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New magnetic transition above the percolation threshold: Disordered Ising model on a Cayley tree

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Magnetic phase transitions in a dilute Ising spin system on a bounded Cayley tree are dominated by the spins lying at the surface in the thermodynamic limit. An exact analytic treatment incorporating explicitly the spins at the surface of a site-disordered Cayley tree is developed for constant nearest-neighbor exchange interactions, and the existence of a new type of magnetic transition at low temperatures is demonstrated. This transition is characterized by the absence of spontaneous magnetization, by a divergence of the magnetic susceptibility at the percolation threshold p_c as well as at $(p_c)^{1/2}$, and by a divergence of all higher-order zero-field correlation functions at p_c and at higher concentrations forming a discrete set. The divergence of the susceptibility at p_c marks the onset of ferromagnetic ordering in the central region of the Cayley tree. We also calculate explicitly the leading nonregular magnetic field dependence of the magnetization at high concentrations of spins, and show that the transition is effectively of infinite order. The possible relevance of our results to the transition in a diluted two-dimensional *xy* model is noted.

I. INTRODUCTION

A new type of magnetic phase transition has been found recently for a spatially ordered (pure) ferromagnetic Ising model on a Cayley tree.[1,2] This transition is characterized by the absence of spontaneous magnetization per spin, a divergence of the magnetic susceptibility[3,4] and of all higher zero-field derivatives (correlation functions) of the free energy,[2] respectively, at temperatures forming a discrete set extending from $T=0$ to the Bethe-Peierls critical temperature (T_{BP}) for ferromagnetism in the central region of the Cayley tree. These unusual properties arise from the fact that the majority of atomic spins on a large Cayley tree lie at the surface, since the ratio of the number of surface spins to the total number of spins tends to $1-1/K$ (K is the connectivity of the lattice) in the thermodynamic limit.[1,3] In particular, the divergence of successive zero-field derivatives of the free energy marks the onset of successively higher-order correlations of the spins lying at the boundary. This is shown indirectly by the fact that when the Cayley tree is treated implicitly as an unbounded lattice, it has only an ordinary ferromagnetic transition at the temperature $T=T_{\text{BP}}$,[5] as expected.

The transition occuring below T_{BP} on a bounded Cayley tree has been referred to as a transition of continuous order.[2,6] This is because, for $T<T_{\text{BP}}$, the leading nonanalytic contribution in the free energy in the presence of an applied field of magnitude $h\equiv|h|\to 0$ varies as $|h|^{\kappa(T)}$, where $\kappa(T)$ smoothly increases from 1 to ∞ as T varies from zero to T_{BP}. This implies that the order of the transition in an Ehrenfest classification varies continuously from 1 at $T=0$ to ∞ at T_{BP}.

In the present paper we study for the first time a similar phase transition of continuous order occurring in a dilute (i.e., site-disordered) Ising model on a Cayley-tree lattice in the vicinity of the percolation threshold at low temperatures. In this model the exchange interaction $J>0$ between nearest-neighbor magnetic atoms on the lattice is assumed to be constant, but a fraction $1-p$ of the lattice sites are occupied at random by nonmagnetic atoms. As is well known, such a system displays a ferromagnetic transition when the fraction p of Ising spins is increased beyond the percolation threshold p_c, provided all the sites are regarded as equivalent, i.e., the system is treated effectively as unbounded. This type of treatment describes, therefore, the transition taking place among the spins in the central region of an actual (bounded) Cayley tree. Of course, the neglect of the boundary is justified if one is using the Cayley-tree approximation to simulate the properties of a real three-dimensional lattice, where the fraction of surface spins is indeed small and has a negligible influence on the average bulk properties.

In general, however, the phase transition of a disordered Ising model on a bounded Cayley tree may be expected to be dominated by the spins at the surface, just as in the corresponding ordered case.[1-3] On the other hand, as far as the transition in the central region of the Cayley tree is concerned, the properties (e.g., critical indices) of the pure Ising model near T_{BP}, and those of the site-disordered Ising model near p_c at $T=0$, are known to be quite different.[7] Thus, one may expect similar qualitative differences to exist between the thermodynamic phase transitions below T_{BP} in an ordered Cayley tree and above p_c in a disordered tree at $T=0$, when the distinct nature of the surface sites is taken into account.

Part of the motivation for the present work arose also from the recent study of a phase transition of continuous order in the spatially ordered two-dimensional *xy* model by Zittartz.[8] This study reveals some remarkable similarities between the transition of the *xy* model and that of the Ising model on a bounded Cayley tree. This suggests that the bounded Cayley tree gives to some extent a model description of phase transitions in two-dimensional systems with continuous symmetry (where usual second-order phase transitions are ruled out), just as the Ising model on an unbounded Cayley tree provides a simple picture of ferromagnetic transitions in three-dimensional systems.

In order to study the dilute Ising model on a bounded Cayley tree we first generalize the formalism of Refs. 1 and 2 to the case of a disordered lattice. The free energy of a random system is defined in a probabilistic sense and must be averaged over the probability distribution of the logarithm of the partition function. Our study of this probability distribution will be based on the moment-expansion technique introduced recently in the context of critical properties of the dilute Ising model on an unbounded Cayley tree.[9]

The paper is organized as follows. In Sec. II we describe the general formalism for the study of the configuration-averaged free energy. In Sec. III we discuss the detailed calculation of the averaged free energy in the form of an expansion in powers of the applied magnetic field. This enables us to demonstrate the absence of spontaneous magnetization and to study the poles (critical points) in the susceptibility, and in the higher-order derivatives of the free energy. In order to study the nonanalytic part of the free energy above $p=p_c$ (and hence the order of the transition) one cannot rely on an expansion in powers of the magnetic field. Therefore, we proceed in Sec. IV to discuss an exact calculation of the averaged free energy in the limit of high concentrations of atomic spins, where the nonanalytic terms dominate. Some concluding remarks are presented in Sec. V.

II. GENERAL FORMULATION

We start from the usual Hamiltonian for a dilute spin-$\frac{1}{2}$ Ising model

$$\mathcal{H}=-J\sum_{\langle ij\rangle}\epsilon_i\epsilon_j\sigma_i\sigma_j-h\sum_i\epsilon_i\sigma_i\,,\quad J>0 \tag{1}$$

where $\sigma_i=\pm1$ and the symbol $\langle ij\rangle$ means that the summation is restricted to pairs of nearest neighbors. The parameters ϵ_i are independent random variables equal to unity if there is an atomic spin at site i and are equal to zero otherwise. The ferromagnetic nearest-neighbor exchange interaction J is assumed to be constant. The second term in Eq. (1) describes the interaction of the spins with a uniform external magnetic field parallel to the z axis and of magnitude $B=h/\mu_{\mathrm{B}}$ (μ_{B} is the Bohr magneton).

It is convenient to introduce the following notations to describe the finite-Cayley-tree lattice. Consider a reference site denoted by 0 on a Cayley tree of connectivity K. The site 0 has $z=K+1$ nearest neighbors denoted by 0_{α_1}, $\alpha_1=1,2,\ldots,z$, and each of the neighbors 0_{α_1} is connected to K further neighbors going outwards (i.e., disregarding the central site 0) and denoted by $0_{\alpha_1\alpha_2}$, $\alpha_2=1,2,\ldots,K$. Looking outwards into the network, each of the sites $0_{\alpha_1\alpha_2}$ is again connected to K nearest neighbors, $0_{\alpha_1,\alpha_2,\alpha_3}$, $\alpha_3=1,2,\ldots,K$, and so on. By analogy with the notation of Eq. (1) we introduce a random variable $\epsilon_{0_{\alpha_1\alpha_2\cdots\alpha_l}}$ equal to unity if there is an Ising spin at site $0_{\alpha_1\cdots\alpha_l}$ and equal to zero otherwise. Thus, the probability distribution $p_\epsilon(\epsilon_{0_{\alpha_1\cdots\alpha_l}})$ of $\epsilon_{0_{\alpha_1\cdots\alpha_l}}$ is

$$p_\epsilon(\epsilon_{0_{\alpha_1\cdots\alpha_l}})=p\,\delta(\epsilon_{0_{\alpha_1\cdots\alpha_l}}-1)+(1-p)\,\delta(\epsilon_{0_{\alpha_1\cdots\alpha_l}})\;, \tag{2}$$

where p denotes the fraction of sites occupied by Ising spins. Following Eggarter[1] we define an n-generation branch as a particular graph connecting sites on the Cayley-tree lattice. A graph with only one site i is called a one-generation branch. A graph obtained by connecting a reference site 0 to K nearest neighbors, 0_{α_1}, $\alpha_1=1,2,\ldots,K$, is called a two-generation branch. A graph obtained by connecting each of the second-generation sites, 0_{α_1}, of a two-generation branch to K further nearest-neighbors, $0_{\alpha_1\alpha_2}$, $\alpha_2=1,2,\ldots,K$, is called a three-generation branch, etc. Finally, an n-generation branch is a graph with n layers of successive generations of neighbors such that any site $0_{\alpha_1\cdots\alpha_l}$ in the lth layer ($l<n$) is connected to K nearest neighbors of the $(l+1)$th layer. Since an n-generation branch results from attaching K $(n-1)$-generation branches to a central site, one can immediately write down a set of nonlinear recursion relations expressing the partition function of a distribution of spins on the sites of an n-generation branch, in terms of the partition functions of the corresponding K $(n-1)$-generation branches. We call $Z_{i,n}^{\epsilon_i,\pm}=Z_{i,n}^{\pm}$, $\epsilon_i=1$, the partition function of an n-generation branch starting from a reference site i if the latter is occupied by a spin-up

(+) or a spin-down (−) magnetic atom, respectively. Similarly, we denote by

$$Z_{i,n}^{0}=2Z_{i,n}^{\epsilon_i,\pm}=2Z_{i,n}^{0,+}=2Z_{i,n}^{0,-},\quad \epsilon_i=0$$

the partition function of an n-generation branch starting from a site i, if the latter is occupied by a nonmagnetic atom. Since for a Cayley tree the traces over spins in the various branches leaving from a site 0 are independent, we have

$$Z_{0,n+1}^{+}=e^{\beta h}\prod_{\alpha=1}^{K}\left(e^{\beta\epsilon_{0_\alpha}J}Z_{0_\alpha,n}^{\epsilon_{0_\alpha},+}+e^{-\beta\epsilon_{0_\alpha}J}Z_{0_\alpha,n}^{\epsilon_{0_\alpha},-}\right)\,, \tag{3a}$$

$$Z_{0,n+1}^{-}=e^{-\beta h}\prod_{\alpha=1}^{K}\left(e^{-\beta\epsilon_{0_\alpha}J}Z_{0_\alpha,n}^{\epsilon_{0_\alpha},+}+e^{\beta\epsilon_{0_\alpha}J}Z_{0_\alpha,n}^{\epsilon_{0_\alpha},-}\right)\,, \tag{3b}$$

$$Z_{0,n+1}^{0}=\prod_{\alpha=1}^{K}\left(Z_{0_\alpha,n}^{\epsilon_{0_\alpha},+}+Z_{0_\alpha,n}^{\epsilon_{0_\alpha},-}\right),\quad \beta=(k_BT)^{-1}\,, \tag{3c}$$

for a fixed configuration of spins, i.e., a given set of ϵ_i values. The partition function $\mathfrak{Z}_{0,n+1}$ of a finite Cayley tree, with $n+1$ generations of successive neighbors surrounding the central site 0, is then given by

$$\mathfrak{Z}_{0,n+1}=p(\mathfrak{Z}_{0,n+1}^{+}+\mathfrak{Z}_{0,n+1}^{-})+(1-p)\,\mathfrak{Z}_{0,n+1}^{0}\,, \tag{4}$$

where $\mathfrak{Z}_{0,n+1}^{+}$ is obtained by hanging $z=K+1$ n-generation branches onto the site 0 assumed to be occupied by a spin-up atom, and $\mathfrak{Z}_{0,n+1}^{-}$ and $\mathfrak{Z}_{0,n+1}^{0}$ are similarly defined. In the thermodynamic limit $n\to\infty$, one clearly has[2]

$$\mathfrak{Z}_{0,n+1}^{+}\sim\mathfrak{Z}_{0,n+1}^{-}\sim\tfrac{1}{2}\mathfrak{Z}_{0,n}^{0} \tag{5}$$

so that

$$\mathfrak{Z}_{0,n+1}\sim\mathfrak{Z}_{0,n+1}^{+}\,,\quad n\to\infty\,. \tag{6}$$

Also, if $\mathcal{F}_{n+1}$ denotes the free energy of a Cayley tree with $n+1$ generations of neighboring sites (total number of sites $\mathcal{N}_{n+1}$), and F_{n+1} denotes that of an $(n+1)$-generation branch with

$$N_{n+1}=(K^{n+1}-1)/(K-1)$$

sites, the free energy per site is given by

$$f(T,h)=\lim_{n\to\infty}\frac{\mathcal{F}_{n+1}}{\mathcal{N}_{n+1}}=-\frac{1}{\beta}\lim_{n\to\infty}\left(\frac{\ln\mathfrak{Z}_{0,n+1}}{\mathcal{N}_{n+1}}\right)=\lim_{n\to\infty}\frac{F_{n+1}}{N_{n+1}}\,. \tag{7}$$

From the analogs of Eqs. (4) and (6) for the partition function $Z_{0,n+1}$ of an $(n+1)$-generation branch we then obtain

$$f(T,h)=-\frac{1}{\beta}(1-K^{-1})\lim_{n\to\infty}(K^{-n}\ln Z_{0,n+1}^{+})\,, \tag{8}$$

for a fixed configuration of occupied sites. The study of the free energy per site thus reduces to that of the quantity $\ln Z_{0,n+1}^{+}$ for an $(n+1)$-generation branch. The latter must be suitably averaged over the spatial configurations of the spins.

For the purpose of solving the recurrence relations (3) to calculate the average free energy per site, it is convenient to make the following transformations. First, using explicitly the fact that the random variables ϵ_{0_α} take the values 0 and 1, we rewrite the Eq. (3a) in the equivalent form

$$\begin{aligned}\ln Z_{0,n+1}^{+}&=\beta h+\sum_{\alpha=1}^{K}\ln Z_{0_\alpha,n}^{+}\\&\quad+\sum_{\alpha=1}^{K}\ln\left((1-\epsilon_{0_\alpha})\frac{Z_{0_\alpha,n}^{0,+}}{Z_{0_\alpha,n}^{+}}+\epsilon_{0_\alpha}\right)\\&\quad+\sum_{\alpha=1}^{K}\ln\left[2(1-\epsilon_{0_\alpha})+\epsilon_{0_\alpha}\left(e^{\beta J}+e^{-\beta J}\frac{Z_{0_\alpha,n}^{-}}{Z_{0_\alpha,n}^{+}}\right)\right]\,.\end{aligned} \tag{9}$$

Next we define the quantity $G_{i,n}=Z_{i,n}^{-}/Z_{i,n}^{+}$, giving the probability that the spin at the origin i of an n-generation branch is down for a given configuration of occupied sites, divided by the probability, that it is up for the same configuration. We note in passing that if we had placed the magnetic field antiparallel to the z axis we would be led to introduce $\tilde{G}_{i,n}=G_{i,n}^{-1}$ instead of $G_{i,n}$ as our basic variable. The final results for the free energy would be identical to those obtained by assuming the field parallel to the z axis, as expected. Taking now the ratios of Eqs. (3a) and (3b) and of Eqs. (3a) and (3c), respectively, and again making use explicitly of the values 0 and 1 of ϵ_{0_α}, we get

$$G_{0,n+1}=\lambda\prod_{\alpha=1}^{K}\left(1-\epsilon_{0_\alpha}+\epsilon_{0_\alpha}\frac{1+\nu G_{0_\alpha,n}}{\nu+G_{0_\alpha,n}}\right)\,, \tag{10}$$

and

$$\frac{Z_{0,n+1}^{0}}{Z_{0,n+1}^{+}}=\lambda^{1/2}\prod_{\alpha=1}^{K}\left(1-\epsilon_{0_\alpha}+\epsilon_{0_\alpha}\frac{1+G_{0_\alpha,n}}{\nu^{1/2}+(\nu^{-1})^{1/2}G_{0_\alpha,n}}\right)\,, \tag{11}$$

where

$$\nu=e^{2\beta J}\,, \tag{12a}$$

$$\lambda=e^{-2\beta h}\,. \tag{12b}$$

Equation (10) which defines the variables $G_{i,m}$ as random variables related to the parameters ϵ_j was, in fact, the starting point for studying the critical properties of the ferromagnetic transition taking place in the central region of the Cayley tree.[7,9] In the present treatment, this latter transition is masked by the effect of the spins at the surface which dominate the critical behavior as discussed in Sec. I.

We now substitute Eqs. (10) and (11) in Eq. (9), and since our further analysis will be restricted to low temperatures such that $k_B T << J$, we expand the resulting equation through linear order in $(\nu^{-1})^{1/2} = e^{-\beta J}$. After some manipulations, using the fact that ϵ_i takes the values 0 and 1, as well as the expressions $Z_{0,1}^{\pm} = e^{\pm\beta h}$, we obtain

$$\ln Z_{0,n+1}^{+} = \beta h + \sum_{\alpha_1=1}^{K} \ln Z_{0_{\alpha_1},n}^{+} + \sum_{\alpha_1=1}^{K} \ln\left[(1-\epsilon_{0_{\alpha_1}})\left((1-\delta_{n,1})e^{-\beta h}\prod_{\alpha_2=1}^{K}[1-\epsilon_{0_{\alpha_1\alpha_2}}+\epsilon_{0_{\alpha_1\alpha_2}}e^{-\beta J}\right.\right.$$
$$\left.\left.\times(1+G_{0_{\alpha_1\alpha_2},n-1})]+e^{-\beta h}\delta_{n,1}\right)+\epsilon_{0_{\alpha_1}}e^{\beta J}\right]+O(\nu^{-1}) \; . \tag{13}$$

Equation (13) may now be solved in terms of the variables $G_{i,n}$ by means of successive recursions. The final result is

$$\ln Z_{0,n+1}^{+} = \beta h\left(\frac{K^{n+1}-1}{K-1}\right) + \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=1}^{K} \ln[(1-\epsilon_{0_{\alpha_1,\alpha_2,\ldots,\alpha_n}})e^{-\beta h}+\epsilon_{0_{\alpha_1,\alpha_2,\ldots,\alpha_n}}e^{\beta J}]$$
$$+\sum_{\alpha_1,\ldots,\alpha_{n-1}}^{K} \ln\left((1-\epsilon_{0_{\alpha_1,\ldots,\alpha_{n-1}}})e^{-\beta h}\prod_{\alpha_n=1}^{K}[1-\epsilon_{0_{\alpha_1,\ldots,\alpha_n}}+\epsilon_{0_{\alpha_1,\ldots,\alpha_n}}e^{-\beta J}(1+G_{0_{\alpha_1,\ldots,\alpha_n},1})]+\epsilon_{0_{\alpha_1,\ldots,\alpha_{n-1}}}e^{\beta J}\right)$$
$$+\sum_{\alpha_1\ldots\alpha_{n-2}}^{K} \ln\left((1-\epsilon_{0_{\alpha_1,\ldots,\alpha_{n-2}}})e^{-\beta h}\prod_{\alpha_{n-1}=1}^{K}[1-\epsilon_{0_{\alpha_1,\ldots,\alpha_{n-1}}}+\epsilon_{0_{\alpha_1,\ldots,\alpha_{n-1}}}e^{-\beta J}\right.$$
$$\left.+(1+G_{0_{\alpha_1,\ldots,\alpha_{n-1}},2})]+\epsilon_{0_{\alpha_1,\ldots,\alpha_{n-2}}}e^{\beta J}\right)+\cdots$$
$$+\sum_{\alpha_1=1}^{K}\ln\left((1-\epsilon_{0_{\alpha_1}})e^{-\beta h}\prod_{\alpha_2=1}^{K}[1-\epsilon_{0_{\alpha_1\alpha_2}}+\epsilon_{0_{\alpha_1\alpha_2}}e^{-\beta J}(1+G_{0_{\alpha_1\alpha_2},n-1})]+\epsilon_{0_{\alpha_1}}e^{\beta J}\right)+O(\nu^{-1}) \; . \tag{14}$$

The average free energy is obtained from Eqs. (8) and (14) by averaging over the probability distribution of the random variables ϵ_i and $G_{i,m}$. This means that any function

$$g(\epsilon_1,\ldots,\epsilon_j,\; G_{1,m},\ldots G_{j,m})$$

of independent random variables $\epsilon_1,\ldots,\epsilon_j$ and $G_{1,m},\ldots,G_{j,m}$ in Eq. (14) is replaced by the average value

$$\langle g\rangle = \int\prod_{i=1}^{j} d\epsilon_i\, dG_{i,m}\, p_\epsilon(\epsilon_i)p_G(G_{i,m})g(\epsilon_1,\ldots,\epsilon_j,G_{1,m},\ldots,G_{j,m}) \; ,$$

where $p_G(G_{i,m})$ denotes the probability distribution of $G_{i,m}$. Before studying this probability distribution in various cases we perform explicitly the average over the variables ϵ_i, using Eq. (2). After a little algebra the average of Eq. (14) is found to be of the form

$$\langle \ln Z_{0,n+1}^{+}\rangle = 1/(K-1)K^{n+1}-1-(1-p)K(K^n-1)]\beta h + K/(K-1)[K^n-1-(1-p)(K^n-K)]p\beta J + p(1-p)$$
$$\times\sum_{m=2}^{n}K^m\int dG_{0_{\alpha_1,\ldots,\alpha_m},n-m+1}\,p_G(G_{0_{\alpha_1,\ldots,\alpha_m},n-m+1})\ln(1+G_{0_{\alpha_1,\ldots,\alpha_m},n-m+1}) \; . \tag{15}$$

The average free energy per site $\langle f(t,h)\rangle$ is given by Eq. (8) with $\ln Z_{0,n+1}^{+}$ replaced by the average value in Eq. (15).

The probability distribution of the independent variables $G_{i,m}$ must be determined self-consistently from Eq. (10) in such a way that the probability distribution of $G_{0_\alpha,m}$, when used on the right-hand side of Eq. (10), yields the same distribution for $G_{0,n+1}$

on the left-hand side. In order to determine the explicit form of $p_G(G_{i,m})$ in various limiting cases it is convenient to proceed in two steps. Define

$$R_{0_{\alpha},n} = 1 - \epsilon_{0_{\alpha}} + \epsilon_{0_{\alpha}}(1 + \nu G_{0_{\alpha},n})/(\nu + G_{0_{\alpha},n})$$
$$= 1 - \epsilon_{0_{\alpha}} + \epsilon_{0_{\alpha}} G_{0_{\alpha},n} + O(\nu^{-1}) \ , \qquad (16)$$

so that Eq. (10) reads

$$G_{0,n+1} = \lambda \prod_{\alpha=1}^{K} R_{0_{\alpha},n} \ . \qquad (17)$$

Since the Eqs. (10) and (16) are, in fact, valid for any site in the interior of a generation branch, we have

$$R_{0_{\alpha_1},n} = 1 - \epsilon_{0_{\alpha_1}} + \epsilon_{0_{\alpha_1}} \lambda \prod_{\alpha_2=1}^{K} R_{0_{\alpha_1\alpha_2},n-1} + O(\nu^{-1}) \ . \qquad (18)$$

The variables $R_{i,m}$ (like $G_{i,m}$) are independent random variables and from Eq. (18) we find that their probability distribution $p_R(R_{i,m})$ obeys the self-consistency condition

$$p_R(R_{0_{\alpha_1},n}) = \int d\epsilon_{0_{\alpha_1}} \, p_\epsilon(\epsilon_{0_{\alpha_1}}) \int \cdots \int \prod_{\alpha_2=1}^{K} dR_{0_{\alpha_1\alpha_2},n-1} \, p_R(R_{0_{\alpha_1\alpha_2},n-1})$$
$$\times \delta\left(R_{0_{\alpha_1},n} - 1 + \epsilon_{0_{\alpha_1}} - \epsilon_{0_{\alpha_1}} \lambda \prod_{\alpha_2=1}^{K} R_{0_{\alpha_1\alpha_2},n-1} \right) \ . \qquad (19)$$

This equation may be easily converted into an algebraic equation for the moments,

$$F_{s,m} = \int_{-\infty}^{\infty} dR_{i,m} \, R_{i,m}^{s} \, p_R(R_{i,m}) \ , \qquad (20)$$

of $p_R(R_{i,m})$. Indeed, using Eq. (2) we obtain

$$F_{s,n} = 1 - p + p\lambda^s F_{s,n-1}^{K} + O(\nu^{-1}) \ , \quad s = 1, 2 \ldots \ . \qquad (21)$$

Similarly, from the self-consistency relation for the probability distribution $p_G(G_{o,n})$ which follows from Eq. (17), we get

$$M_{s,n} = \lambda^s F_{s,n-1}^{K} \ , \qquad (22)$$

where

$$M_{s,m} = \int_{-\infty}^{\infty} dG_{i,m} \, G_{i,m}^{s} \, p_G(G_{i,m}) \ , \qquad (23)$$

denotes the sth-order moment of $p_G(G_{i,m})$. Furthermore, by comparing Eqs. (21) and (22) we have

$$M_{s,n} = 1/p\,(F_{s,n} - 1 + p) \ . \qquad (24)$$

The probability distribution $p_G(G_{i,n})$ is then defined as usual by the Fourier transform of the characteristic function $\phi_n(k)$,

$$p_G(G_{i,n}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-ikG_{i,n}} \phi_n(k) \ , \qquad (25)$$

where

$$\phi_n(k) = 1 + \sum_{s=1}^{\infty} \frac{(ik)^s}{s!} M_{s,n} \ . \qquad (26)$$

Note that the ranges of integration in Eqs. (20) and (23) reduce effectively to the interval $(0^-, \infty)$ since $G_{i,m}$ and $R_{i,m}$ are intrinsically positive.

Finally, the average magnetization $\langle m \rangle$ and the susceptibility per site χ are defined by

$$\langle m \rangle = -\mu_{\mathrm{B}} \partial \langle f(T,h) \rangle / \partial h \ , \qquad (27)$$

$$\chi = -\mu_{\mathrm{B}}^2 \frac{\partial^2 \langle f(T,h) \rangle}{\partial h^2} \Big|_{h=0} \ . \qquad (28)$$

In Secs. III and IV the nature of the phase transition on the bounded Cayley tree at low temperatures is analyzed by calculating $\langle f(T,h) \rangle$, $\langle m \rangle$, and χ explicitly in various limiting cases.

III. LOW-FIELD EXPANSION AND SINGULARITIES IN THE AVERAGED FREE ENERGY

Like in the ordered lattice case ($p = 1$) we expect the free energy of the dilute Ising model on a Cayley tree to be regular for $h = 0$. The phase transition which takes place at low temperatures ($k_B T << J$) when the concentration of Ising spins is increased beyond the percolation threshold, will show up through various singular dependences of the free energy on the applied magnetic field. Due to this singular behavior a simple iteration of the equations of Sec. II, in powers of the magnetic field, is expected to converge only over a restricted range of low concentrations at any given order in the field. As usual, such an iteration enables one to discuss the zero-field susceptibility above the lowest critical point defined as the highest concentration at which this quantity becomes infinite. However, as in the pure lattice case,[2] it turns out that there exists an infinite set of additional critical points corresponding to singularities of all higher-order derivatives of the free energy, respectively.

We first determine the probability distribution $p_G(G_{i,m})$ to second order in h. In order to solve Eq. (21) we require the value of $F_{s,1}$, for $h = 0(\lambda = 1)$. In fact, the probability distribution of the variable $R_{i,1}$ may be easily found exactly. For a one-generation branch involving a single site i we have, by definition, $G_{i,1}=e^{-2\beta h}=\lambda$ and from Eq. (16)

$$R_{i,1}=1-\epsilon_i+\epsilon_i\lambda+O(\nu^{-1}) \ . \tag{29}$$

A self-consistency condition similar to Eq. (19) then yields

$$p_R(R_{i,1})=(1-p)\delta(R_{i,1}-1)+p\delta(R_{i,1}-\lambda) \ . \tag{30}$$

This shows that $F_{s,1}=1$ for $h=0$ and hence, from Eq. (21), $F_{s,n}=1$ for $h=0$. We now write the moment $F_{s,n}$ in terms of linear and quadratic corrections in the field

$$F_{s,n}=1+F_{s,n}^{(1)}+F_{s,n}^{(2)}+\cdots \ , \tag{31}$$

and by expanding Eq. (21) through order h^2 we obtain a pair of linear recurrence relations for $F_{s,n}^{(1)}$ and $F_{s,n}^{(2)}$, respectively. After resumming the explicit solutions obtained by successive recursions of these linear relations we get

$$F_{s,n}^{(1)}=2ps\beta h f_n^{(1)} \ , \tag{32a}$$

$$f_n^{(1)}=(1-p^nK^n)/(pK-1) \ , \tag{32b}$$

$$F_{s,n}^{(2)}=2ps^2\beta^2h^2f_n^{(2)} \ , \tag{33a}$$

$$\begin{aligned}f_n^{(2)}=&\left(1-\frac{2pK}{pK-1}+\frac{K(K-1)p^2}{(pK-1)^2}\right)\frac{(pK)^{n-1}-1}{pK-1}\\&+\frac{2pK(p-1)}{(pK-1)^2}(n-1)(pK)^{n-1}+\frac{p^2K(K-1)}{(pK-1)^3}\\&\times p^nK^n[(pK)^{n-1}-1]+(pK)^{n-1} \ .\end{aligned} \tag{33b}$$

Then, by performing the summation in Eq. (26) for $\phi_n(k)$, using Eqs. (24), (31), (32a), and (33a), and taking the Fourier transform we obtain the normalized probability distribution

$$\begin{aligned}p_G(G_{i,n})=&\ \delta(G_{i,n}-1)-2\beta hf_n^{(1)}\delta'(G_{i,n}-1)\\&-2\beta^2h^2f_n^{(2)}[\delta'(G_{i,n}-1)-\delta''(G_{i,n}-1)]\\&+O(h^3) \ .\end{aligned} \tag{34}$$

Finally we insert Eq. (34) in Eq. (15) to evaluate $\langle \ln Z_{0,n+1}^{+}\rangle$ explicitly. After a change of summation variables we obtain from Eqs. (8) and (15) in the thermodynamic limit

$$\begin{aligned}\langle f(T,h)\rangle=&-ph-p^2J-\frac{p}{\beta}(1-p)\ln2-h\frac{p(1-p)(K-1)}{pK-1}\sum_{t=1}^{\infty}(K^{-t}-p^t)-\frac{\beta h^2}{2}p(1-p)(K-1)\\&\times\left[\frac{1}{pK-1}\left(1-\frac{2pK}{pK-1}+\frac{K(K-1)p^2}{(pK-1)^2}\right)\sum_{t=1}^{\infty}\left(\frac{p^t}{pK}-K^{-t}\right)+\frac{2(p-1)}{(pK-1)^2}\sum_{t=1}^{\infty}(t-1)p^t\right.\\&\left.+\frac{p(K-1)}{(pK-1)^3}\sum_{t=1}^{\infty}[(p^2K)^t-Kp^{t+1}]+\frac{1}{pK}\sum_{t=1}^{\infty}p^t\right] \ .\end{aligned} \tag{35}$$

For $p^2K<1$ the various geometric series in Eq. (35) may be summed explicitly and, after some rearrangements, we find

$$\langle f(T,h)\rangle=-p^2J-\frac{p}{\beta}(1-p)\ln2-\frac{\beta h^2p(K-1)}{2(1-pK)}\left(\frac{1}{K-1}(1+2p-pK)-\frac{p^2(1+pK)}{1-p^2K}\right)+O(h^3) \ , \quad p<1 \ , \quad p^2K<1 \ . \tag{36a}$$

On the other hand, for $p=1$ Eq. (35) reduces to

$$\langle f(T,h)\rangle=-J-h \ . \tag{36b}$$

Equations (36a) and (36b) reveal a number of interesting features.

(i) In the cases $h=0$, p arbitrary, and $h\neq0$, $p=1$ the free energy is an analytic function of T, p, and h, respectively. In particular, Eq. (36b) coincides with the low-temperature limit of Eggarter's result for $h=0$[1] and the first term of Eq. (36a) is just the average interaction energy per spin at $T=0$, $h=0$.

(ii) Except for $p=1$, there is no term linear in h in the free energy which implies absence of spontaneous magnetization, when the concentration $p<1$ is varied across, e.g., the percolation threshold, $p_c=1/K=1/(z-1)$. We emphasize that this conclusion remains valid for *all* values $p<1$ since in this range the linear term in Eq. (35) converges. This implies that the phase transition is dominated by the spins at the surface since a ferromagnetic transition is known to occur in the central part of the Cayley tree when p exceeds the value p_c.[7,9]

(iii) The average susceptibility given by Eq. (28) has a simple pole singularity not only at $p=p_c$ but also at the concentration $p=p_{c2}=(z-1)^{-1/2}$, and according to Eq. (35), it is formally infinite for all higher concentrations. The pole at $p=p_c$ corresponds, of course, to the onset of ferromagnetic ordering of the spins in the central region of the Cayley tree. This is in fact, the only vestige of this transition in the averaged free energy for a bounded Cayley tree. The pole at $p=p_{c2}$ corresponds to ordering of the spins at the surface which is not accompanied by spontaneous magnetization. The critical concentration p_{c2} is the analog of the "surface critical temperature" at which the average susceptibility of an ordered Ising model ($p=1$) on a Cayley tree is found to diverge.[3,4] We note, incidentally, that in the case of the ordered Cayley tree the average susceptibility remains finite at the Bethe-Peierls critical temperature for ferromagnetic ordering in the bulk.[2,3]

(iv) Like in the case of the second-order contribution in $\langle f(T,h)\rangle$, we may locate the poles of the higher-order terms from the general structure of the linear recurrence relations for the corrections $F_{s,n}^{(l)}$, $l>2$, to the moments $F_{s,n}$. Firstly, the form of Eq. (21) shows that the correction $F_{s,n}^{(l)}$ at any order h^l in the field, involves terms proportional to various powers of the quantity $(1-pK)^{-1}$, similar to those occurring in Eqs. (32b) and (33b). As in Eq. (36a), this dependence on $(1-pK)^{-1}$ leads to a *simple* pole in the contribution of order h^l in the free energy $\langle f(T,h)\rangle$, and is associated with the ferromagnetic ordering of the spins in the bulk region. In addition $F_{s,n}^{(l)}$ has a term proportional to $(pK)^{l\mathrm{n}}$ [see Eqs. (32b) and (33b) for the cases $l=1,2$] which leads to contribution in $\langle f(T,h)\rangle$ proportional to $(h=|h|)$,

$$|h|^l\sum_{t=1}^{\infty}K^{-t}(pK)^{lt}=-\frac{|h|^l p^l K^{l-1}}{p^l K^{l-1}-1}$$

$$(\text{ for } p^l K^{l-1}<1)\ , \tag{37}$$

as is evident from the general structure of Eqs. (8) and (15) after averaging over distributions of the form of Eqs. (2) and (34) with the lth-order term included. This shows that the free energy has simple-pole infinities at the concentrations

$$p_{cl}=\frac{1}{K^{(l-1)/l}}\ ,\quad l=2,3,\ldots \tag{38}$$

where $p_{c\infty}=p_c=1/K$ is nothing but the percolation threshold. We note that, while for the ordered Ising model the existence of contributions involving odd powers of h is ruled out by the general structure of the free energy for arbitrary fields, this is not so at the higher orders in the present site-disordered case. The divergence at p_c corresponds to the onset of correlations of the spins at the surface described by the lth-order derivative (l-point correlation function) of the free energy with respect to the field h.

For $p>p_{cl}(l\geqslant 2)$ the Eq. (35) ($l=2$) and the left-hand side of Eq. (37) cease to converge. This indicates that the free energy is a nonanalytic function of $|h|$ for $p>p_c$, where an expansion in powers of $|h|$ has thus only a formal meaning, as it is invalid. Actually, however, the form of Eq. (37) for the l-th order contribution suggests that the free energy is composed of a regular (analytic) part and of a nonanalytic part. In particular, for $p_{cl}<p<p_{c\,l-1}$ the free energy has in general, $l-2$ regular terms proportional to $|h|^2$, $|h|^3$, ..., $|h|^{l-1}$, respectively, and an irregular part associated with the formally divergent terms of order $|h|^l$, $|h|^{l+1}$,.... The question arises then, whether it is possible to modify the expansion of $\langle f(T,|h|)\rangle$ in powers of $|h|$ so as to remove the divergencies of the terms proportional to $|h|^m$, $m\geqslant l$, for $p>p_{cl}$. Of course, this amounts, in principle, to determining the actual form of the nonanalytic part of the free energy.

A procedure which enables one to find the form of the leading nonanalytic term in the free energy in the case of the ordered Cayley tree has been proposed by Zittartz and Müller-Hartmann.[2] Its extension to the present problem amounts to making use of the fact that, for $pK>1$ the recursion process giving rise to a leading term proportional to $|h|^l(pK)^{nl}$ in $F_{s,n}^{(l)}$, is approximately valid up to a value n_0 of n such that

$$|h|^l(pK)^{n_0 l}=O(1)$$

or

$$n_0\simeq-\ln|h|/\ln(pK)\ . \tag{39}$$

Following Zittartz and Müller-Hartmann, one might argue, therefore, that the leading nonanalytic contribution to the free energy in the range $p>p_{cl}$ is properly determined by cutting off the summations on the left-hand side of Eq. (37) at an upper limit $t\simeq n_0$ for all orders $m\geqslant l$. This would yield a nonanalytic contribution of the form

$$\begin{aligned}|h|^m\sum_{t=1}^{n_0}(p^mK^{m-1})^t&\simeq\frac{|h|^m(p^mK^{m-1})^{n_0}}{1-p^{-m}K^{-m+1}}\\&=\frac{|h|^{\kappa(p)}}{1-p^{-m}K^{-m+1}}\ ,\end{aligned} \tag{40}$$

where

$$\kappa(p)=\ln K/\ln(pK)\ , \tag{41}$$

for any order $m\geqslant l$. From this expression one sees immediately that $l-1<\kappa(p)<l$ for $p_{cl}<p<p_{c\,l-1}$, which shows that the nonanalytic term (40) is of higher order than the $l-2$ regular terms, proportional to $|h|^2$,..., $|h|^{l-1}$, respectively. In particular, it follows that when

$p > p_{c_2}$ [$\kappa(p) < 2$], this nonanalytic contribution dominates all regular terms.

The above discussion of nonanalytic terms in the free energy using the cutoff procedure of Zittartz and Müller-Hartmann indicates that in the whole range of concentration $p_c < p < 1$, the free energy has contributions of the form $|h|^{\kappa(p)}$, where the exponent $\kappa(p)$ increases continuously from unity at $p = 1$ to infinity at $p = p_c$. Thus, for $|h| \to 0$ the transition above p_c would be an ordinary first-order transition at $p = 1$ (ferromagnetic transition) and a transition of "infinite order" at $p = p_c$. Moreover, the smooth variation of $\kappa(p)$ as a function of p would suggest that one is dealing with a transition whose order varies continuously as a function of concentration.

In Sec. IV we present an exact calculation of the free energy for $p \to 1$, which avoids expanding in powers of the magnetic field from the outset. This calculation leads to an unambiguous identification of the form of the leading nonanalytic terms and shows that the above conclusions concerning nonanalyticities in the free energy are only partially correct. We find that while the free energy has indeed a contribution of the form of Eq. (40), the latter is not the leading nonanalytic term, which turns out to be proportional to $\ln|h|$ and implies a transition of infinite order (at least in a range of concentrations near $p = 1$). Furthermore, this treatment will give us a clue for correcting *a posteriori* the procedure of Zittartz and Müller-Hartmann to obtain the correct form for the leading nonanalytic term in the present case.

IV. HIGH-CONCENTRATION LIMIT

For concentrations $p > p_{c2}$ the free energy involves nonanalytic contributions from every order of a formal expansion in powers of h. As shown in Sec. III, the expansion of Eq. (21) in powers of h, and its solution by successive recursions, breaks down in this concentration range for sufficiently large n. In this case it is necessary to solve the recursion relations (21) in their actual nonlinear form in order to treat correctly the nonanalytic behavior of the free energy. An analytic treatment of this problem is possible in the high-concentration limit $p \to 1$, where we may expand Eq. (21) in powers of $1-p$. Writing

$$F_{s,n} = F^0_{s,n} + F^1_{s,n} + O((1-p)^2) \quad , \tag{42}$$

where $F^0_{s,n}$ is of zeroth order in $1-p$ and $F^1_{s,n}$ is linear. Starting the recursions from the values

$$F^0_{s,1} = \lambda^s, \quad F^1_{s,1} = (1-p)(1-\lambda^s) \quad ,$$

which follow from Eq. (30), we obtain after some algebra

$$F^0_{s,n} = \lambda^{-[s/(K-1)](1-K^n)} \quad , \tag{43}$$

$$\begin{aligned} F^1_{s,n} = (1-p)\Bigg[&\sum_{m=1}^{n-2} K^{n-m-1} \left(\lambda^{-[s/(K-1)]K^{m+1}} - \lambda^{-s/(K-1)}\right) \lambda^{[s/(K-1)]K^n} + 1 - \lambda^{[s/(K-1)](K^n-1)} \\ &+ K^{n-1}(1-\lambda^s)\,\lambda^{[sK/(K-1)](K^{n-1}-1)}\Bigg] + O((1-p)^2) \quad . \end{aligned} \tag{44}$$

By inserting these results in Eqs. (24) and (26) and performing the summation in the resulting expression for the characteristic function, we obtain for the Fourier transform

$$\begin{aligned} p_G(G_{l,n}) = &\left\{1 + \frac{K(1-p)}{K-1}(1-K^{n-1})\right\} \delta\!\left(G_{l,n} - \lambda^{(K^n-1)/(K-1)}\right) + (1-p)K^{n-1} \\ &\times \delta\!\left(G_{l,n} - \lambda^{K(K^{n-1}-1)/(K-1)}\right) + (1-p)\sum_{m=2}^{n-1} K^{n-m}\delta\!\left(G_{l,n} - \lambda^{(K^n-K^m)/(K-1)}\right)\theta(n-2) + O((1-p)^2) \quad . \end{aligned} \tag{45}$$

Using Eq. (45) to calculate $\langle \ln Z^+_{0,n+1} \rangle$ given by Eq. (15), we obtain for the average free energy to order $(1-p)^2$, in the thermodynamic limit,

$$\begin{aligned} \langle f(T,h) \rangle = &-p^2 J - ph - \frac{p}{\beta}(1-p)K(K-1) \\ &\times \Bigg[\frac{1}{K}\left\{1 + \frac{K(1-p)}{K-1}\right\} \sum_{l=1}^{\infty} K^{-l} \ln\!\left(1 + \lambda^{(K^l-1)/(K-1)}\right) - \frac{(1-p)}{K(K-1)} \sum_{l=1}^{\infty} \ln\!\left(1 + \lambda^{(K^l-1)/(K-1)}\right) \\ &+ \frac{1-p}{K^2} \sum_{l=1}^{\infty} \ln\!\left(1 + \lambda^{(K^l-K)/(K-1)}\right) + \frac{1-p}{K} \sum_{l=3}^{\infty}\sum_{k=2}^{l-1} K^{-k} \ln\!\left(1 + \lambda^{(K^l-K^k)/(K-1)}\right) \Bigg] + O((1-p)^3) \quad . \end{aligned} \tag{46}$$

Before proceeding with the explicit evaluation of this exact expression we note that, for $p=1$, it coincides with Eq. (36b) as it should. It follows by inspection that for $p \neq 1$ Eq. (46) is valid for $h \neq 0$ only. On the contrary Eq. (35) is valid down to $h \rightarrow 0$ provided $p^2K < 1$. This difference is not surprising, since for p sufficiently close to unity the condition $p^2K < 1$ is violated. This shows that Eqs. (36a) and (46) apply in different ranges of concentration. The form of Eq. (46) indicates that it will be difficult to get accurate detailed results for very small values of the field h, as this requires a numerical summation of large numbers of terms.

In order to extract the leading nonanalytic contributions in the free energy expression (46) one may perform approximate summations of the various infinite series using the Euler–Mac Laurin formula. Actually, a more relevant quantity to discuss is the average magnetization per site. From Eqs. (27) and (46) we obtain

$$\langle m\rangle = \mu_B p - 2\mu_B p(1-p)K\Bigg[\bigg[\bigg(K^{-1}+\frac{1-p}{K-1}\bigg)\sum_{l=1}^{\infty}(1-K^{-l})\{1+\exp[b(K^l-1)]\}^{-1} - \frac{1-p}{K(K-1)}\sum_{l=1}^{\infty}(K^l-1)$$

$$\times\{1+\exp[b(K^l-1)]\}^{-1} + \frac{1-p}{K}\sum_{l=1}^{\infty}(K^{l-1}-1)\{1+\exp[b(K^l-K)]\}^{-1}$$

$$+\frac{1-p}{K}\sum_{l=3}^{\infty}\sum_{k=2}^{l-1}(K^{l-k}-1)\{1+\exp[b(K^l-K^k)]\}^{-1}\Bigg] + O((1-p)^3,\nu^{-1}) \ , \qquad (47)$$

where

$$b=\frac{2\beta|h|}{K-1} \ . \qquad (48)$$

In using the Euler–Mac Laurin formula we have chosen to retain the following leading contributions[10]

$$\sum_{l=1}^{n-1} a_l \simeq \int_0^n a(l)\,dl - \tfrac{1}{2}[a(0)+a(n)] + \tfrac{1}{12}[a'(n)-a'(0)] \ . \qquad (49)$$

The nonanalytic behavior of $\langle m\rangle$ arises from l values such that the arguments of the exponenials in Eq. (47) are larger than unity. Therefore, the form of the nonanalytic field dependence of various terms (but not their precise magnitude) may be obtained by neglecting unity compared to the exponentials in the terms to be summed in Eq. (47). This leaves us with the following distinct sums to be performed

$$S_0=\sum_{l=1}^{\infty} e^{-bK^l} \ , \qquad (50)$$

$$S_{\pm 1}=\sum_{l=1}^{\infty} K^{\pm l}e^{-bK^l} \ , \qquad (51)$$

$$S_2=\sum_{l=3}^{\infty}\sum_{k=2}^{l-1}\frac{K^l-K^k}{K^k}e^{-b(K^l-K^k)} \ . \qquad (52)$$

The evaluation of Eqs. (50) and (51) using Eq. (49) is straightforward. After expanding the results for small fields we get

$$S_0=-\ln b/\ln K + O(b^0=1) + O(b) \ , \qquad (53)$$

$$S_1=1/b\ln K + O(b^0=1) \ , \qquad (54)$$

$$S_{-1}=O(b^0=1)+O(b\ln b) \ . \qquad (55)$$

The series S_1 which converges only for $b \neq 0$, and which also enters in the following evaluation of S_2, gives the leading nonanalytic field dependence of $\langle m\rangle$ at order $(1-p)^2$. In evaluation S_2 it is convenient to use K^l-K^k as a new variable when applying the Euler formula to perform the summation over k. In this way we obtain

$$S_2=\sum_{l=3}^{\infty}\Bigg((\ln k)^{-1}(1-bK^l)e^{-bK^l}[Ei(b)-Ei(bK^l)] - \frac{K^l-K}{K}e^{-b(K^l-K)} + (\ln K)^{-1}(K^l e^{-b(K^l-1)}-1)$$

$$-\tfrac{1}{2}[(K^l-1)e^{-b(K^l-1)}+(K-1)e^{-bK^l(1-K^{-1})}] + \tfrac{1}{12}\ln K[-Ke^{-bK^l(1-K^{-1})}+bK^l(1-K^{-1})$$

$$\times e^{-bK^l(1-K^{-1})}+(K^l-bK^l+b)e^{-bK^l-1)}]\Bigg) \ , \qquad (56)$$

where $Ei(x)$ is the exponential integral.[10] The only summation in Eq. (56) which requires some further approximation is

$$S_2^1=-(\ln K)^{-1}\sum_{l=3}^{\infty}[(1-bK^l)e^{-bK^l}Ei(bK^l)+1] \ , \qquad (57)$$

which we approximate by (with γ the Euler's constant)

$$S_2^1 \simeq -(\ln K)^{-1} \sum_{l=3}^{l_0-1} (1-bK^l)e^{-bK^l} \times [\gamma + \ln(bK^l) + \ldots] - (\ln K)^{-1}(l_0-2)$$

$$-(\ln K)^{-1} \sum_{l=l_0}^{\infty} \left[(1-bK^l)\frac{f(bK^l)}{bK^l}+1\right] , \tag{58}$$

where l_0 is defined by

$$bK^{l_0}=1 , \tag{59}$$

i.e.,

$$l_0 = -\ln b/\ln K ,$$

and where the quantity $f(bK^l)$ may be replaced by unity for our purposes,[10] in which case the last term of Eq. (58) reduces to $K[(1-K)\ln K]^{-1}$. In the first term of Eq. (58) we expand the exponential and use Eq. (49) to perform the summation over l. After some algebra we get

$$S_2^1 = \frac{\ln b}{\ln K}\left(\frac{\gamma+1}{\ln K}+\frac{5}{2}\right)+\frac{1}{2}\left(\frac{\ln b}{\ln K}\right)^2 + O(b^0=1)+O(b\ln b) . \tag{60}$$

Finally, by collecting the various contributions in Eq. (56) we obtain

$$S_2 = \frac{1}{b}\left(\frac{1}{(\ln K)^2}-\frac{1}{K\ln K}-\frac{1}{2\ln K}+\frac{1}{12}\right)-\frac{1}{2}\left(\frac{\ln b}{\ln K}\right)^2+\frac{\ln b}{\ln K}\left[\frac{1}{2}+\frac{K}{2}\left(1+\frac{\ln K}{6}\right)\right]+O(b^0=1)+O(b\ln b) . \tag{61}$$

By using the approximation stated after Eq. (49), and inserting the above explicit values for S_0, $S_{\pm 1}$, and S_2 in Eq. (47) for $\langle m\rangle$ we find

$$\langle m\rangle = 2\mu_B p(1-p)\left\{\frac{\ln b}{\ln K}+\frac{1-p}{\ln K}\left[\left(\frac{1}{K-1}+\frac{1}{2}-\frac{1}{\ln K}-\frac{\ln K}{12}\right)\frac{1}{b}+\frac{1}{2}\frac{(\ln b)^2}{\ln K} - \left(\frac{3}{2}+\frac{K}{2}-\frac{K+1}{K-1}+\frac{K\ln K}{12}\right)\ln b\right]\right\}+O(b^0=1)+O(b\ln b) . \tag{62}$$

Since $1/b$ varies more rapidly than any finite power of $\ln b$, we may drop the terms proportional to $\ln b$ and to $(\ln b)^2$ in the contribution of order $(1-p)^2$, so that finally

$$\langle m\rangle \simeq 2\mu_B p(1-p)\left[\frac{\ln b}{\ln K}+\frac{(1-p)}{\ln K}\left(\frac{1}{K-1}+\frac{1}{2}-\frac{1}{\ln K}-\frac{\ln K}{12}\right)\frac{1}{b}\right] . \tag{63}$$

We recall that Eq. (63) is valid only for $p \neq 1$ and $h \neq 0$ as shown, in particular, by the above explicit evaluation. However, since we know from the study of Sec. III that $\lim_{p\to 1}\langle m\rangle = \mu_B$ and $\lim_{h\to 0}\langle m\rangle = 0$ for $p \neq 1$, we may attempt to adjust Eq. (63) to a slightly more general expression of the form

$$\langle m\rangle = D(p)e^{[1/\delta(p)]\ln b}+C(p)e^{-\gamma(p)/b} , \tag{64}$$

which incorporates the above limiting values if we assume $D(1)=\mu_B$ and $1/\delta(1)=C(1)=\gamma(1)=0$. Thus, using the following expressions for $p \to 1$,

$$D(p)=\mu_B[1+O(1-p)] , \tag{65a}$$

$$1/\delta(p)=\alpha_1(1-p)+O((1-p)^2) , \tag{65b}$$

$$C(p)=\mu_B[1-p+O((1-p)^2)] , \tag{65c}$$

$$\gamma(p)=\beta_1(1-p)+O((1-p)^2) , \tag{65d}$$

and expanding Eq. (64) through $O(1-p)$ we get by comparison with Eq. (63)

$$\alpha_1 = 2p(\ln K)^{-1} , \tag{66a}$$

$$\beta_1 = -\frac{2p}{\ln K}\left(\frac{1}{K-1}+\frac{1}{2}-\frac{1}{\ln K}-\frac{\ln K}{12}\right) . \tag{66b}$$

Finally, we note that while the first term of Eq. (63) corresponds to a contribution in the free energy varying as $|h|^{1+1/\delta(p)}$ (which is similar to the leading nonanalytic term found at finite temperature in the case $p=1)^2$, the second term corresponds to a nonanalytic term in $\langle f(T,h)\rangle$ proportional to $\ln|h|$ which dominates at small fields. The existence of this $\ln|h|$ term implies that all zero-field derivatives of the free energy are infinite. Hence, the transition is of infinite order, at least in the range $p \sim 1$.

In closing this section we wish to show how the approximate procedure introduced by Zittartz and Müller-Hartmann[2] for the ordered Ising model must be modified in order to yield the leading nonanalytic term in $\langle m \rangle$ proportional to $1/b$. We focus on the expression S_1 in Eq. (51) which gives rise to the $1/b$ term. A straightforward application of the cutoff procedure of Ref. 2 would amount to formally expanding Eq. (51) in powers of b, and to evaluate the resulting expression at any given order b^m by summing the contribution $K^l(-bK^l)^m/m!$ over l in a range where we have $bK^l < 1$, that is, for l varying from 1 to $l_0 \simeq -\ln b/\ln K$. In fact, this would yield an incorrect result because the terms to be summed in Eq. (51) have a maximum as a function of l for $K^l = 1/b$. A correct approximate evaluation of Eq. (51)) by series expansion consists, therefore, in writing the individual terms in the form of expansions in powers of $K^l - 1/b$, i.e.,

$$K^l e^{-bK^l} = \frac{1}{e}\left[\left(K^l - \frac{1}{b}\right) + \frac{1}{b}\right] \sum_{m=a}^{\infty} \frac{[-b(K^l - 1/b)]^m}{m!} \quad . \tag{67}$$

An approximate value of S_1 is then obtained by restricting the summation over l in Eq. (51), to an interval of width unity for $K^l - 1/b$ around the position of the maximum at $1/b$. Thus, for this estimate l is allowed to vary from $l_0 = \ln b/\ln K$ to $l_1 = \ln(1 + 1/b))/\ln K$. In particular, if one keeps only the zeroth-order term in Eq. (67) (i.e., the term independent of $K^l - 1/b$) one obtains for S_1

$$S_1 = 1/eb + O(b^0 = 1) \quad , \tag{68}$$

which is of the same form as Eq. (54), as expected. This shows that one must be extremely careful in using the results for the free energy in a range of concentrations where an expansion in powers of the magnetic field is valid, to make predictions about the nonanalytic behavior in a range where such an expansion breaks down and has only a formal meaning.

V. CONCLUDING REMARKS

The new type of magnetic transition which we have found at low temperatures for a system with randomly distributed Ising spins on a Cayley tree, with constant nearest-neighbor interactions, is characterized by the absence of spontaneous magnetization, and the onset of correlations described by the higher-order derivatives of the free energy above the percolation threshold. Spontaneous magnetization appears only in the ordered limit, $p = 1$. These features result from the topology of the Cayley tree which implies that the critical behavior is dominated by the spins lying at the surface in the thermodynamic limit.[1,3] More precisely, we have shown that the lth-order derivative (correlation function) of the free energy diverges at the percolation threshold $p = p_c$, as well as at a higher concentration $p = p_{cl} < p_{cl-1} < p_{c2}$, defined by Eq. (38). For p greater than any of the thresholds (38), the free energy involves nonanalytic contributions and a detailed treatment in the range $p \sim 1$, shows that the leading nonanalytic term is proportional to $\ln|h|$. Therefore, the transition occurring for $p < 1$ and for $h \to 0$ at low temperatures, is an infinite order transition in the sense of the Ehrenfest classification. On the contrary, the order of the transition found in the case of an ordered Cayley tree varies continuously from 1 at $T = 0$ to infinity at the Bethe-Peierls transition temperature.[2]

Phase transitions with absence of long-range ferromagnetic order are expected to occur in various spatially ordered two- and one-dimensional systems of continuous symmetry, and in particular, in the two-dimensional xy model studied recently by Zittartz and others.[8] In fact, as shown by Zittartz, the transition in the two-dimensional xy model is similar in many respects to that found for the ordered Ising model on a Cayley tree. It is thus tempting to speculate that the above results provide a qualitative description of some features of the magnetic transition of a disordered two-dimensional xy model. However, because of the absence of closed loops on a Cayley tree it is not clear what would be, e.g., the topological "defect" analogous to a vortex pair, which plays a central role in the exact treatment of the ordered two-dimensional xy model.[11]

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