}$$

to obtain an alternative expression for $\mathcal{E}$, i.e.,

$$\mathcal{E} = \frac{1}{\pi}\int dz\, d\bar{z}\, \{|(\bar{\partial} - i\bar{A})\phi|^2 + [-i(\partial\bar{A} - \bar{\partial}A) + \tfrac{1}{4}(|\phi|^2 - 1)]^2\} - \frac{i}{2\pi}\int dz\, d\bar{z}\, (\partial\bar{A} - \bar{\partial}A) \ . \tag{2.21}$$

The first term on the right-hand side of Eq. (2.21) is the integral of two positive semidefinite quantities. It vanishes if the equations

$$(\bar{\partial} - i\bar{A})\phi = 0 \tag{2.22}$$

and

$$\partial\bar{A} - \bar{\partial}A + \tfrac{1}{4}i(|\phi|^2 - 1) = 0 \ , \tag{2.23}$$

are satisfied. The second term is $e/2\pi$ times the total magnetic flux, i.e., n. Thus for $\lambda = 1$, $\mathcal{E}$ is bounded below by n, the bound being saturated when Eqs. (2.22) and (2.23) are satisfied. For negative n an analogous procedure shows that $\mathcal{E}$ is bounded by $-n$ and that we have $\mathcal{E} = -n$ if

$$(\partial - iA)\phi = 0, \quad \partial\bar{A} - \bar{\partial}A - \tfrac{1}{4}i(|\phi|^2 - 1) = 0 \ .$$

From the work of Ref. 7 one knows that the first-order Eqs. (2.22) and (2.23) can be solved for arbitrary (positive) n in a radially symmetric configuration. This means that n vortices occupying the same position are in equilibrium, with zero interaction energy, for $\lambda = 1$. The values $\mathcal{E}(\lambda = 1, n) = n$ are reproduced to high precision by our variational procedure. This can be used as a test of the accuracy of the numerical method.

Since no analytical solutions of Eqs. (2.22) and (2.23) are known for two (or more) vortices at arbitrary separation, one cannot conclude that $\lambda = 1$ vortices do not interact. Our numerical analysis, however, shows that the lower bound on the energy is reached for $\lambda = 1$, $n = 2$ to within a relative error of order 2×10^{-4} — equal to the expected accuracy of the approximation. Thus our results indicate that Eqs. (2.22) and (2.23) do indeed have solutions for arbitrary locations of the two vortices, implying that the interaction energy vanishes in this case.

When $\lambda \neq 1$, the method outlined above can still be used to derive a lower bound on the energy, albeit less strong: in particular, it can not be a minimum.

For $\lambda > 1$ we simply note that the last term in the energy functional can be written

$$\mathcal{E}_3 = \frac{1}{8\pi}\langle(|\phi|^2-1)^2\rangle + \frac{\lambda^2-1}{8\pi}\langle(|\phi|^2-1)^2\rangle \ , \quad (2.24)$$

where the notation $\langle f\rangle = \int \frac{i}{2}(dzd\bar{z})f = \int d^2x f$ is used. The first term in the right-hand side of Eq. (2.24) can be combined with the other two terms in $\mathcal{E}$ and a bound derived as for $\lambda=1$. We thus obtain

$$\mathcal{E}(\lambda \geq 1) \geq |n| + \frac{\lambda^2-1}{8\pi}\langle(|\phi|^2-1)^2\rangle \geq |n| \ . \quad (2.25)$$

For $\lambda < 1$ we rewrite the first term of the energy functional,

$$\mathcal{E}_1 = \frac{1}{2\pi}\int dz\, d\bar{z}\,[|(\partial - iA)\phi|^2 + (\bar{\partial} - i\bar{A})\phi|^2] \quad (2.26)$$

as

$$\mathcal{E}_1 = (1-\lambda)\mathcal{E}_1 + \lambda\mathcal{E}_1 \ . \quad (2.27)$$

$\lambda\mathcal{E}_1$ can be combined with the other two terms in $\mathcal{E}$ and a bound can be derived following a procedure analogous to the one used for $\lambda=1$, but this time we add and subtract a term

$$(\lambda i/2\pi)\int dz\, d\bar{z}\,(\partial\bar{A} - \bar{\partial}A) \ .$$

The final result is

$$\mathcal{E}(\lambda \leq 1) \geq \lambda|n| + (1-\lambda)/\pi \times \langle |(\partial - iA)\phi|^2 + |(\bar{\partial} - i\bar{A})\phi|^2\rangle \geq \lambda|n| \ . \quad (2.28)$$

These lower bounds are indicated in Fig. 1.

Using a variational method which we shall describe in detail in Sec. III, we have determined the constrained minimum of $\mathcal{E}$ with two vortices kept at a fixed separation for $\lambda=0.7$, $\lambda=1$, and $\lambda=1.3$. The results are presented in Table II and an interpolation of the energy curves is displayed in Fig. 2. One notes that the two vortices attract each other at all separations for $\lambda=0.7$, repel each other for $\lambda=1.3$, and as mentioned before, do not interact for $\lambda=1$.

III. VARIATIONAL METHOD

The Euler-Lagrange equations (2.15) and (2.16) are solved by fields which minimize the energy functional. In a variational approximation, one introduces a set of field configurations depending on a number of variational parameters V_i, $i=1,N$, evaluates $\mathcal{E}$ as functions of the V_i and minimizes the ensuing expression. We have performed this computation both for a rotationally symmetric configuration of fields with vorticity one and two, and for a configuration of two vortices kept at a finite separation. The results of the first computation are used in the second.

To find numerically the minimum of $\mathcal{E}$ in a rotationally symmetric configuration we start from the *Ansatz* of Eq. (2.17) and further approximate the functions $f(r)$ and $a(r)$ as follows:

$$f(r) = 1 + e^{-\lambda\rho}\sum_{l=0}^{n}(f_l\rho^l/l!) \ , \quad (3.1)$$

TABLE I. Energies for a single vortex and two superimposed vortices as functions of λ. (Units are explained in the text.)

λ	$E(\lambda, n=1)$	$E(\lambda, n=2)$
0.5	0.757 42	1.391 29
0.6	0.813 05	1.526 27
0.7	0.864 40	1.653 37
0.8	0.912 30	1.774 07
0.9	0.957 36	1.889 36
1.0	1.000 00	2.000 00
1.1	1.040 53	2.106 55
1.2	1.079 21	2.209 45
1.3	1.116 25	2.309 05
1.4	1.151 81	2.405 67
1.5	1.186 39	2.499 53

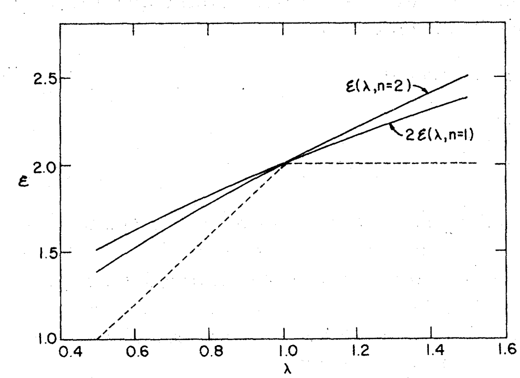

FIG. 1. Energy of two superimposed vortices, $\mathcal{E}(\lambda, n=2)$, and twice the energy of a single vortex, $2\mathcal{E}(\lambda, n=1)$, as functions of λ. The dashed line represents the bound derived in the text. At $\lambda=1$ a crossover occurs, at which point the bound is saturated.

$$a(r)=1+e^{-\rho}\sum_{l=0}^{n}(a_l\rho^l/l!) \quad . \tag{3.2}$$

f_0, a_0, and a_1 are set equal to -1, f_1 is set equal to $-\lambda$ for vorticity two, and other expansion coefficients are the variational parameters. Expressions (3.1) and (3.2) are chosen so as to reproduce the correct behavior of the fields at the origin and the correct approach to their asymptotic values. Otherwise, the choice of the expansion is guided by the convenience of the numerical analysis.[9] Insertion of the *Ansatz* into the formula for the energy produces a polynomial of the fourth order in the variational parameters with coefficients which can be all evaluated analytically. In the actual computation we have set $n=10$. The coefficients of the polynomial (taking into account the symmetry of the indices) then form a set of less than 10^4 numbers, which can be handled very easily by a large computer. The search for a minimum has been carried out approximating the quartic $\mathcal{E}$ by a quadric tangent to it at the current values of the variational parameters. The minimum of the quadric is then chosen as the new value of the variational parameters. The process converges extremely rapidly (typically in four or five iterations, in spite of the large dimensionality of parameter space) to the minimum of $\mathcal{E}$. Clearly, it is the physical nature of the problem which makes the surface $\mathcal{E}(f_i,a_i)$ concave and well behaved, and allows for a very efficient variational procedure. The steps for the minimization are expressed by the following equations, where V_i stand for the collection of variational parameters:

$$\mathcal{E}(V_i)=\mathcal{E}_0+\sum_i\mathcal{E}_i^{(1)}V_i+\sum_{i\geq j}\mathcal{E}_{ij}^{(2)}V_iV_j+\sum_{i\geq j\geq k}\mathcal{E}_{ijk}^{(3)}V_iV_jV_k+\sum_{i\geq j\geq k\geq l}\mathcal{E}_{ijkl}^{(4)}V_iV_jV_kV_l \tag{3.3}$$

$$\mathcal{G}_i^{(m)}=\frac{\partial\mathcal{E}}{\partial V_i}\Big|_{V_i=V_i^{(m)}} \tag{3.4}$$

$$\mathcal{H}_{ij}^{(m)}=\frac{\partial^2\mathcal{E}}{\partial V_i\partial V_j}\Big|_{V_i=V_i^{(m)}} \tag{3.5}$$

$$\mathcal{E}_{\text{approx}}^{(m)}(V_i)=\mathcal{E}(V_i^{(m)})+\sum_i\mathcal{G}_i^{(m)}(V_i-V_i^{(m)})+\frac{1}{2}\sum_{i,j}\mathcal{H}_{ij}^{(m)}(V_i-V_i^{(m)})\times(V_j-V_j^{(m)}) \quad , \tag{3.6}$$

$$V_i^{(m+1)}=V_i^{(m)}-\sum_j(\mathcal{H}^{(m)^{-1}})_{ij}\mathcal{G}_j^{(m)} \quad . \tag{3.7}$$

The results of this numerical computation are exhibited in Table I and Fig. 1. We shall comment on the error in the approximation in Sec. IV and here only anticipate that we believe the numbers quoted in the tables to be accurate to all decimal figures.

We turn now to the major goal of this work, the computation of the energy of two vortices kept at a separation d. It is crucial to use a variational *Ansatz* capable of reproducing the expected physical features of the system; in particular, the field configuration should reduce to that of two rotationally symmetric vortices if the separation parameter becomes large with respect to (twice) the range of a single vortex.

To formulate an *Ansatz* we first fix the phase of the matter field: this amounts to a choice of gauge. In the single-vortex configuration the phase factor of ϕ is

$$e^{i\theta}=(z/\bar{z})^{1/2} \quad . \tag{3.8}$$

The conformal transformation

$$z=\tilde{z}^2-\left(\frac{d}{2}\right)^2 \quad , \tag{3.9}$$

defines a 1 to 2 mapping between points in the z plane and points in the $\tilde{z}$, with the properties that the origin in the z plane has for images the points $\tilde{z}=\pm\frac{1}{2}d$ and that, as the argument of z varies by 2π at large distance ($>\frac{1}{2}d$) from the origin, the argument of $\tilde{z}$ undergoes a rotation by 4π. For the two-vortex configuration we choose then the phase factor of ϕ to be the transform of expression (3.8) and set

$$\phi(z,\bar{z})=\left\{\left[z^2-\left(\frac{d}{2}\right)^2\right]\Big/\left[\bar{z}^2-\left(\frac{d}{2}\right)^2\right]\right\}^{1/2}f(z,\bar{z}) \tag{3.10}$$

with real f. f must vanish at $z=\pm\frac{1}{2}d$, behaving there as

$$f=O\left[\left(z\mp\frac{d}{2}\right)\left(\bar{z}\mp\frac{d}{2}\right)\right]^{1/2} \quad . \tag{3.11}$$

This guarantees that ϕ has two zeros of the appropriate type at the location of the vortices.

Further specializing the *Ansatz* we demand that f consists of an asymptotic term capable of reproducing a configuration of separated vortices, plus a correction, containing the variational parameters. Also, we want to take advantage of the information already ob-

TABLE II. The energy of two vortices at a separation d for $\lambda=0.7$, 1.0 and 1.3. (Units are explained in the text.)

d	$\lambda=0.7$	$\lambda=1.0$	$\lambda=1.3$
0	1.653	2.000	2.309
1	1.653	2.000	2.308
2	1.656	2.000	2.299
3	1.665	2.000	2.276
4	1.680	2.000	2.254
5	1.696	2.000	2.242
6	1.710	2.000	2.236
7	1.718	2.000	2.234
8	1.723	2.000	2.233
∞	1.728	2.000	2.232

tained for vanishing separation. We set therefore

$$f(z,\bar z)=\omega f^{(1)}\left(|z-\frac{d}{2}|\right)f^{(1)}\left(|z+\frac{d}{2}|\right)+(1-\omega)\frac{|z^2-(d/2)^2|}{|z^2|}\times f^{(2)}(|z|)+\delta f(z,\bar z)\;, \qquad (3.12)$$

where ω is a weight factor and $f^{(1)}$, $f^{(2)}$ are the functions given by Eq. (3.1) with the variational parameters previously determined for a single vortex and two superimposed vortices. The function $f^{(1)}$ approaches one exponentially when its argument becomes larger than $1/\lambda$. For $d >> 2/\lambda$, the product

$$f^{(1)}\left(|z-\frac{d}{2}|\right)f^{(1)}\left(|z+\frac{d}{2}|\right)$$

then reduces to the function

$$f^{(1)}\left(|z-\frac{d}{2}|\right)\quad\left[f^{(2)}\left(|z+\frac{d}{2}|\right)\right]\,.$$

in the right- (left-) half plane and the first term on the right-hand side of Eq. (3.12), with $\omega=1$, reproduces an asymptotic configuration of separated vortices. In the second term the factor $|z^2-(d/2)^2|/|z^2|$ has the effect of replacing the double zero of $f^{(2)}$ at the origin with two zeros at $z=\pm\frac{1}{2}d$; for $d=0$ and $\omega=0$ this second term reproduces the field of two superimposed vortices. δf contains the variational

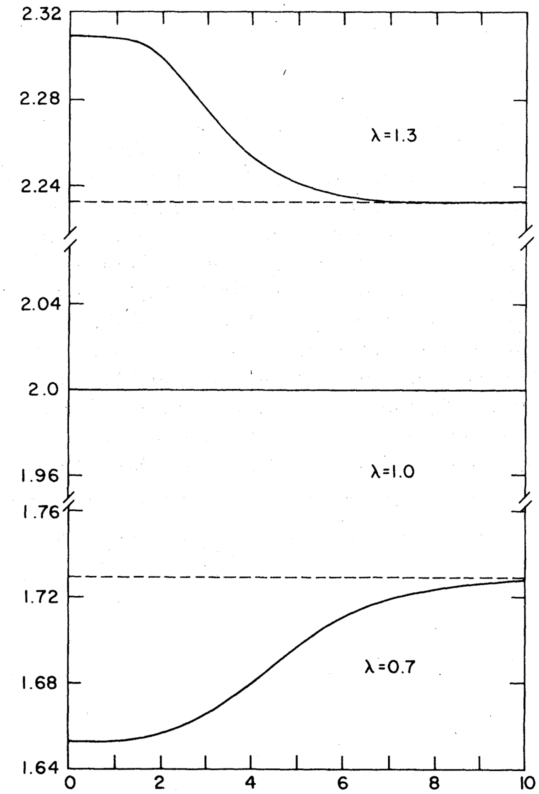

FIG. 2. Energy of a two-vortex field configuration for $\lambda=0.7$, 1.0 and 1.3 as a function of the inter-vortex separation. The dashed lines correspond to asymptotic values.

parameters, and is expanded as follows:

$$\delta f(z,\bar z)=\left|z^2-\left(\frac{d}{2}\right)^2\right|(\cosh\lambda|z|)^{-1}\times\sum_{i=0}^{n}\sum_{j=0}^{i}f_{ij}\frac{(z\bar z)^i}{2}\left[\left(\frac{z}{\bar z}\right)^j+\left(\frac{\bar z}{z}\right)^j\right]\,. \qquad (3.13)$$

The first factor on the right-hand side of Eq. (3.13) ensures that ϕ vanishes at $z=\pm\frac{1}{2}d$; the second factor introduces an exponential cut-off, consistent with the analytic behavior of ϕ at the origin; the double summation is an expansion into powers of $|z|$ and cosines of arg z, which takes into account the symmetry of the field configuration.

To formulate an *Ansatz* for A we add the potentials of a single vortex at $z=\frac{1}{2}d$ and a single vortex at $z=-\frac{1}{2}d$, with weight factor ω, and of two vortices superimposed at the origin with weight factor $(1-\omega)$

plus a correction term. We have

$$A=\omega\left[-\frac{i}{2z-d}a^{(1)}\left(\left|z-\frac{d}{2}\right|\right)-\frac{i}{2z+d}a^{(1)}\left(\left|z+\frac{d}{2}\right|\right)\right]$$

$$-(1-\omega)\frac{i}{z}a^{(2)}(|z|)+\delta a(z,\bar{z}) \ , \tag{3.14}$$

where $a^{(1)}$ and $a^{(2)}$ are the a functions [see Eq. (3.2)] previously determined for the symmetric configurations with $n=1,2$. δa is given by

$$\delta a(z,\bar{z})=\frac{1}{\cosh|z|}[za^{\mathrm{I}}(z,\bar{z})+\bar{z}a^{\mathrm{II}}(z,\bar{z})] \ , \tag{3.15}$$

where the real functions a^{I} and a^{II} contain the variational parameters and are expanded as follows:

$$a^{\mathrm{I(II)}}=\sum_{i=0}^{n}\sum_{j=0}^{i}a_{ij}^{\mathrm{I(II)}}\frac{1}{2}[(z\bar{z})^{i}]\left[\left(\frac{z}{\bar{z}}\right)^{j}+\left(\frac{\bar{z}}{z}\right)^{j}\right] \ . \tag{3.16}$$

Asymptotically A behaves as $A=O(-i/z)$, which agrees with the behavior $A=O(-iz/[z^2-(\frac{1}{2}d)^2])$ inferred from Eq. (2.10). There are no further constraints on A, other than it should be regular. In particular, whereas the potential vanishes at the location of the vortices in the rotationally symmetric configuration, it does not have to vanish at $z=\pm\frac{1}{2}d$ in the nonsymmetric configuration.

Once the *Ansätze* for ϕ and A are formulated, the variational search for a minimum of $\mathcal{E}$ is performed following the same procedure outlined for the symmetric vortices. ω is in a sense also a variational parameter, but we have chosen for it the value that minimizes the energy obtained from the *Ansatz* without variational corrections (i.e., with $\delta f=\delta a=0$). As expected, this optimal ω is small for small separations but quickly approaches 1 as d increases past a value of ≈ 2. In the actual computation we have truncated the expansions of δf, a^{I}, and a^{II} at a power $|z|^6$, thus keeping 18 variational parameters, apart for a few tests performed with a larger number of parameters. Whenever possible, the coefficients of the polynomial expansion of $\mathcal{E}$ were evaluated analytically, being otherwise obtained by numerical integration. The results for the minimum energies as function of the inter-vortex separation are given in Table II for $\lambda=0.7$, 1, and 1.3 and shown, interpolated, in Fig. 2. The profiles of three gauge-invariant quantities — the energy density, the magnitude of the matter field and the magnetic field — are displayed in Fig. 3 for $\lambda=1$, and four values of the separation between the vortices.

IV. DISCUSSION

In this last section we shall comment briefly on the accuracy of the numerical computation and on some of the physical implications of the results. The evaluation of the minimum energy can be in error because of the limitation in the space of trial functions, inherent to the method, and of possible numerical approximations in the calculation of the energy functional. In the rotationally symmetric configuration all of the coefficients in the polyomial expansion of $\mathcal{E}(V_i)$ are evaluated analytically and only the first source of error can be relevant. We then estimate the accuracy of the computation comparing the numerical results

$$\mathcal{E}(\lambda=1,\quad n=1)=1.000\,000\,13 \ ,$$

$$\mathcal{E}(\lambda=1,\quad n=2)=2.000\,004\,35 \ ,$$

with the values $\mathcal{E}=1$ and 2, which can be derived analytically for $\lambda=1$. We see that in both cases the first five decimals are correct and, extrapolating to all values of λ considered, we estimate the error to be in the sixth decimal figure.

In the computation for nonsymmetric configurations, some of the coefficients in the polynomial expansion of $\mathcal{E}$ are evaluated by numerical integration. We have checked the accuracy of the procedure by doubling the number of points in the grid and have found the results stable up to the fourth decimal digit.

The error attributable to the truncation of the number of variational parameters is then estimated assuming that the lower bound $\mathcal{E}=2$ for $\lambda=1$, $n=2$ is actually reached by the true minimum. The numerical computation gives 2.000 212, 2.000 922, 2.000 459, 2.000 102, and 2.000 082, for $\lambda=1$ and separations $d=1$, 2, 3, 4, and 5, respectively. If the correct value is $\mathcal{E}=2$, the maximum error is encountered for $d=2$, and is less than $\Delta=5\times10^{-4}$ in relative magnitude. Since $\mathcal{E}=2$ is a lower bound for the energy, the error cannot exceed Δ. To check whether the deviation from $\mathcal{E}=2$ is consistent with the variational approximation we have repeated the variational search for $\lambda=1$, $d=2$ with an enlarged space of test functions: terms in $|z|^6$ have been included in the expansion of Eqs. (3.13) and (3.16), for a total of 24 variational parameters. The minimization then produces the value $\mathcal{E}=2.000\,576$ (versus 2.000 922 with 18 variational parameters). This makes it plausible that the deviation from 2 is due entirely to the truncation.

Our results for $\lambda=0.7$ and $\lambda=1.3$ show quite clearly how vortices attract and repel each other for values of the coupling constant, which are, respectively, less and greater than the critical value $\lambda=1$. As mentioned in the Introduction, such a behavior has been found asymptotically[4] and also at zero separa-

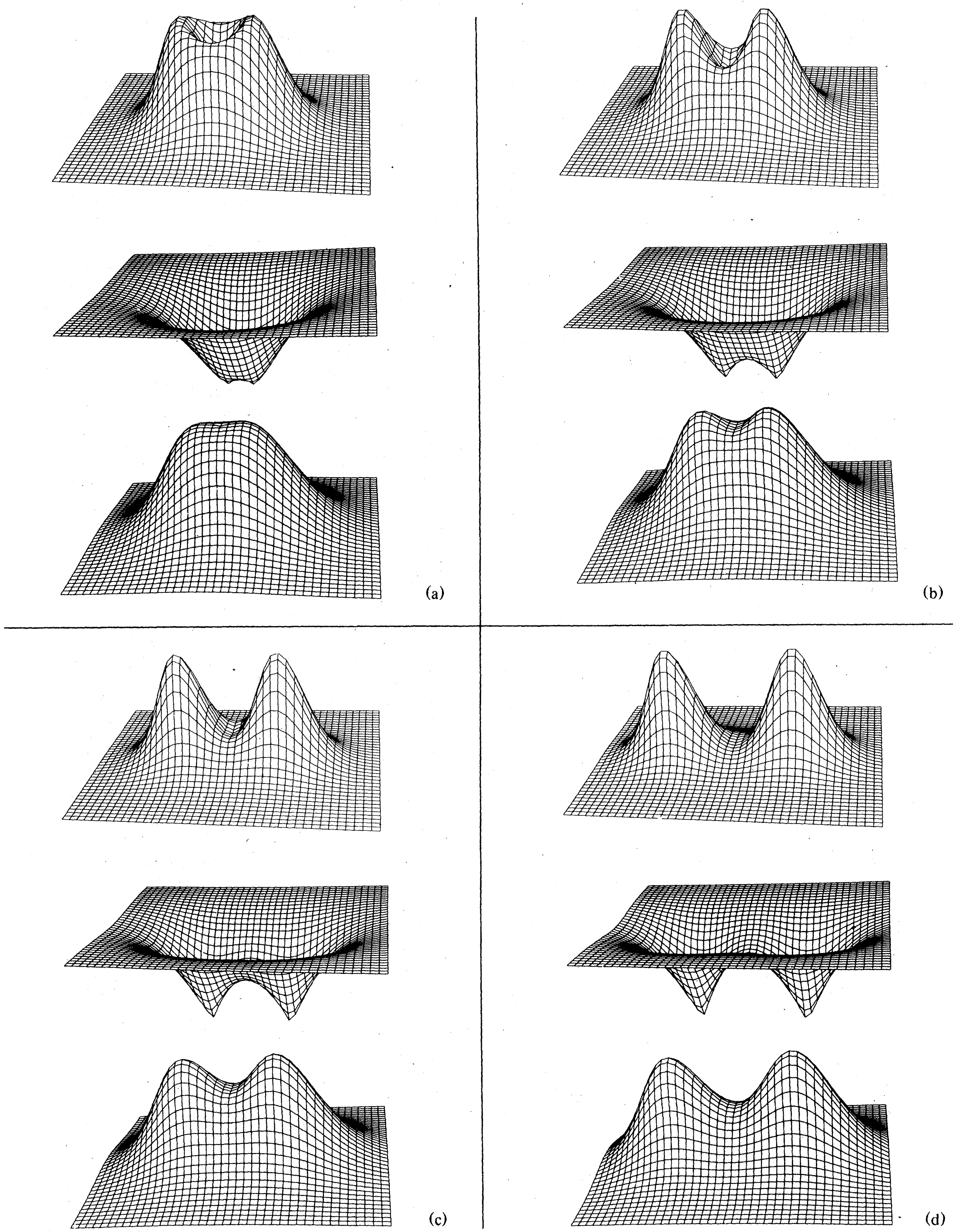

FIG. 3. Energy density (upper), magnitude of the matter field (middle) and magnetic field (lower) of two vortices for $\lambda = 1$ and separation $d = 1$, 2, 3, and 4 [(a) – (d), respectively].

tion from stability studies.[5] The numerical computation presented here interpolates nicely between these extreme cases.

We find most interesting the results obtained for $\lambda = 1$. With this value of the coupling constant one can derive the first-order Eqs. (2.22) and (2.23).[6] If the fields solve these equations, then the energy equals the lower bound, additive in the vorticity, and vortices do not interact. However, the existence of solutions with arbitrary inter-vortex separation is neither obvious nor straightforward to establish. Our work for $\lambda = 1$ is tantamount to a numerical determination of the solutions. We observe that, following a procedure already used in different context,[10] Eqs. (2.22) and (2.23) can be reduced to a single second-order equation for a superpotential function χ. To this end, we set

$$A = i\partial\psi \ , \quad \bar{A} = -i\bar{\partial}\psi \ . \tag{4.1}$$

(This is equivalent to a choice of gauge, as it implies $\partial\bar{A} + \bar{\partial}A = 0$, or $\partial_\mu A^\mu = 0$, for the electromagnetic potential.) Eq. (2.22) then becomes

$$(\bar{\partial} - \bar{\partial}\psi)\phi = e^{\psi}\bar{\partial}(e^{-\psi}\phi) = 0 \ , \tag{4.2}$$

and reduces to the statement that the function $f = e^{-\psi}\phi$ is analytic. Inserting

$$\phi(z,\bar{z}) = e^{\psi(z,\bar{z})} f(z) \tag{4.3}$$

into Eq. (2.23) we obtain

$$\partial\bar{\partial}\psi = \tfrac{1}{8}(e^{2\psi} f\bar{f} - 1) \ , \tag{4.4}$$

and by the further substitution,

$$\psi = \chi - \tfrac{1}{2}\ln f - \tfrac{1}{2}\ln\bar{f} \ , \tag{4.5}$$

we arrive at the final equation for χ

$$\partial\bar{\partial}\chi = \tfrac{1}{8}(e^{2\chi} - 1) \ . \tag{4.6}$$

The boundary conditions follow from the expression for the matter field,

$$\phi = e^{\chi}\frac{f}{|f|} \ ; \tag{4.7}$$

χ should go to zero at infinity and should behave as $\ln|z - z_i|$ at the position of the vortices, where $f(z)$ vanishes like $z - z_i$. Our results indicate that Eq. (4.6) has solutions with two singularities (vortices) at arbitrary separation, and one may conjecture more generally that solutions will exist with any number of vortices at arbitrary locations in the plane. The analytic verification of this conjecture is left, however, as an interesting, open problem.

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[8]The indices $i,j = 1,2,3$ label spatial components. Summation over repeated indices is implied throughout.

[9]In particular, the parametrization is not constrained to take into account the parity of the expansions for f and a near the origin. The minimization produces values for the parameters which do satisfy this constraint within the error inherent in the numerical approximation.

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