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Series-expansion analysis of critical-temperature shifts of finite Ising lattices with "self-consistent" boundary conditions

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Finite Ising $N \times N$ ($N = 1$–5) and $N \times \infty$ ($N = 1$–8) systems with one of the three self-consistent extended mean-field boundary conditions all around and on both sides, respectively, are studied. The boundary conditions used are: (*a*) extended mean-field boundary condition using average, abbreviated EA (in which each boundary spin takes some average value); (*b*) extended mean-field boundary condition using probability, abbreviated EP (in which each boundary spin takes the values ± 1 with certain probabilities) and (*c*) extended Bethe-Peierls, EBP (in which each boundary spin is assumed to be acted on by some effective field). The self-consistent equations are then derived by letting the average magnetization of the internal spins of the system be equal to that of the boundary spins. The finite systems considered therefore represent some high-order systematic generalization of the familiar molecular-field and Bethe-Peierls approximations. The equations for the critical temperature are formulated graphically in terms of some high-temperature expansion series which are calculated to the ninth and tenth orders. Two of the series in the equations of $N \times \infty$ for $N \to \infty$ are directly related to the layer and local susceptibilities considered by Binder and Hohenberg for the Ising infinite half plane. Critical temperatures obtained numerically with the use of truncation and extrapolation techniques. Results on the systems with EA and EP indicate that the shifts of their critical temperatures can be described by a power law $N^{-\lambda}$ where $\lambda = 1/\gamma_1$ and γ_1 ($\approx 11/8$) is the critical exponent of the layer susceptibility. This is different from the value $\lambda = 1$ which is believed true for two-dimensional finite systems with periodic or free boundary conditions. No conclusive remark can be made to systems with EBP through they suggest some even higher value (≥ 2).

I. INTRODUCTION

In most theoretical studies on the Ising model, only bulk properties are of interest. For this purpose, periodic boundary conditions in all directions are often first assumed and then the thermodynamic limit is later taken. For example, they are the lines followed in most of the exact solutions in two dimensions and also widely used in the applications of series expansion to two- and three-dimensional systems. The use of this periodic boundary condition, though artificial, is to make the calculations much easier mathematically.

Both periodic and the more realistic free (i.e., open) boundary conditions have been extensively used in the investigations of finite-size effect. One of the most important and sensitive quantities to the full effect of boundaries or surfaces is the critical temperature, especially its relative shift. The critical temperature depends on the finite size and the boundary conditions. In some finite systems where a sharp phase transition does not occur and the rounding effect often takes over, we can still define the so-called quasi-critical temperature but not wholly unambiguously; examples are the maximum in the specific heat or the zero-field susceptibility, and the point of maximum slope in the internal-energy curve using periodic or free boundary conditions. For those using the "pure phase" ($\pm$) boundary conditions which are recently used by Abraham and Martin-Löf,[1] there is one more definition for this quasi-critical temperature which can also be identified as the point of maximum slope in the nonvanishing average magnetization curve. The values obtained from all these various definitions need not be the same. We will consider in this work some self-consistent boundary conditions of a type similar to the one first used by Binder and Müller-Krumbhaar[2] in their Monte Carlo studies of some finite classical Heisenberg systems. A finite system with one of these boundary conditions exhibits mean-field behavior and therefore provides a better approximation to an infinite lattice than the usual free or periodic boundary conditions or some combination of both, in particular, with respect to the average magnetization. The critical temperature can here be defined as the point in the magnetization curve below which it is finite and above which it is zero.

Most of the analyses on finite Ising systems have been numerical, either by the high-temperature series expansion or the recent Monte Carlo simulation. The finite systems studied have been $N \times N$ and $N \times \infty$ in two dimensions, and $N \times N \times N$ and $N \times \infty \times \infty$ in three dimensions. The ultimate aim has been not only to determine the critical

temperature T_N for individual values of N, but also to estimate, as far as possible, the form of the dependence of T_N on N. More explicitly, we are usually interested in the shift exponent of T_N, given by the value of λ in the following asymptotic equation:

$$(T_N - T_c)/T_c \sim \text{const}/N^{\lambda}, \tag{1.1}$$

where T_c is the bulk critical temperature and the constant may be positive or negative depending mainly on the imposed boundary conditions. This form is believed to be generally true for large finite systems in two and three dimensions although Onsager[3] showed exactly in his classic paper that the asymptotic form for T_N of $N \times \infty$ Ising system with periodic boundary conditions on both sides is given by

$$(T_N - T_c)/T_c \sim \text{const}(\ln N/N^2). \tag{1.2}$$

In other studies, Ferdinand and Fisher[4] analytically considered an $N \times N$ system with toroidal boundary conditions using Kaufman's exact expression for the partition function. Au-Yang and Fisher[5] investigated an $N \times \infty$ system with free boundary conditions on both sides by using finite-size scaling theory. The authors[6] have recently studied $N \times 1$ systems with two extended mean-field boundary conditions on the sides and at the ends by the modified transfer-matrix method. Numerically, Landau[7] has discussed $N \times N$ systems both with toroidal boundary conditions and with free boundary conditions all around by a Monte Carlo method; all these studies on the two-dimensional systems say that $\lambda = 1$. But we will show some evidence for some finite systems with some self-consistent mean-field boundary conditions for which $\lambda \neq 1$. For the three-dimensional finite system, Allan[8] studied the values of λ for both $N \times \infty \times \infty$ film systems with free and periodic boundary conditions on the two surfaces using the standard high-temperature series-expansion techniques. His results were later reanalyzed and extended by himself,[9] and Capehart and Fisher.[10] Ritchie and Fisher[11] investigated a similar problem with the Heisenberg film systems. In a Monte Carlo study of $N \times N \times N$ simple cubic Ising lattice with free boundary conditions all around, Binder's first prediction[12] for λ was later corrected by Binder and Hohenberg,[13] and this was confirmed by Landau[14] who also investigated the corresponding system with periodic boundary conditions.

In this work, we are also interested in the determination of the value for λ of some two-dimensional finite Ising systems imposed with some self-consistent boundary conditions. However, unlike the usual interest in the finite-size studies, our interest has little to do with the interpretations of experimental data. Instead, the study of the asymptotic behavior for $N \to \infty$ has been made important because of conclusions about bulk critical behavior which can be drawn from the Monte Carlo studies[15] in which only relatively small finite systems can be used. Some extrapolation procedure to deduce the bulk properties may be made possible with the knowledge of this finite-size effect. On the other hand, due to the smallness of the systems, the studies and comparison of various boundary conditions are useful. This enables us to choose some suitable boundary conditions with which the system simulates closely to the infinite system and yet only a reasonable amount of computing time is needed. Moreover, the finite systems with the extended mean-field boundary conditions are some high-order generalization of certain familiar closed-form approximations. These constitute the aims of our work.

The systems under investigation are $N \times N$ and $N \times \infty$ using one of the three boundary conditions. Of these, two have been recently introduced by Bolton and Johnson.[16] The last one is defined with the spirit of the Bethe-Peierls approximation. A series-expansion method is proposed to derive the equations for the critical temperature of these finite systems and then obtain their critical temperatures by numerical methods. We will describe the boundary conditions and the systems in Sec. II and present the series expansion formulation in Sec. III. Some numerical techniques to obtain the critical temperatures will be presented in Sec. IV. Estimations of the values of λ will be made in Sec. V. We also briefly discuss the connection of some series in the critical-temperature equation with some standard bulk high-temperature expansion series of the half plane.[21]

II. BOUNDARY CONDITIONS AND FINITE SYSTEMS

As mentioned in the Introduction, we will consider two self-consistent mean-field boundary conditions previously introduced by Bolton and Johnson using the average (EMFBCA) which we abbreviate to EA, and using the probability (EMFBCP) which we abbreviate to EP. In EA, each spin just outside the system (called a boundary spin) is given some average value s with $s \leq 1$. Then s is determined by equating it to the average magnetization of the spins of the system (called internal spins). Obviously, EA is closely related to the molecular-field approximation.

In EP, each boundary spin takes the values ± 1 with such probabilities that their average is s. We define the probability distribution function Q for each boundary spin, say s_b, by

$$Q(s_b)=\tfrac{1}{2}(1+ss_b). \tag{2.1}$$

As pointed out in Bolton and Law,[6] there is no unique way to treat the boundary spins in EP. We always have to calculate the partition function by averaging over the internal spins. We can then either do the boundary spins averaging immediately after calculating the partition function or we take the free energy and then do the averaging. In this work, we will not consider this second choice and the symbol EP used throughout will always refer to the first choice. Both this EP and EA have recently been applied in Monte Carlo simulations[16,17] and analytic investigations of finite-size effect.[6]

We will also consider another self-consistent mean-field boundary condition which we abbreviate to EBP (extended Bethe-Peierls). In this boundary condition, each boundary spin is assumed to be acted on by some effective field which is again determined self-consistently by letting the average magnetization of all boundary spins to be the same as that of all internal spins. It is clear that this boundary condition is connected to the spirit of the Bethe-Peierls approximation.

We will apply all these boundary conditions defined above to the finite systems which are $N\times N$ systems with one boundary condition all around or $N\times\infty$ systems with one of them on both sides. Explicitly, let $N\times N$ internal spins be labeled $\{s_i\}$ for $i=1,2,3,\ldots,N\times N$, each of which takes $+1$ or -1. Among these, we call those on one of the four edges (not the corners) and corners of $N\times N$ spins as edge spins and corner spins denoted by $\{s_e\}$ and $\{s_c\}$, respectively. Each edge spin s_e has one boundary spin denoted by s_{be} and each corner spin s_c has two boundary spins denoted by s_{bc1} and s_{bc2}. For convenience, these boundary spins are collectively represented by $\{s_b\}$, where b may be be, $bc1$, or $bc2$. They are defined according to the boundary conditions. The positive interaction energy J acts between each nearest-neighbor pair of internal spins and also between each edge (or corner) spin and its boundary spin (or spins). As usual, in order to study the $N\times\infty$ system analytically or numerically, we first assume that the system is of $N\times N^*$ with periodic boundary conditions imposed in the direction which was initially of infinite extent. We later let N^* go to infinity to meet the required situation. This $N\times N^*$ system can still be defined as above except there is no corner here. We observe that there are actually $(N+4)\times N$ and $(N+2)\times N^*$ total number of all spins (including $4N$ and $2N^*$ total number of boundary spins) in these $N\times N$ and $N\times N^*$ systems, respectively. We denote the partition functions by $Z_{(N+4)N}$ and $Z_{(N+2)N^*}$, respectively. The average magnetization of all the internal spins always comes into the definitions of these boundary conditions, and we denote it by s. Consider firstly $N\times N$ systems. In the following, we write down explicitly the total energy E of a configuration denoted by $(\{\};\ldots)$ in the presence of some field, the partition function and the self-consistent equation for s of this $N\times N$ system with the self-consistent boundary condition shown.

With EA: $(\{s_i\};s)$,

$$E=-J\sum_{\langle ij\rangle}s_is_j-Js\sum_e s_e-2Js\sum_c s_c-mH\sum_i s_i, \tag{2.2}$$

$$Z_{(N+4)N}=\sum_{\{s\}}e^{-\beta E}, \tag{2.3}$$

$$s=\left\{\frac{\partial}{\partial h}\left(\frac{\ln Z_{(N+4)N}}{N\times N}\right)\right\}_{h=0}, \tag{2.4}$$

where $\sum_{\langle ij\rangle}$, $\sum_e$, $\sum_c$, and $\sum_i$ mean that only nearest-neighbor pair, edge, corner, and all internal spins are to be taken, respectively; m is the magnetic moment per spin and H is an external magnetic field acting on every internal spin; $\sum_{\{s\}}$ denotes the sum over all configurations of the internal spins; $\beta\equiv 1/k_BT$, k_B is the Boltzmann's constant and T is the temperature; $\{\}_{h=0}$ means that $h(\equiv\beta mH)=0$ is to be taken in the bracket after doing the calculation, if any, in it; moreover, we use the notation $K\equiv\beta J$ and, for convenience, further adopt the unit such that $J/k_B=1$ and therefore $K=1/T$.

With EP: $(\{s_i\};\{s_b\})$,

$$E=-J\sum_{\langle ij\rangle}s_is_j-J\sum_e s_es_{be}-J\sum_c s_c(s_{bc1}+s_{bc2})-mH\sum_i s_i, \tag{2.5}$$

$$Z_{(N+4)N}=\sum_{\{s\}}\sum_{\{s_b\}}\prod_b Q(s_b)e^{-\beta E}, \tag{2.6}$$

$$s=\left\{\frac{\partial}{\partial h}\left(\frac{\ln Z_{(N+4)N}}{N\times N}\right)\right\}_{h=0}, \tag{2.7}$$

where $\sum_{\{s_b\}}$ and Π_b denote the sum over all configurations of and the product over all the boundary spins; and $Q(s_b)$ is given in (2.1).

With EBP: $(\{s_i\};\{s_b\};H_{\text{eff}})$,

$$E=-J\sum_{\langle ij\rangle}s_is_j-J\sum_e s_es_{be}-J\sum_c s_c(s_{bc1}+s_{bc2})-mH_{\text{eff}}\sum_b s_b-mH\sum_i s_i, \tag{2.8}$$

$$Z_{(N+4)N} = \sum_{\{s\}} \sum_{\{s_b\}} e^{-\beta E}, \tag{2.9}$$

$$s = \left\{ \frac{\partial}{\partial h} \left(\frac{\ln Z_{(N+4)N}}{N \times N} \right) \right\}_{h=0} = \left\{ \frac{\partial}{\partial h_{\text{eff}}} \left(\frac{\ln Z_{(N+4)N}}{4N} \right) \right\}_{h=0}, \tag{2.10}$$

where H_{eff} is some effective field acting on every boundary spin and $h_{\text{eff}} \equiv \beta m H_{\text{eff}}$. Hence, the self-consistent equations are given by (2.4), (2.7), and (2.10) for $N \times N$ systems with EA, EP, and EBP, respectively. For those of EA and EP, it is readily seen that they are of the form of

$$s = f(s), \tag{2.11}$$

where $f(s)$ is an odd function of s, i.e., $f(-s) = -f(s)$, in zero field. This is a usual result of the molecular-field approximation. The critical temperature can then be defined as the point where the gradients of the curves $y = s$ and $y = f(s)$ vs s are equal at the origin. That is, if we expand $f(s)$ in powers of s, we get

$$f(s) = a(T)s + b(T)s^3 + c(T)s^5 + \cdots, \tag{2.12}$$

where $a(T), b(T), c(T), \ldots$ are some functions of T. Hence, the critical point is given by a solution of the critical temperature equation

$$a(T_c) = 1. \tag{2.13}$$

Moreover, retaining the terms up to the third order in s on the right-hand side of (2.12) immediately gives us the value of the critical exponent for the average magnetization β of $\frac{1}{2}$, compared with the exact Yang value of $\frac{1}{8}$.[18] For the $N \times N$ system with EBP, we get equations analogous to (2.11) with (2.12). Hence, the behavior is the same. The similar procedure and remarks can also be applied to the $N \times N^*$ cylindrical systems defined above. We will not consider the details here. It is quite clear that the 1×1 systems with EA and EBP are exactly the standard molecular-field approximation and Bethe-Peierls approximation in two dimensions, respectively. Hence, all the above-mentioned finite systems (i.e., $N \times N$ and $N \times \infty$) and some systems[6,16,17] studied previously with EA represent some high-order systematic generalization of these approximations. No matter how large the systems are, as long as N is finite, they all exhibit mean-field behavior very close to the critical temperature. The region in which these finite systems give this wrong prediction about the infinite lattice gets narrower and narrower as N increases. Also, the critical points are improved. We are here interested only in this latter problem.

III. SERIES-EXPANSION FORMULATION FOR THE CRITICAL-TEMPERATURE EQUATION

In this section, we consider in detail the derivation of the equations for the critical temperature of both $N \times N$ and $N \times \infty$ systems with one of mean-field boundary conditions defined previously. Though the equations can be obtained by using self-consistent equations for the average magnetization in zero field such as those in (2.11)–(2.13), we here derive these equations directly by looking at the expansion of the partition functions in the presence of some small fields and using (2.4), (2.7), and (2.10).

We take the $N \times N$ system with EA as an example. Since the critical temperature equation is obtained by retaining only the linear term in s of the expansion on the right-hand side of Eq. (2.4) in zero external field H, we need the expansion of $Z_{(N+4)N}$ in small s and h, and retain only terms linear in s, h, and sh. To do this, we use the following identities for the exponential factors in the partition function and do the expansion:

$$\exp(Ks_i s_j) = (1 + w s_i s_j)\cosh K \tag{3.1}$$

with $w = \tanh K$;

$$\exp(Kss_e) = (1 + t_1 s_e)\cosh Ks \tag{3.2}$$

with $t_1 = \tanh Ks \approx Ks$ in limit of small s;

$$\exp(2Kss_c) = (1 + t_2 s_c)\cosh 2Ks \tag{3.3}$$

with $t_2 = \tanh 2Ks \approx 2Ks$ in limit of small s;

$$\exp(hs_i) = (1 + s_i \tanh h)\cosh h \tag{3.4}$$

$\approx 1 + hs_i$ in limit of small h. Using these for all factors in the partition function and the spirit of the standard high-temperature series expansion,[19] it is not hard to see that the terms in the expansion of $Z_{(N+4)N}$ can be represented by certain allowable graphs. That is, to the order linear in s, h, and sh,

$$Z_{(N+4)N} = \cosh^{\alpha} K \cosh^{\beta} Ks \cosh^{\gamma} 2Ks \cosh^{N \times N} h \times \sum_{\{s\}} \prod_{\text{all}} (1 + w s_i s_j)(1 + t_1 s_e)(1 + t_2 s_c) \times (1 + s_i \tanh h) \tag{3.5}$$

$$\approx 2^{N \times N} \cosh^{\alpha} K (A_0 + P_1 Ksh), \tag{3.6}$$

where α is the number of internal bonds; β is the number of edge spins; γ is the number of corner spins (= 4 for square lattice); and Π_{all} denotes that the product is taken over all bonds and all internal sites. We observe that terms linear in s or h alone vanish on the right-hand side of (3.5),

leaving nonzero contributions from terms of sh. The reason behind this can be easily realized by noting either (2.3) or (3.5) for $Z_{(N+4)N}$ which is an even function of either s or h alone, i.e., due to the symmetry of the total energy under the transformation $s_i \to -s_i$ for all internal sites (including the edge and corner spins). Using (2.4) and (3.6), we get the following equation for the critical temperature for $N \times N$ system with EA all around:

$$(K_N/N^2)(P_1/A_0) = 1 . \tag{3.7}$$

We now demonstrate that the equations for the critical temperature of the $N \times N$ systems with EA and EP are closely related. This equation with EP can be obtained in a similar way to that with EA, except now the partition function is given in (2.6). We note that all boundary spin variables can be eliminated by first summing over the boundary spins. For example, take a term in the summation $\sum_{\{s_b\}}$ associated with an edge spin s_e and its boundary spin s_{be}. Then,

$$\sum_{s_{be}=\pm 1} \frac{1}{2}(1 + s s_{be}) e^{K s_e s_{be}} = \cosh K + s s_e \sinh K . \tag{3.8}$$

After all these boundary summations and again retaining the terms up to linear order in s, h, and sh in the expansion of small s and h, the partition function can be written

$$Z_{(N+4)N} \approx 2^{N \times N} \cosh^{\alpha} K (A_0 + P_1 sh \tanh K) , \tag{3.9}$$

where A_0 and P_1 here are exactly the same as those in Eq. (3.6). Therefore,

$$[(\tanh K_N)/N^2](P_1/A_0) = 1 . \tag{3.10}$$

Equations (3.7) and (3.10) clearly show the close connection between the equations for systems with EA and EP. In fact, by the same procedure as above, it can be shown that this relationship holds in general for any finite systems using these two boundary conditions. That is, to obtain the critical temperature equation of a corresponding system with EP, we just replace the multiplicative factor K_N in that of system with EA by $\tanh K_N$. We will also see that these two systems behave rather similarly.

We can use the same approach used above to derive the equation for the $N \times N$ system with EBP. As before, it can be obtained by expanding the partition function in small h_{eff} and h as we use the second and the last terms in (2.10) directly. It is clear that we need the expansion to the orders h_{eff}, h, $h_{\text{eff}} h$ for $h \neq 0$ and to h_{eff}, h_{eff}^2 for $h = 0$. For the same reasoning as before, it is obvious that terms linear in h_{eff} or h alone vanish. Explicitly, for $h \neq 0$

$$Z_{(N+4)N} \approx 2^{(N+4)N} (A_0 + P_1' h_{\text{eff}} h) \cosh^{\alpha+\beta} K , \tag{3.11}$$

and for $h = 0$

$$Z_{(N+4)N} \approx 2^{(N+4)N} (1 + \tfrac{1}{2} h_{\text{eff}}^2)^{4N} (A_0 + P_{11}' h_{\text{eff}}^2) \times \cosh^{\alpha+\beta} K , \tag{3.12}$$

where α, β denote the total number of internal bonds, and bonds between edge (or corner) spins and their boundary spins, respectively; A_0 in both these expansions is the same as that in (3.6) and (3.9). Using (2.10), (3.11), and (3.12), we get

$$(2P_1' - NP_{11}')/2N^2 A_0 = 1 . \tag{3.13}$$

In summary, we have derived the general equations for the critical temperature of $N \times N$ systems with EA, EP, and EBP, as given in (3.7), (3.10), and (3.13), respectively. It is clear that for the direct use of these equations, we have to calculate quantities such as A_0, P_1, P_1', and P_{11}' explicitly in the equations. It is understood that terms in these quantities in (3.6), (3.9), (3.11), and (3.12) arise from contributions of certain allowable graphs. For example, terms in A_0 contain exactly the same type of Ising graphs used in the expansion of the zero-field partition function by the series method. That is, they are closed polygons embedding now on an $N \times N$ lattice with free boundary conditions all around. On the other hand, the contributions to P_1, P_1', and P_{11}' all come from certain magnetic graphs (i.e., a magnetic graph is one which has two and only two free ends). In P_1, the allowable graphs are those having one end at an edge or a corner and the other end elsewhere at any internal site. We will see later that this quantity is related to the definition of the boundary susceptibility. In P_1', they are those having one end at a boundary spin and the other end on any of the internal spins. Obviously, these graphs differ from those in P_1 only in that graphs in P_1 have one end on an edge or a corner spin (of the internal spins) rather than a boundary spin. Hence, P_1' and P_1 are related by the following relation:

$$P_1' = w P_1 . \tag{3.14}$$

On the other hand, graphs in P_{11}' are those having two different boundary spins for the two ends of magnetic graphs. Similarly, we can calculate first a quantity P_{11} defined such that allowable graphs are those having one edge and a corner spin, two different edge spins or two corner spins (may be the same) for the two ends. Then P_{11}' is related to P_{11} by

$$P_{11}' = w^2 P_{11} . \tag{3.15}$$

Therefore, the derivations of the equations for

the critical temperature reduce to the combinatorial problem of counting the total number of ways of placing certain allowable graphs on the $N \times N$ lattice. We did this by manual counting. The calculations were performed to the orders of w^9 and w^{10} with the use of the so-called elementary graphs (i.e., those having one end at some edge and the other end elsewhere at any internal spin) to w^9 to facilitate the counting procedure. These elementary graphs were arranged in such a way that they can also be used in the explicit calculations of $N \times \infty$ systems. We now apply the critical temperature equations directly to small $N \times N$ systems (for $N = 2-5$) with EA, EP, and EBP. We will also mention the problem associated with the direct application of these equations to an $N \times N$ system when N is large.

To illustrate how this series approach works explicitly for small $N \times N$, we consider the system with EA for simplicity. First, we represent graphs in A_0 and P_1 by the following entities. $\bar{ij}$: This represents the nearest-neighbor bond link between i and j, and has a weight w. + : This represents an edge spin and has a weight t_1 or Ks in limit of small s. × : This represents a corner spin and has a weight t_2 or $2Ks$ in limit of small s. ○ : This represents a magnetic field element acting on every internal site and has a weight $\tanh h$ or h in limit of small h. We give in Fig. 1 a few typical allowable graphs in A_0 and P_1. In passing, we note that figures b, c, y, z, etc., of these Ising graphs are not taken into account in the so-called self-avoiding-walk (SAW) approximation which is slightly easier but we do not consider this approximation here. With these entities, we can obtain all terms in A_0 and P_1 to the order of w^9 with the use of the elementary graphs by exact enumerations of all allowable graphs to this order on the lattice. As a simple illustration, we consider 2×2 with EA in Figs. 2 and 3; where it has only four corner spins and no edge spin. The number in square brackets,

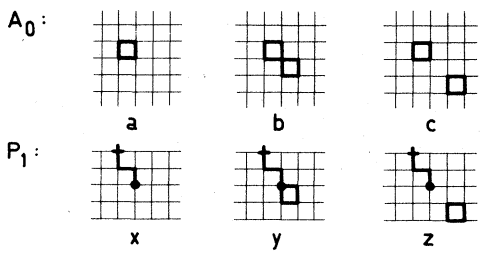

FIG. 1. Some allowable graphs in A_0 and P_1.

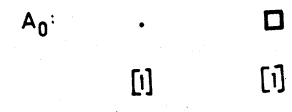

FIG. 2. Graphs in A_0 for 2×2 system.

denotes the total number of ways of embedding the graph sketched above it onto the finite lattice. The total number of the internal bonds is four. Therefore, all possible graphs on the lattice have been counted and the final result given by

$$(\tfrac{1}{4}K_2)[(8+16w+16w^2+16w^3+8w^4)/(1+w^4)]=1 \tag{3.16}$$

is exact and equivalent to Eq. (2.4) of Bolton and Johnson,[16] as expected. This is readily generalized to the corresponding system with EP. By this method, we can also derive the exact results for $N \times 1$ systems obtained previously by the authors.[6] For $N \geq 3$ of $N \times N$, the final equations are not expected to be exact as graphs of order higher than w^9 exist. It is clear that A_0, P_1, and P_{11} are all power series in w and hence can be written individually as $\sum_{n=0} C_n w^n$. We summarize our explicit results for this coefficient C_n $(n = 0-9)$ for $N \times N$ systems $(N = 2-5)$ in Table I. We recall that P_1' and P_{11}' in (3.13) are related to P_1 and P_{11}, respectively, by (3.14) and (3.15). We have so far considered rather small $N \times N$ systems $(N = 2-5)$. We consider briefly here the problem associated with the direct applications of Eqs. (3.7), (3.10), and (3.13) when N is large. Take a large $N \times N$ system with EA. A_0 and P_1 are of the following form:

$$A_0 = 1 + (N^2 - 2N + 1)w^4 + (2N^2 - 6N + 4)w^6 + (\tfrac{1}{2}N^4 - 2N^3 + \tfrac{15}{2}N^2 - 23N + 22)w^8 + \cdots, \tag{3.17}$$

$$P_1 = (4N) + (12N - 8)w + (28N - 32)w^2 + (76N - 112)w^3 + (4N^3 - 8N^2 + 200N - 368)w^4 + \cdots . \tag{3.18}$$

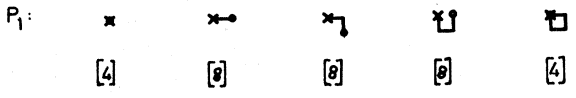

FIG. 3. Graphs in P_1 for 2×2 system.

TABLE I. Series coefficients of A_0, P_1, and P_{11} for $N\times N(N=2-5)$ systems.

n \ N	A_0				P_1				P_{11}			
	2	3	4	5	2	3	4	5	2	3	4	5
0	1	1	1	1	8	12	16	20	4	4	4	4
1	0	0	0	0	16	28	40	52	16	16	20	24
2	0	0	0	0	16	52	80	108	16	26	24	28
3	0	0	0	0	16	104	192	268	16	64	64	64
4	1	4	9	16	8	200	544	932	4	132	196	210
5	0	0	0	0	0	256	1112	2032	0	160	556	720
6	0	4	12	24	0	312	2032	5008	0	196	888	1436
7	0	0	0	0	0	312	3488	10700	0	192	1640	3408
8	0	7	50	157	0	300	5984	23668	0	176	2560	7568
9	0	0	0	0	0	132	9216	46260	0	80	4604	18256

It is obvious that, for large N, the direct application of the equation for the critical temperature as given in (3.7) will fail because terms of higher orders will dominate and calculations to the order of w^9 are certainly not enough. However, this problem is not important for small $N\times N$ ($N=2$–5) where we can apply these equations directly. For very large N, we then have to look at the logarithmic expansion of the partition function in powers of N as we do below for the $N\times N^*$ system where $N^*\to\infty$.

We now follow the above procedure to the $N\times N^*$ systems with EA, EP, and EBP on the sides. As mentioned, there are no corners. There are $N\times N^*$ and $2N^*$ total number of internal spins and boundary spins, respectively. Therefore, we replace $Z_{(N+4)N}$, $N\times N$, and $4N$ by $Z_{(N+2)N^*}$, $N\times N^*$, and $2N^*$, respectively in (2.2)–(2.10). We still have the expansions of the partition function of the form in (3.6) for the $N\times N^*$ system with EA, in (3.9) for that with EP, and in (3.11) and (3.12) for that with EBP except we again replace $4N$ in (3.12) by $2N^*$. Since we are interested in letting $N^*\to\infty$ eventually, we derive the equation for the critical temperature by examining the logarithmic expansion of the partition function for large N^*. More explicitly, take the $N\times N^*$ system with sides EA. It is easily seen that A_0 and P_1 can be written

$$A_0 = 1 + a_1 N^* + a_2 N^{*2} + \cdots , \tag{3.19}$$

$$P_1 = b_1 N^* + b_2 N^{*2} + b_3 N^{*3} + \cdots , \tag{3.20}$$

where $a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$ are polynomials of w only. We can rewrite the partition function as

$$Z_{(N+2)N^*} \approx 2^{N\times N^*}[1 + (a_1 + b_1 hKs)N^* + (a_2 + b_2 hKs)N^{*2} + \cdots]\cosh^{\alpha} K , \tag{3.21}$$

Taking logarithms on both sides and by the direct logarithmic expansion in N^*, we get

$$\frac{\ln Z_{(N+2)N^*}}{N\times N^*} \approx \text{other terms} + b_1 hKs/N + \cdots . \tag{3.22}$$

Using (2.4), we obtain the following equation for the critical temperature of $N\times\infty$ with sides EA:

$$K_N b_1/N = 1 , \tag{3.23}$$

where N may be small or even one. Note that b_1 is the coefficient of N in the polynomial of P_1. The equation for the corresponding system with EP is again obtained by replacing the multiplicative factor K_N in (3.23) by $\tanh K_N$.

To obtain the equation for the critical temperature of $N\times\infty$ system with EBP on sides, we follow similar procedure as above by first treating an $N\times N^*$ for N^* large. A_0, P_1', and P_{11}' involved here have similar forms to those in (3.19) and (3.20) except we replace b_i (for $i\geq 1$) in (3.20) by b_i' and b_{ii}', respectively, for P_1' and P_{11}'. We finally obtain

$$(b_1' - N b_{11}')/N = 1 . \tag{3.24}$$

We note that terms in A_0, P_1', and P_{11}' contain the same types of graphs as those for $N\times N$ systems, except now the embedding is on an $N\times N^*$ cylindrical system rather than an $N\times N$ square system. However, it is true that we still have

$$b_1' = w b_1 \tag{3.25}$$

similar to that in (3.14). Again, we can relate P_{11}' to P_{11} by

$$b'_{11} = w^2 b_{11} \tag{3.26}$$

as in (3.15) by defining P_{11} such that graphs are those having any two different edge spins for the two ends and P_{11} is given by (3.20) except that b_i is replaced by b_{ii}. We have used our elementary graphs to calculate explicitly all the terms exactly in b_1, b_{11} to w^9 and hence the series $b'_1 - N b'_{11}$ in (3.24) using the relations in (3.25) and (3.26) for these $N \times \infty$ systems where $N = 1-10$. They are given in Table II. We note that the equations with these series are expected to be exact to these orders. For example, using (3.23) for $N \times \infty$ systems with EA, it has been checked for $N = 1, 2$ where the exact results can be both obtained by other methods. The agreement for the result of $2 \times \infty$ which has been derived by using a transfer-matrix method and a first-order perturbation indicates to some extent the accuracy of the embedding calculations to the order of w^9 with the use of elementary graphs.

IV. NUMERICAL TECHNIQUES AND EXTRAPOLATION

We have previously obtained the equations for the critical temperature of $N \times N$ and $N \times \infty$ systems using EA, EP, and EBP, as those given in (3.7), (3.10), (3.13), (3.23), and (3.24) where the series involved are given in Tables I and II using the relations in (3.14), (3.15), (3.25), and (3.26). We recall that only a limited number of terms (i.e., to the order of w^9 or w^{10}) were calculated. We here propose two different techniques to derive the critical temperatures from these data of $N \times N$ and $N \times \infty$ systems. We are interested in $N \times N$ for $N = 2-5$ and $N \times \infty$ for $N = 1-8$.

For $N \times N$ systems, we propose here some truncating procedure which allows us to locate quite accurately the critical temperatures of these small systems. As an illustration, take 4×4 system with EA all around. Explicitly, the equation for the critical temperature is given by (3.7), or

$$\frac{K_4}{16} \frac{16 + 40w + 80w^2 + 192w^3 + 544w^4 + 1112w^5 + 2032w^6 + 3488w^7 + 5984w^8 + 9216w^9}{1 + 9w^4 + 12w^6 + 50w^8} = 1 \tag{4.1}$$

where A_0 and P_1 are given in Table I. We now treat the series on the left-hand side of (4.1), i.e., P_1/A_0 as a truncated series of w and hence obtain a series of truncated equations from, say, order w^3–w^9. By solving numerically for each individual equation, we obtain $T_4(r)$, the solutions of the truncated equations for $r = 3-9$. It is clear that the result for $T_4(r)$ gets closer to the exact critical temperature (i.e., T_4 which is unknown here) as r increases. Moreover, we observe a rapid convergence indicating that the higher-order terms in (4.1) become less important in determining the solution. We find the approximate critical temperature by plotting $T_4(r)$ vs $1/r$ (see Fig. 4) for $r = 3-9$ using an extrapolation procedure. The intersection of the extrapolated curve with the line $r = 24$ gives T_4 approximately, where the value 24 is the maximum number of internal bonds. This line was used because we do not know what the number of bond is for the largest allowable graph embedding on the lattice. Presumably, if all allowable graphs for A_0 and P_1 up to this largest one were taken into account in the equation, the solution would be the exact T_4. However, the difference in the final results caused by this different choice for the line is expected to be insignificant. The result for T_4 found on Fig. 4 is 3.104 ± 0.006. We note that the regularity of the points plotted indicates the validity of the truncation method used here. The lines used for 3×3 and 5×5 are, respectively, $r = 12$ and $r = 40$. We can readily extend this technique to small $N \times N$ systems with EP or EBP. We remark that this procedure is not necessary for 2×2 as all graphs can be taken into account. Moreover, this method becomes less accurate as N increases meaning that exact data for terms of order higher than w^9 are needed. We now summarize in Table III the results for these small $N \times N$ systems obtained by this truncation and extrapolation procedure, together with some existing results for $N = 1, 2$. The exact results quoted for 3×3 with EA and EP are derived by exact calculations. These compare with the results: $T_3 \approx 3.2483 \pm 0.0001$ for EA and 3.1977 ± 0.0001 for EP obtained from the above extrapolation techniques. These agreements strengthen the accuracy of our T_3 of 3×3 system with EA compared with a slightly different value (≈ 3.2433) given exactly by Bolton and Gruen who also got

TABLE II. Series coefficients of b_1, b_{11}, and $b_1' - Nb_{11}'$ for $N \times \infty$ ($N = 1$–10) systems. Note: blank entries in a row represent the same figure as in the nearest column.

n \ N	b_1 1	2	3	4	5	6	7	8	9	10	b_{11} 1	2	3	4	5	6	7	8	9	10	$b_1' - Nb_{11}'$ 1	2	3	4	5	6	7	8	9	10
0	2																				0									
1	4	6									1	0									2									
2	4	12	14								4	3	2								3	6								
3	4	24	36	38							4	6	3	2							0	6	8	6	4	2	0	−2	−4	−6
4	4	40	80	96	98						4	12	10	5	4						0	12	27	30	28	26	24	22	20	18
5	4	64	164	232	252	254					4	20	26	16	9	8					0	16	50	76	78	74	70	66	62	58
6	4	92	304	512	616	640	642				4	32	56	50	28	19	18				0	24	86	168	207	206	198	190	182	174
7	4	120	544	1068	1448	1596	1624	1626			4	46	102	124	86	48	37	36			0	28	136	312	476	526	516	498	480	462
8	4	148	920	2104	3216	3848	4048	4080	4082		4	60	180	272	250	152	94	81	80		0	28	238	572	1018	1308	1365	1338	1302	1266
9	4	152	1484	3992	6816	8940	9920	10180	10216	10218	4	74	302	530	620	462	268	186	171	170	0	28	380	1016	1966	2936	3390	3432	3362	3282
10											4	76	492	984	1384	1294	844	510	400	383	0	4	578	1872	3716	6168	8044	8692	8677	8518

$T_4 \approx 3.114 \pm 0.009$, $T_5 \approx 3.074 \pm 0.089$, and $T_{10} \approx 2.758 \pm 0.044$ for EA using a Monte Carlo method. We note that our results in Table III for T_4 and T_5 are in accordance with their results. We also observe that the results for systems with EBP are much closer to the exact value ($\approx 2.269\,185$) than those of EA or EP in which the results are approximately the same.

We now proceed to find the critical temperatures of $N \times \infty$ by an extrapolation technique. Again, only a limited number of terms (to w^9 or w^{10}) in the series (see Tables I and II) of the equations [see (3.23) and (3.24)] using relations (3.25) and (3.26) are known. We apply an extrapolation method to deduce the coefficients of higher-order terms by assuming that they follow the apparent behavior of the first dozen or so known coefficients. This is in the same spirit as that adopted in the standard series expansion. However, we are interested, for each $N \times \infty$, in the values of these higher-order coefficients in the series, but not in their asymptotic behavior as in the series method. As an example, take again the $N \times \infty$ system with side EA. The coefficients a_r ($r = 0$–9) for b_1 ($\equiv \sum_{r=0}^{9} a_r w^r$) are given in Table II. We plot the ratio a_r/a_{r-1} vs $1/r$ of $r = 3$–9 for $N = 1$–8. Two limiting lines are drawn within reasonable range through these points. The upper- and lower-limit values of the coefficients of terms higher than w^9, say from w^{10}–w^{20} (i.e., a_r for $r = 10$–20), are estimated from these lines. Now, using all these coefficients to w^{20} and solving the equation, we finally obtain the critical temperature of a given $N \times \infty$ system. Again, this extrapolation procedure faces increasing difficulty as N increases, mainly because more exact coefficients are required. As an illustration, we present here a sample of the ratio plot for $4 \times \infty$ in Fig. 5, where we also plot the points from the exact result of $2 \times \infty$ system to show how they behave in an exact solution. We again remark that terms of higher order have less and less influence on the final result as the order increases. Other than the straightforward generalization to $N \times \infty$ system with EP, we can also extend the method to that with EBP. We consider $N = 1$–7 and the ratio plots of the series $b_1' - Nb_{11}'$ given in Table II are found to be oscillating. Instead we use the r-root plot, i.e., $(a_r)^{1/r}$ vs $1/r$ of this series to estimate the coefficients of the higher-order terms from w^{11}–w^{20}. We show a sample of $4 \times \infty$ in Fig. 6. We again obtain the critical temperatures by solving the equations to this order. Results for these $N \times \infty$ systems with sides EA, EP, and EBP are summarized in Table IV. These results again demonstrate that results of the systems with EBP are closer to the exact value than those with EA or EP. The

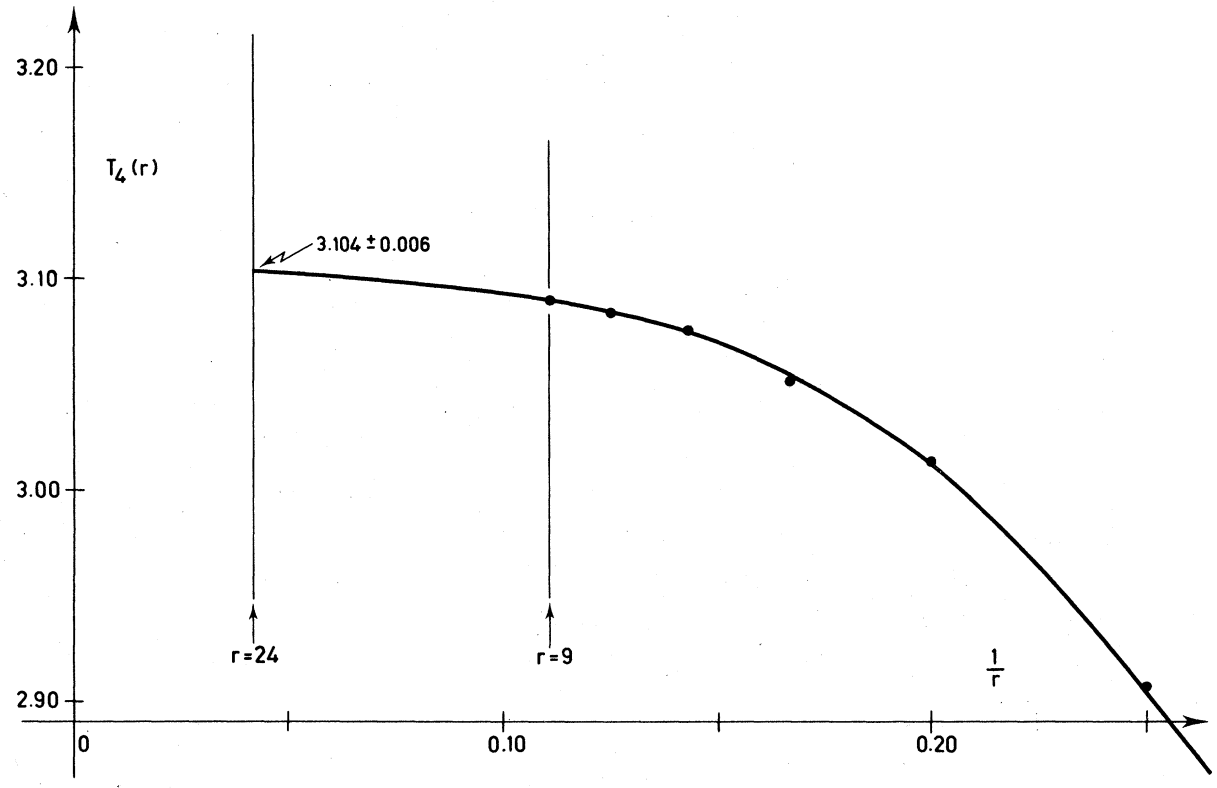

FIG. 4. Plot of $T_4(r)$ against $1/r$ for 4×4 system with EA boundary conditions.

reason for this is that the Bethe-Peierls method incorporates some information about the nearest-neighbor correlation coefficient.

V. RESULTS AND DISCUSSIONS

We have summarized the results for T_N of $N\times N$ ($N=1-5$) and $N\times\infty$ ($N=1-8$) using EA, EP, and EBP in Tables III and IV, respectively. Before we study the asymptotic size dependence of these results [i.e., the value of λ defined in Eq. (1.1)], we make a few remarks on the physical meaning of the series in the equations for the critical temperature especially those of $N\times\infty$. It is obvious that the $N\times\infty$ system in the thermodynamic limit $N\to\infty$ is equivalent to the infinite half plane considered by McCoy and Wu,[20] and Binder and Hohenberg.[21] It can be readily shown that, from the types of allowable graphs in P_1 and P_{11} of $N\times\infty$, the series b_1 and b_{11} given in Table II for $N\to\infty$ are related to the so-called layer (χ_1) and local ($\chi_{1,1}$) susceptibilities in the following and we also quote their values directly from our results in Table II:

TABLE III. Summary of T_N of $N\times N$ ($N=1-5$) systems.

N	EA	EP	EBP
1	4.000 000 [a]	3.915 230 [b]	2.885 390 [a]
2	3.499 621 [b]	3.436 877 [b]	2.830 932 [b]
3	3.248 269	3.197 700	2.7614 ± 0.0010
4	3.104 ± 0.006	3.062 ± 0.006	2.715 ± 0.010
5	3.005 ± 0.015	2.980 ± 0.015	2.670 ± 0.030

[a] Molecular-field and Bethe-Peierls approximations.
[b] Exact calculation (see Ref. 16).

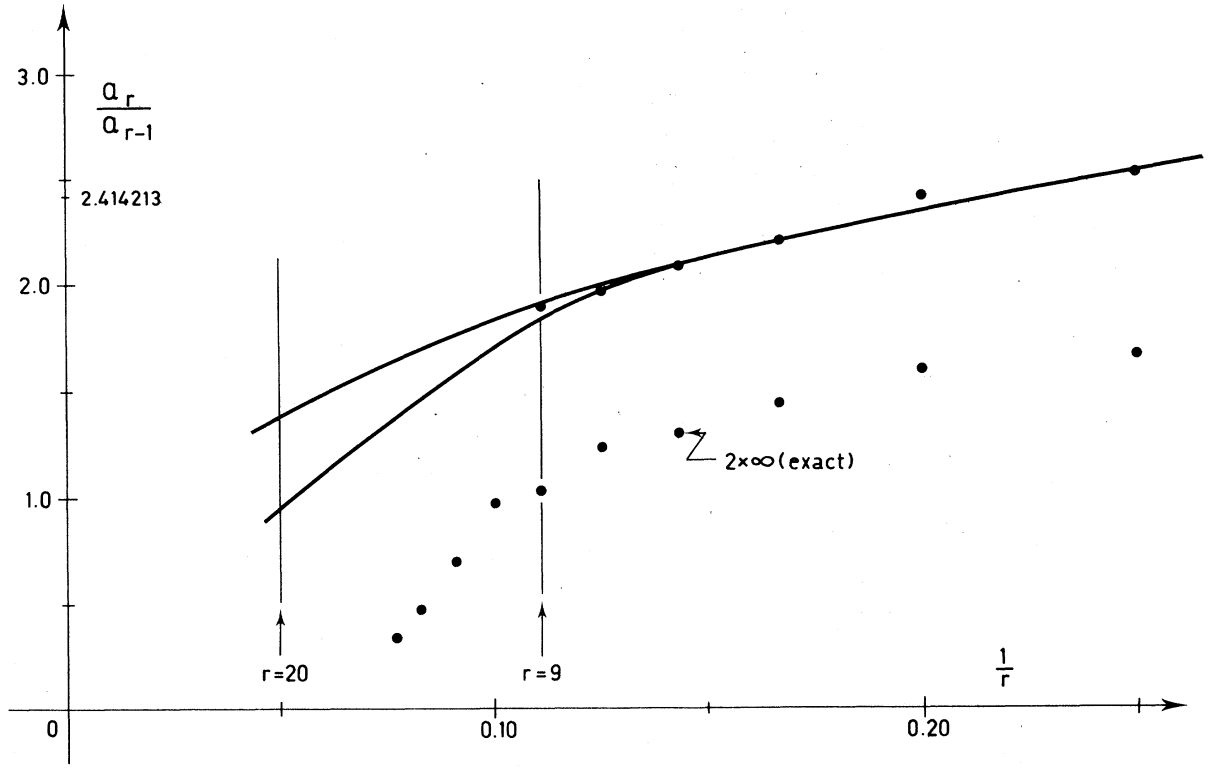

FIG. 5. Ratio plots for $4\times\infty$ and $2\times\infty$ systems with EA boundary conditions.

$$\chi_1 = \tfrac{1}{2}b_1 \tag{5.1}$$

$$= 1 + 3w + 7w^2 + 19w^3 + 49w^4 + 127w^5 + 321w^6 + 813w^7 + 2041w^8 + 5109w^9 \tag{5.2}$$

and

$$\chi_{1,1} = 1 + b_{11} \tag{5.3}$$

$$= 1 + 2w + 2w^2 + 4w^3 + 8w^4 + 18w^5 + 36w^6 + 80w^7 + 170w^8 + 382w^9 \tag{5.4}$$

to the order of w^9. χ_1 differs from that of Binder and Hohenberg[21] who have calculated χ_1 and $\chi_{1,1}$ to w^{10} in the coefficients of w^7 and w^9 which are given by 815 and 5117, respectively, in their calculations. However, the series $\chi_{1,1}$ here agree exactly with theirs to this order w^9. The origin of the discrepancy for χ_1 is unclear. We do not mean that our results are more accurate, but the differences, in our view, are small and should not affect our final results obtained by the truncation and extrapolation techniques used previously. Moreover, Binder and Hohenberg also conclude that the critical exponents denoted by γ_1 and $\gamma_{1,1}$ of χ_1 and $\chi_{1,1}$ are given by $\frac{11}{8}\pm 0.05$ and 0 (or logarithmic divergence found from exact calculations[20]), respectively.

The authors recently[6] have studied the value of λ from the exact results of some $N\times 1$ systems using either EA or EP on both sides and at both ends. It was found that $\lambda = 1$ which agree with those of other studies.[4,5,7] Assuming this and a certain asymptotic form of T_N, Bolton and Gruen[17] located the bulk critical temperature by extrapolating the results obtained from small $N\times N$ systems with EA all around. However, the exact results on the $N\times 1$ systems cannot draw very definite conclusions on either $N\times N$ or $N\times\infty$ systems using these boundary conditions. This is because the $N\times 1$ systems are large in only one dimension and the bulk critical temperatures are known[6] to be different from the Onsager value, as expected. We will see that $\lambda = 1$ does not hold for the two-dimensional finite systems considered in this work.

Since the values of N considered for these $N\times N$ and $N\times\infty$ systems are very small, some modified form was used to find λ instead of using the asymptotic form given in (1.1), as usually done.[8-11] In each case, plots of $(T_N - T_c)^{-1/\lambda}$ vs N were

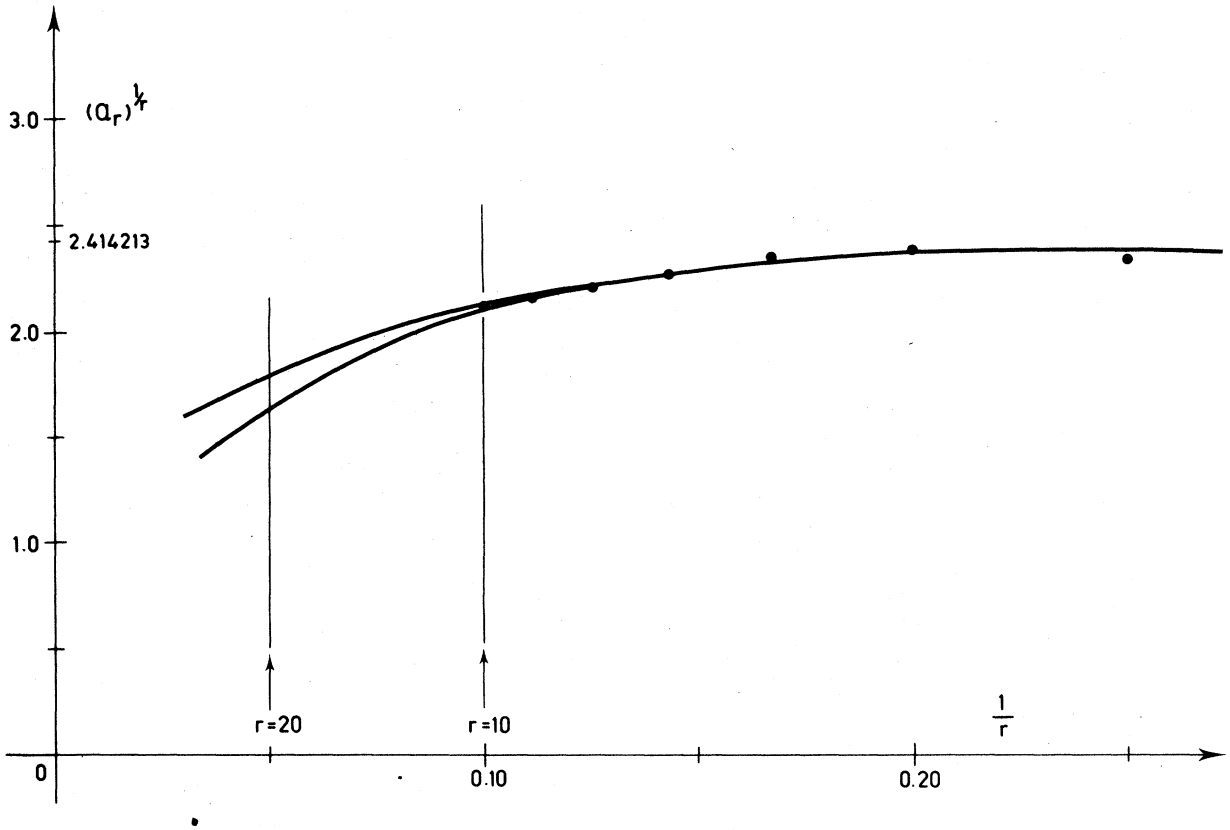

FIG. 6. Root plot for $4 \times \infty$ system with EBP boundary conditions.

carried out for various trial values of λ, where the results are given in Tables III and IV, and T_c is the Onsager value. The value which gives the most linear plot for the larger values of N provides the favored estimate for λ. It is expected that the results for λ obtained from $N \times N$ systems are less conclusive than those from $N \times \infty$ systems in which more accurate T_N and more values of N are considered. Moreover, the finite-size effect is greater in the former cases due to the existence of the corners, especially for these small systems. Indeed, the estimations of λ for both $N \times N$ systems with EA and EP are quite difficult, and were found to be $\lambda \approx 0.60 \pm 0.20$ in both cases though they should not be considered conclusive. The results for $N \times \infty$ systems with sides EA and EP also behave similarly and a good straight-line fit to all points ($2 \leq N$) can be found with $\lambda = 0.75$. Trial values of λ differing by ± 0.07 clearly display curvature in opposite senses. They can both be represented approximately by

TABLE IV. Summary of T_N of $N \times \infty$ ($N = 1$–8) systems.

N	EA	EP	EBP
1	3.526 445 [a]	3.465 907 [a]	2.885 390 [b]
2	3.155 924 [b]	3.111 801 [b]	2.754 434
3	2.970 68 ± 0.000 41	2.935 62 ± 0.000 46	2.672 52 ± 0.000 81
4	2.854 6 ± 0.002 0	2.825 2 ± 0.002 2	2.611 6 ± 0.003 2
5	2.772 5 ± 0.003 5	2.746 7 ± 0.003 7	2.559 2 ± 0.005 0
6	2.713 6 ± 0.004 7	2.690 6 ± 0.005 0	2.515 3 ± 0.005 9
7	2.671 3 ± 0.008 2	2.650 8 ± 0.008 8	2.481 ± 0.013
8	2.638 0 ± 0.009 5	2.619 0 ± 0.010 2	

[a] See Ref. 6.
[b] Exact calculations.

$$T_N = T_c + a/(N+0.60)^{0.75}, \tag{5.5}$$

where

$a \approx +1.84$ for EA

$\approx +1.74$ for EP.

This representation is found to yield the values of T_N to within 1% down to $N=1$ in both cases. Moreover, it can be shown that no reasonable straight-line fit to all the available points ($2 \leq N$) for these $N \times N$ and $N \times \infty$ systems with EA, EP, and EBP can be found with $\lambda = 1$. The above results for these systems with EA and EP indicate that $\lambda = 1/\gamma_1$ (≈ 0.70–0.75), where γ_1 is the critical exponent of layer susceptibility[21] mentioned before. However, this does not appear to hold for systems with EBP. A much higher value (i.e., $\lambda \geq 2.0$) is likely and no definite conclusion can be made. We assemble the values of λ from the literature into Table V.

A general point remains to be analyzed. Finite-size scaling theory has been used by Fisher[9,22] to discuss the way in which the critical temperature of a finite-sized sample would approach T_c as the size diverges to infinity. We give a brief description for, say, the free boundary condition. Consider an $N \times N$ cluster; its free energy per spin F_{N^2} can be approximated asymptotically (as long as the temperature considered is not too close to T_c) as

$$N^2 F_{N^2}(T) \simeq N^2 F_\infty(T) + 4N F^\times(T) + \cdots, \tag{5.6}$$

where $F_\infty(T)$ and $F^\times(T)$ are the bulk and boundary free energy per spin, respectively. The scaling theory suggests that the shift exponent λ is given by $1/\nu$, which is equal to 1 and 1.56 in two and three dimensions, respectively. Evidence for this is given in our Table V. The three boundary conditions discussed in this paper are at first sight of the same kind as the free boundary condition in that a surface term in the free energy has to be inserted. However, our values of λ are substantially less than unity. An explanation of the apparent inconsistency is that the boundary conditions EA and EP imply that the parameter s which acts as a boundary field has to be included in the free energy. It is assumed that

$$N^2 F_{N^2}(T,s) \simeq N^2 F_\infty(T) + 4N F^\times(T,s) + \cdots, \tag{5.7}$$

where we expect that $F^\times(T,s) = F^\times(T)$ as $s = 0$. Since every physical property involves differentiation of the free energy for the whole system we can for instance write the internal energy per spin as proportional to

$$\frac{\partial F_{N^2}(T,s)}{\partial T} \simeq \frac{\partial F_\infty(T)}{\partial T} + \frac{4}{N}\frac{\partial F^\times(T,s)}{\partial T} + \frac{4}{N}\frac{\partial F^\times(T,s)}{\partial s}\frac{\partial s}{\partial T} + \cdots \tag{5.8}$$

and we see that there is an extra term compared to Eq. (5.6). Since this term contains $\partial s/\partial T$ and this can be large, it seems that there is a larger boundary correction in the EA and EP than in the free boundary condition and this implies a correspondingly smaller value for λ. In other words, the crossover region for a finite system with EA or EP would be larger than that with the free boundary condition. A possible physical interpretation of this behavior of EA and EP is that the spins near the boundary are correlated strongly with the boundary value s and the amount of a cluster available for the asymptotic behavior outside the crossover region is reduced. In other words, the effective size of a cluster is less than the real size. It therefore appears that the EA and EP are unfavorable in comparison with the free and periodic boundary conditions. However, this is certainly not so when we are interested in the zero-field magnetization, as pointed out

TABLE V. Shift exponents λ for finite Ising systems.

	$N \times N$ with boundary conditions all around				$N \times \infty$ with sides boundary conditions			
2-D	Periodic	Free	EA or EP	EBP	Periodic	Free	EA or EP	EBP
	1[a]	1[b]	0.60 ± 0.20	Not clear	$\sim \frac{1nN}{N^2}$[c]	1[d]	0.75 ± 0.07	Not clear
	$N \times N \times N$ with boundary conditions all around				$N \times \infty \times \infty$ with surfaces boundary conditions			
3-D	Periodic		Free		Periodic		Free	
	$1/\nu_3$[e] ≈ 1.56		$1/\nu_3$[e]		2.0 ± 0.1[f]		$1/\nu_3$[f]	

[a] Exact calculation (Ref. 4).
[b] Monte Carlo simulation (Ref. 7).
[c] Exact calculation (Ref. 3).
[d] Exact calculation (Ref. 5).
[e] Monte Carlo simulation (Ref. 14).
[f] Series expansion (Refs. 9 and 10).

in the Introduction. For the EBP, the above argument does not seem to hold because our results suggest a value $\lambda > 1$.

In summary, we have demonstrated how the equations for the critical temperature of small $N \times N$ ($N = 1-5$) and $N \times \infty$ ($N = 1-8$) systems with one of the self-consistent boundary conditions (i.e., EA, EP, and EBP) can be formulated in terms of some high-temperature expansion series which were calculated to ninth and tenth orders. The series involved in $N \times \infty$ for $N \to \infty$ are connected to the layer and local susceptibilities considered by Binder and Hohenberg.[21] The equations for the systems with EP and EA are shown to be related. Their results are similar and are not as good as those with EBP which is more complicated. The critical temperatures were all obtained numerically with the use of truncation and extrapolation techniques. They do not show $\lambda = 1$ but those of EA and EP systems indicate that $\lambda = 1/\gamma_1$ while that of EBP is not consistent with this value. From the nature of the numerical methods used, longer series are certainly preferable. They not only sharpen the precision of the estimated critical temperatures and also enable larger systems to be studied. In this way, more precise value of λ can be made, especially for the $N \times N$ systems. We feel that these series techniques can be further extended to treat larger systems and three-dimensional systems with the use of computer counting.

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