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Critical slowing down of energylike perturbation in the Glauber-Ising model

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A real-space renormalization-group analysis of Glauber's equation of motion for a two-dimensional Ising system on a triangular lattice is carried out up to the second order of the cumulant approximations. The study of the relaxation of the energy gives the dynamic exponent $\Delta_E = 2.09$.

I. INTRODUCTION

The Glauber-Ising model[1] is presumably the simplest model exhibiting nontrivial dynamical behavior near the critical point. Thus, although this model can describe the time-dependent behavior of a real system only at the limit of high temperature,[2] it becomes one of the most studied dynamical models. The Glauber model has an exact solution only in the one-dimensional (1-D) case.[1] In two dimensions, it has become a subject for mean-field and initial-decay-rate approximations.[3] It has been studied using high-temperature expansions[4–6] and by Monte Carlo simulations.[7–9] Lately, the renormalization-group (RG) approach using the expansion technique around four and six dimensions was adapted to the study of the critical dynamics[10] of the continuum-spin version of the Glauber model[11] and some generalizations of it. Following the vast amount of interest in the Glauber model, one would expect a wide agreement with respect to the value of the exponents describing the singular behavior near the critical point. However, this is not the case. The approach towards equilibrium, near the critical point, is described by a time scale τ, $\tau_A \sim |T - T_c|^{-\Delta_A}$, where A is an index describing the relaxing physical quantity and the perturbation from equilibrium, Δ is the dynamical index which characterizes the singularity of τ as the temperature T approaches the critical temperatures T_c. The only exact results, concerning the Glauber model (except of course its solution in one dimension) are a few inequalities which the dynamical exponents fulfill.[12,13] These inequalities unlike the inequalities between the static exponents, are not expected to reduce to equalities in the general case. Thus, the inequalities can only put bounds to the real exponents which have to be found by some approximation. Restricting ourselves from now on to the two dimensional (2-D) case, one can find in the literature different estimates to Δ_M, measuring the relaxation of the magnetization disturbance. The lower bound of Δ_M is $\Delta_M \geq \Delta_M' = \gamma = 1.75$, where Δ_M' is the initial-time decay exponent.[14] The high-temperature expansion gives[4,5] $\Delta_M = 2.0$. and[5] 2.125. The numerical Monte Carlo simulations give even wider range of values: 1.85 (Ref. 8) and 2.30 (Ref. 9). The situation becomes even worse when one examines the estimates for Δ_E, which measures the decay of an energy perturbation. This ranges from[7,9] 0.3 to[4,8] 2.0.

In recent years the real-space renormalization-group method[16] has been generalized to the treatment of time-dependent problems.[17–23] The value of Δ_M obtained by this technique[21] $\Delta_M = 2.07$ (in the second order of the cumulant approximation[16]), seems to be a reasonable estimate. This encourages us to calculate Δ_E by the same technique and to the same order of approximation. The calculation of Δ_E, which is the subject of this paper, gives the value $\Delta_E = 2.09$. The difference between Δ_E and Δ_M is of the order of the uncertainty of the calculations. Thus we get $\Delta_E \sim \Delta_M$, which supports the idea that these two exponents should be equal.[13]

The paper is organized as follows. In Sec. II we review the Glauber model and the static RG. The extension to the time-dependent RG is discussed. In Sec. II we perform the transformation using the cumulant expansion[16] up to the second order of the intercell potential. From the transformation of the master equation we find the critical exponents Δ_E and z_E. In Sec. IV we discuss the results and comment on the generality of the method.

II. MODEL, METHOD, AND NOTATIONS

A. Glauber-Ising model

The Glauber-Ising model[1] describes the time-dependent behavior of a large interacting spin system whose equilibrium is determined by an Ising Hamiltonian. The system is brought to a constrained equilibrium state. Then at $t = 0$ the constraint is re-

moved and the system relaxes towards the final equilibrium via an interaction with a heat bath. The heat bath is not treated explicitly in the model, and in the relaxation neither the magnetization nor the energy of the system is conserved. During the relaxation only one spin is allowed to flip at a time, with a transition probability rate $w_j(\{\sigma\})$. This procedure can be described by an empirical master equation for the spin probability distribution $P(\{\sigma\};t)$ and a bare time scale τ of a spin system $\{\sigma=\pm1\}$,

$$\frac{\tau dP(\{\sigma\};t)}{dt} = -\sum_j w_j(\sigma_1,\ldots,\sigma_j,\ldots,\sigma_n)P(\sigma_1,\ldots,\sigma_j,\ldots,\sigma_n;t) + \sum_j w_j(\sigma_1,\ldots,-\sigma_j,\ldots,\sigma_n)P(\sigma_1,\ldots,-\sigma_j,\ldots,\sigma_n;t)$$
$$\equiv L(\sigma)P(\sigma,t) \equiv -\sum_j (1-p_j)w_j(\sigma)P(\sigma,t) . \tag{2.1}$$

where p_j is a spin-flip operator

$$p_j f(\sigma_1,\ldots,\sigma_j,\ldots,\sigma_n) = f(\sigma_1,\ldots,-\sigma_j,\ldots,\sigma_n)$$

and the transition probability satisfies the detailed balance, which ensures the ergodicity of the system

$$(1-p_j)w_j(\sigma)P_e(\sigma) = 0 . \tag{2.2}$$

The final equilibrium state $P_e(\sigma) \equiv P(\sigma, t=\infty)$ is given by a reduced Ising-like Hamiltonian $\bar{H}$,

$$P_e(\sigma) = \exp(-H/k_BT)/Z \equiv e^{\bar{H}}/Z , \tag{2.3}$$

where the partition function Z normalizes P_e,

$$\sum_{\{\sigma\}} P_e = 1 . \tag{2.4}$$

The reduced Hamiltonian $\bar{H}$ is composed from extensive functions of spins $S_a(\sigma)$,

$$\bar{H}(\sigma) = \sum_a K_a S_a(\sigma) . \tag{2.5}$$

For example, in the usual nearest-neighbors (nn) Ising-Hamiltonian (2.5) should read $\bar{H}(\sigma) = K_2 S_2$, where

$$S_2 = \sum_{\langle ij\rangle} \sigma_i\sigma_j ,$$

i and j are nn. Further on we shall use also the following operators: $S_3(\sigma) = \sum_{\langle ik\rangle} \sigma_i\sigma_k$, where i and k are next-nearest neighbors (nnn) and, $S_4(\sigma) = \sum_{\langle il\rangle} \sigma_i\sigma_l$, where i and l are third neighbors (tn).

The master equation (2.1), can be written in a slightly different form

$$\tau \frac{dP(\sigma,t)}{dt} = -\mathfrak{L}\phi(\sigma,t) , \tag{2.6}$$

where $\phi(\sigma,t)$ measures the deviation from equilibrium

$$\phi(\sigma,t) \equiv P(\sigma,t)/P_e(\sigma)$$

and L is given by

$$\mathfrak{L} \equiv \sum_i \mathfrak{L}_i , \quad \mathfrak{L}_i = P_e w_i (1-p_i) . \tag{2.7}$$

For further properties of the Liouville operator L (or $\mathfrak{L}$), the reader is referred to Refs. 3, 4, and 12–14. We shall only note that since w_i does not depend on the history of the system, the Glauber model is a Markoffian one. As we shall see, the time-dependent real-space renormalization-group (TRG) transformation does not preserve this property.

The relation (2.2) does not determine w_i uniquely. We will use[20]

$$w_i(\sigma_i) = \left[\frac{P_e(\sigma_1,\ldots,-\sigma_i,\ldots,\sigma_n)}{P_e(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_n)}\right]^{1/2} . \tag{2.8}$$

Later, we shall discuss how our result depends on this choice.

B. Static renormalization-group (RG) transformation

In the application of the RG to the equilibrium state one starts with a probability distribution $P(\sigma)$, which is a functional of all the spins $\{\sigma\}$, $\sigma_i=\pm1$ defined on a lattice. One defines a transformation of this probability distribution to a new probability distribution of a set of spin variables $\{\mu\}$, $\mu_\alpha=\pm1$, on a lattice with the same symmetry but having a larger lattice constant by a factor of b. This transformation is of the form

$$P'(\mu) = \sum_{\{\sigma\}} T(\sigma,\mu)P(\sigma) , \tag{2.9}$$

where T is subject to the following conditions:
(i) It has to preserve the normality of P,

$$\sum_{\{\mu\}} T(\sigma,\mu) = 1 . \tag{2.10}$$

(ii) It has to be non-negative

$$T(\sigma,\mu) \geq 0 .$$

(iii) It should not change the symmetry of the lattice.

The first condition can be satisfied if T is a product over all the new lattice sites α of $\{\mu\}$ of the form

$$T = \prod_\alpha T_\alpha , \quad T_\alpha = \tfrac{1}{2}[1+\mu_\alpha \mathcal{T}_\alpha(\sigma)] , \tag{2.11}$$

while the rest of the conditions have to be satisfied by the particular form of $\mathcal{T}_\alpha$.

According to Eqs. (2.3) and (2.5), $P(\sigma)$ can be represented by a set of interactions $\vec{K}=\{K_a\}$. Similarly, according to condition (iii), $P(\sigma)$ can be described using a similar reduced Hamiltonian

$$\bar{H}'=\sum_a K_a' S_a(\mu) \tag{2.12}$$

and the RG transformation can be considered to be a transformation upon the parameter space K_a,

$$\vec{K}'=\underline{R}\vec{K} \ . \tag{2.13}$$

The fixed point of this transformation,[24] $\vec{K}^*=\underline{R}\vec{K}^*$ is associated with a critical point (or with a zero correlation).

The RG transformation of a time-dependent probability distribution can be done in a similar way. First one has to represent $P(\sigma,t)$ by a time-dependent reduced Hamiltonian

$$\bar{H}(\sigma,t)=\sum_a K_a(t)S_a(\sigma) \ , \tag{2.14}$$

which may require a larger parameter space. The RG transformation [Eq. (2.13)] is parametrized by the set $\{S_a\}$ of operators which are time independent. Hence at a certain time t, which is taken as a parameter, the RG is exactly the static one

$$\vec{K}'(t)=\underline{R}\vec{K}(t) \ , \tag{2.15}$$

where

$$\vec{K}(t)\equiv\{K_a(t)\} \ .$$

C. Time-dependent RG transformation

The dynamical scaling hypothesis[25] suggests that the characteristic time for relaxation behaves as

$$\tau_0\sim\xi^z \tag{2.16}$$

or

$$\tau_0\sim|T-T_c|^{-\Delta} \ , \tag{2.17}$$

where z and Δ are the dynamical exponents, connected via

$$\Delta=\nu z \ , \tag{2.18}$$

where ν is the static exponent[26] which describes the divergence of the correlation length ξ.

The value of z (or Δ) is also a matter of the definition of τ_0, e.g., one can discuss linear and nonlinear relaxation.[27] However, the dominant slow time scale, which enters into the calculation is the time scale of an infinitely small perturbation near the critical point after a long time.[5,15,27] Thus, under these conditions and according to Eq. (2.16), we expect $P'(\mu,t)$ to fulfill an equation of motion similar to Eq. (2.6) in which $\mathcal{L}(\sigma)$ is transformed into $\mathcal{L}'(\mu)$ and the bare time scale τ becomes τ',

$$\tau'=b^z\tau \ , \tag{2.19}$$

where $\tau\xi^z$ is an invariant of the RG transformation.

The TRG is the standard RG transformation [Eq. (2.9)] applied to the master equation (2.6) followed by the time scaling [Eq. (2.19)] which leaves the master equation (2.6) invariant. That is the situation when there is only one mode in the master equation (2.6). However, the real situation is more complicated since there exist transients and the TRG creates memory effects.[19]

To deal with the general situation, we represent ϕ as sum of perturbations

$$\phi=1+\sum_a h_a(t)O_a(\sigma)\equiv 1+(\vec{h}\cdot\vec{O}) \ . \tag{2.20}$$

The master equation (2.6) then reads

$$P_e(\sigma)\frac{d}{dt}[\vec{h}\cdot\vec{O}(\sigma)]=-\mathcal{L}(\vec{h}\cdot\vec{O}(\sigma)) \ . \tag{2.21}$$

Under the RG transformation, $\sum_{\{\sigma\}}T(\sigma,\mu)$ is applied on both sides of Eq. (2.21), and it becomes

$$P_e'(\mu)\frac{d}{dt}[\vec{h}'\cdot\vec{O}(\mu)]=-\mathcal{L}'(\underline{\Omega}\vec{h}\cdot\vec{O}(\mu)) \ , \tag{2.22}$$

where $\vec{h}'=\underline{\Lambda}\vec{h}$ is the static RG transformation of the parameters $\vec{h}$, and the matrix $\underline{\Omega}$ is defined by the transformation of the right-hand side of Eq. (2.21). If $\underline{\Lambda}$ and $\underline{\Omega}$ commute, any common eigenvector of Λ and Ω will leave the master equation (2.6) in an invariant form. The slowest time scale will determine z via Eq. (2.19), while the faster ones play the same role as the static transients.[25] Usually $\underline{\Lambda}$ and $\underline{\Omega}$ do not commute. The main contribution to the right-hand side of Eq. (2.22) after n RG transformations is from the largest eigenvalue of $\underline{\Omega}$, ω_1. Thus as $n\to\infty$ the rescale factor of τ is

$$\omega_1/\lambda_1=b^{-z} \ , \tag{2.23}$$

where λ_1 is the largest eigenvalue of $\underline{\Lambda}$. There is also a contribution to the rescale factor [Eq. (2.23)] from the elements of the matrix which transforms the set of the eigenvectors of $\underline{\Omega}$ into those of $\underline{\Lambda}$. This contribution goes like the nth root of a $O(1)$ number, hence does not contribute at the limit $n\to\infty$. The λ_1 enters into Eq. (2.23) since in the right-hand side of Eq. (2.22) we have h which has to be transformed to $\vec{h}'$. A more detailed discussion of the memory effect, and the dynamical scaling properties which are obtained by the matching to the high-temperature approximation are given elsewhere.[23]

Near equilibrium, ϕ contains a very large set of operators $\{O\}$. In order to calculate $\underline{\Omega}$ and $\underline{\Lambda}$ we need to examine, in principle, the transformation of all of them. But since we need only the largest

eigenvalue of $\underline{\Omega}$ and $\underline{\Lambda}$, it is enough to examine a subspace of the parameter space as long as this subspace is invariant under the RG transformation and includes the most critical modes (the largest eigenvalues).

III. $\underline{\Omega}_E$ IN THE CUMULANT APPROXIMATION

In this paper we study the relaxation of energylike perturbations. Under the term energylike we mean $\{O_\alpha\}$ which are even under spin reversal. The lattice under study is a triangular lattice. The RG we use associates a new spin variable μ_α with a cell α such that each old spin σ_i belongs to one and only one cell and each block consists of three spins (see Fig. 1). The RG transformation [Eq. (2.11)] is

$$T_\alpha(\sigma)=\tfrac{1}{2}(\sigma_1+\sigma_2+\sigma_3-\sigma_1\sigma_2\sigma_3) \ , \tag{3.1}$$

where σ_i, $i=1,2,3$ are the spins which make up the cell α. This is the majority role transformation $\mu_\alpha=\mathrm{sgn}(\sigma_1+\sigma_2+\sigma_3)$, which was suggested in Ref. 16. To evaluate the RG transformation we need a suitable scheme of approximation. We work with the cumulant approximation,[16] which has the advantage of avoiding boundary effects. In this approximation we assume that $\bar{H}$ can be expanded as

$$\bar{H}=\sum_\alpha H_\alpha+V \ , \tag{3.2}$$

where the zero-order Hamiltonian is the interactions within the cells $\sum_\alpha H_\alpha$ (solid lines in Fig. 1) and the intercell interactions (dotted lines in Fig. 1) are taken as perturbations. The RG transformation is performed up to the second order of the cumulant approximation

$$\langle e^V\rangle=\exp[\langle V\rangle+\tfrac{1}{2}\langle\langle(V-\langle V\rangle)^2\rangle+\cdots] \ . \tag{3.3}$$

The reader is referred to Ref. 28 which discusses the convergence of Eq. (3.3) with $\bar{H}$ divided according to Eq. (3.2).

The transformation of the left-hand side of Eq. (2.21) and the calculation of $\underline{\Lambda}$ were discussed in detail in Ref. 16, hence they will not be discussed here. We shall only quote the result. The appropriate parameter space, which is invariant under the RG transformation is $\vec{K}=(K_2,K_3,K_4)$ the fields conjugate to S_2, S_3 and S_4 the nn, the nnn, and tn interactions, respectively. The interactions K_3 and K_4 are of $O(V^2)$ while K_2 is of order $O(1)$ (intracell interaction) and $O(V)$ (intercell interactions) (although it is the same K_2!). The equilibrium Hamiltonian is at the critical point $\vec{K}=\vec{K}^*$ and the disturbance from equilibrium is $\vec{h}=\delta\vec{K}=K(t)-K^*$. $\underline{\Lambda}$ is the linear RG transformation[16]

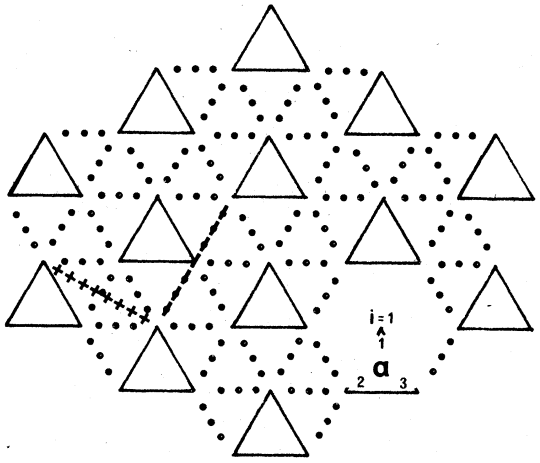

FIG. 1. Spin-cells transformation and the cumulant approximation. The cells are the triangles, whose boundaries represent the contributions to the unperturbed Hamiltonian. The interactions contributing to V appear as dotted lines [nn, $O(V)$], line of crosses [nnn, $O(V^2)$] and broken line (tn, $O(V^2)$]. A σ_i which enters Eq. (3.8) is marked. The nn interactions which do not appear in $P^{(i)}$ were erased.

$$\underline{\Lambda}=\begin{bmatrix}1.8966 & 1.3446 & 0.8964\\ -0.0403 & 0.0 & 0.4482\\ -0.0782 & 0.0 & 0.0\end{bmatrix} \tag{3.4}$$

and

$$\lambda_1=1.7835 \ , \tag{3.5}$$

which corresponds to $\nu=0.950$.[16]

The transformation of the right-hand side of Eq. (2.21) is a little more complicated, although the calculations are of the same nature as the static ones. First we will write the right-hand side of Eq. (2.21) in a more convenient form. As in the calculation of Ref. 16, we shall leave ϕ in an exponential form

$$\phi=\exp\left[\frac{1}{2}\sum_i\sigma_i\left[h_2\sum_j\sigma_j+h_3\sum_k\sigma_k+h_4\sum_l\sigma_l\right]\right]$$
$$\equiv\exp\left[\frac{1}{2}\sum_i\sigma_i E_i\right] \tag{3.6}$$

where σ_j, σ_k, and σ_l are nn, nnn, and tn of σ_i, respectively, and $\vec{h}$ is time dependent. The right-hand side of Eq. (2.21) reads to $O(E_i)$

$$P_e(\sigma)\sum_i w_i(\sigma_i)\sigma_i(e^{E_i}-e^{-E_i}) \ , \tag{3.7}$$

where we used the identity

$$\exp(\sigma_i E_i)-\exp(-\sigma_i E_i)=\sigma_i(\exp E_i-\exp -E_i) \ .$$

Now we perform the RG of Eq. (3.7). We will demonstrate it on one term in the summation over i, and will include in it only the first exponent in parentheses. Since we keep only linear terms in E_i, the renormalization of Eq. (3.7) will be then obtained immediately.

In the calculation of

$$\sum_{\{\sigma\}} T(\sigma,\mu) P_e(\sigma) w_i(\sigma_i) \sigma_i \exp E_i \ , \tag{3.8}$$

we rely on the fact that according to the definition of Eq. (2.8) of w_i, $P_e(\sigma) w_i(\sigma_i)$ is independent of σ_i (Ref. 20). Hence the summation over σ_i can be performed exactly, leading to

$$\mu_\alpha \sum_{\{\sigma\}}{}' \prod_{\beta\neq\alpha} T_\beta(\sigma,\mu) \times \bar{T}_\alpha(\sigma_i) P_e(\sigma) w_i(\sigma_i) \exp E_i \ , \tag{3.9}$$

where

$$\bar{T}_\alpha = \tfrac{1}{2}(1-\sigma_2\sigma_3) \tag{3.10}$$

in which σ_2, σ_3, and σ_i are the three spins included in the α cell. By multiplying P_e by w_i [Eq. (2.8)] all the interactions around the spin σ_i are removed from P_e while the other interactions are unchanged (see Fig. 1). Hence[20] using Eq. (3.2),

$$P_e w_i = \prod_{\beta\neq\alpha} P_\beta P_\alpha^{(i)} e^{V^{(i)}} \ , \tag{3.11}$$

where $P_\beta \equiv (Z)^{-N}\exp\{H_\beta\}$, N is the number of cells and (i) means that all the interactions which contain σ_i are omitted. Substituting Eq. (3.11) Eq. (3.9) becomes

$$\mu_\alpha \sum_{\{\sigma\}}{}' \prod_{\beta\neq\alpha} T_\beta P_\beta \bar{T}_\alpha P_\alpha^{(i)} \exp(V^{(i)}+E_i) = \mu_\alpha Z_0^{N-1} Z_1 \langle \exp(V^{(i)}+E_i)\rangle^\alpha \ , \tag{3.12}$$

where

$$Z_0 = \sum_{\{\sigma\subset\beta\}} T_\beta P_\beta \ , \quad Z_1 = \sum_{\{\sigma_2,\sigma_3\}} \bar{T}_\alpha P_\alpha^{(i)}$$

and

$$\langle A\rangle^\alpha = \sum_{\{\sigma\}} \prod_\gamma F_\gamma(\sigma) A(\sigma) \Big/ \prod_\gamma \sum_{\{\sigma\subset\gamma\}} F_\gamma(\sigma) \ , \tag{3.13}$$

where

$$F_\gamma = \begin{cases} \bar{T}_\alpha P_\alpha^{(i)} \ , & \gamma=\alpha \ , \\ T_\gamma P_\gamma \ , & \gamma\neq\alpha \end{cases} .$$

Using the cumulant expansion we obtain to $O(E_i)$,

$$\langle \exp(V^{(i)}+E_i)\rangle^\alpha = \exp\left[\langle V^{(i)}\rangle^\alpha + \tfrac{1}{2}\langle\langle (V^{(i)}-\langle V^{(i)}\rangle^\alpha)^2\rangle\rangle^\alpha + \langle\langle (E_i-\langle E_i\rangle^\alpha) V^{(i)}\rangle\rangle^\alpha + \tfrac{1}{2}\langle\langle (V^{(i)}-\langle V^{(i)}\rangle^\alpha)^2 E_i\rangle\rangle^\alpha + \langle E_i\rangle^\alpha + \cdots\right] . \tag{3.14}$$

A few of the terms which appear in Eq. (3.14) will be discussed in the Appendix. Here we shall discuss only two terms $\langle V^{(i)}\rangle^\alpha$ and $\langle E\rangle^\alpha$ which demonstrate the main steps of the calculation. $V^{(i)}$ is a sum of terms, of the form $K_a\sigma_j\sigma_l$, where j and l belong to different cells. Hence $\langle V^{(i)}\rangle^\alpha$ will contribute $K_a\langle\sigma_j\rangle^\alpha\langle\sigma_l\rangle^\alpha$. If $j\not\subset\alpha$, $\langle\sigma_j\rangle^\alpha=\mu_\alpha f_1$, where

$$f_1 = [\exp(3K_2)+\exp(-K_2)]/Z_0 \ .$$

If $j\subset\alpha$, $\langle\sigma_j\rangle^\alpha=0$ since the weight function F_α is even under spin reversal in the cell α, while σ_j is odd. Thus

$$\langle V^{(i)}\rangle^\alpha = 2f_1^2 K_2 \sum_{\langle\beta\gamma\rangle} \mu_\beta\mu_\gamma \ , \tag{3.15}$$

where β,γ are nn, $\beta,\gamma\neq\alpha$. Equation (3.15) is connected to the $P_e'(\mu)$ via,[20,21]

$$\exp\langle V^{(i)}\rangle^\alpha = (Z_0)^{-N} P_e'(\mu) w_\alpha(\mu_\alpha) + O(V^2) \ . \tag{3.16}$$

The calculation of $\langle E_i\rangle^\alpha$ is very similar. A typical term $\langle h_a\sigma_j\rangle$ becomes $h_a f_1\mu_\beta$, $\beta\neq\alpha$ or zero if $j\subset\alpha$. The only h_2 which $O(1)$ is associated with spins in α, hence will not contribute to $\langle E_i\rangle^\alpha$. The other terms in E_i give [after summation over i in the α cell, and taking Eq. (3.16) into account]

$$h_2' = (2h_2f_1+3h_3f_1+2h_4f_1)Z_1/Z_0 \ , \quad h_3' = h_3 f_1 Z_1/Z_0 \ , \quad h_4' = 0 \ . \tag{3.17}$$

Equation (3.8) takes into account only half the terms in Eq. (3.7). The other half is obtained by inverting the sign of h. From Eq. (3.17) we see that $\vec{h}'$ is at least of $O(V)$, hence to order $O(V^2)$, terms which do not depend on h, should be taken to $O(V)$. Thus the correction in Eq. (3.16) and the $\langle\langle (V^{(i)}-\langle V^{(i)}\rangle)^2\rangle\rangle^\alpha$ term in Eq. (3.14) will contribute $O(V^3)$ corrections and do not have to be calculated in $O(V^2)$. The other two terms in Eq. (3.14) are discussed in the Appendix. Collecting together their contributions Eqs. (A3)–(A6) and using Eq. (3.17) we find

$$\underline{\Omega} = \frac{Z_1}{Z_0} f_1 \begin{bmatrix} 2+4(1+\bar{f}_2)K_2[1+2K_2(g_1+g_2)]+4K_2(g_1+g_2) & 3 & 2 \\ K_2(g_1+7g_2)+2K_2^2(1+\bar{f}_2)(g_1+7g_2) & 0 & 1 \\ 8K_2^2(1+\bar{f}_2)g_2+4K_2g_2 & 0 & 0 \end{bmatrix} , \tag{3.18}$$

where

$$g_1=1-f_1^2\ ,\quad g_2=f_2-f_1^2\ ,$$

$$f_2=\exp(3K_2)-\exp(-K_2)\ ,\quad Z_1=2\exp(-K_2)\ ,$$

$$Z_0=\exp(3K_2)+3\exp(-K_2)\ ,$$

and

$$\bar{f}_2=-1.$$

At the fixed point value ($K_2^*=0.2789$) $\underline{\Omega}$ has the largest eigenvalue, $\omega_1=0.5317$. Together with Eqs. (2.23) and (3.5) we have

$$z_E=2.203\pm0.03\ , \tag{3.20}$$

where the error was estimated by examining the effect of the $O(V^2)$ terms in Eq. (3.18) on the value of z.

IV. DISCUSSION

In a previous paper[21] we calculated z_M which describes the critical slowing down of the magnetization. We obtained $z_M=2.18$ which differs from z_E by 1%. In their margins of error the two exponents are overlapping. This result seems to support the assumption[13] that the main contribution to the slowing down of the energy comes from the coupling to the order parameter, hence $z_E=z_M$. The prediction $z_E=z_M$ has also been confirmed in the exact solution of the one-dimensional Glauber model,[1] and in the ϵ expansion for the kinetic Ising model.[29]

In order to compare our result to other values in the literature, we have to compute Δ. Using the value $\nu=0.950$ [Eq. (3.5)] which is obtained from the renormalization of the left-hand side of Eq. (2.21), we have

$$\Delta_E=2.09\ ,\quad \Delta_M=2.07\ .$$

This Δ_E seems to be quite close to the value 2.0, [Eq. (4.8)] and it satisfies the inequality[13] $\Delta_E\geq\gamma$.

In order to obtain the value of Δ_E we did not rely upon the dynamical scaling hypothesis,[25] as in previous calculations. To the contrary, our calculations support this hypothesis (the extended one). This can be seen by renormalizing the master equation (2.6) to the high-temperature region, and then to perform a $1/T$ expansion. The result is matched back, and expressed with the unrenormalized parameters. The scaling arguments are similar to those of the statics,[16,24] and are discussed in more detail in Ref. 22. Another advantage of this calculation is that we get by exactly the same approximation the static exponent, and we can examine scaling laws in which both static and dynamical exponents appear.

In our calculation we used a particular form of w_i, given by Eq. (2.8). Most of the other forms of w_i which are found in the literature, as Eq. (117) in Ref. 1, or Eq. (13) in Ref. 4(a) differ from Eq. (2.8) only by a function of the spins around σ_i, which does not depend on σ_i itself. These functions can be expanded in terms of the operators $O_a(\sigma)$ and be included in ϕ before the RG transformation. This procedure contributes a factor of order unity to the nth power of the time rescale factor, and hence will affect the transients connected to the memory effects, but will not affect the asymptotical behavior as $t\to\infty$.

As a last remark on the above calculations, we note that by examining the terms appearing in a solution to the master equation one can find interaction like nnn which are of $O(V)$. In our calculation we assumed that such terms are of $O(V^2)$. However, after few RG transformations these terms will disappear and new $O(V^2)$ terms will appear, from the transformation of O_2. The only one to reproduce itself under the RG to the same order is O_2 which was taken properly into the account.

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APPENDIX

In this Appendix we discuss the evaluation of the high-order contributions to $\underline{\Omega}$ [Eq. (2.22)] which result from (a) $\langle\langle(E_i-\langle E_i\rangle^\alpha)V^{(l)}\rangle^\alpha$ and (b) $\langle\langle(V^{(l)}-\langle V^{(l)}\rangle^\alpha)^2E_i\rangle^\alpha$ in Eq. (3.14). To all the values given here one has to add a factor z_1/z_0 coming from Eq. (3.16) and (3.12).

(a) A typical term in this sum is

$$\langle h_2(\sigma_j-\langle\sigma_j\rangle^\alpha)K_2\sigma_n\sigma_m\rangle^\alpha\ , \tag{A1}$$

where j is a nn of i. In order to have a nonzero contribution to Eq.(A1), one of the spins σ_n or σ_m (e.g., σ_n), has to be in the same cell β as σ_j is. σ_m is in a different cell γ. There are two possibilities:

(i) $\beta=\alpha,\ \gamma\neq\alpha\ \ (\sigma_i\subset\alpha)$,

thus

$$\langle\sigma_j\rangle^\alpha=0\ ;$$

Eq. (A1) becomes

$$h_2K_2f_1\mu_\gamma\times\begin{cases}1, & n=j\ ,\\ \bar{f}_2, & n\neq j\ ,\end{cases} \tag{A2}$$

where γ is a nn of α and

$$\bar{f}_2\equiv\langle\sigma_j\sigma_n\rangle^\alpha\ \ (j,n\subset\alpha;\ j\neq n\neq i)\ .$$

Summation over the different choice of i in the cell α will contribute to $\underline{\Omega}$ the term

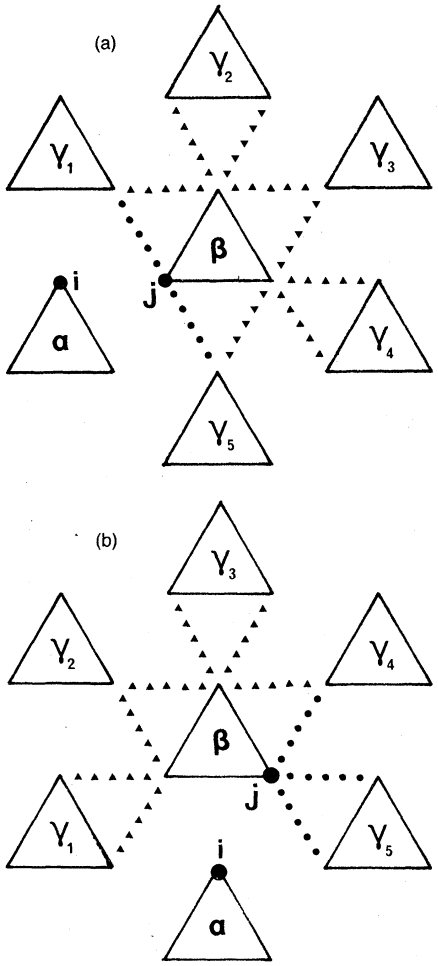

FIG. 2. Contributions to Eq. (A1) from two different orientations of σ_i and σ_j which appear in the disturbance $\sigma_i E_i$ as $h_2\sigma_i\sigma_j$. The interactions $K_2\sigma_n\sigma_m$ which contribute a factor g_1 to Eq. (A4) are marked by dots, and those contributing a factor g_2 are marked by triangles.

$$h_2' = 4f_1K_2(1+\bar{f}_2)h_2 \ ,$$

(ii) $\alpha \neq \beta \neq \gamma$. (A3)

The contributions from two orientations of σ_j with respect to σ_i are given in Figs. 2(a) and 2(b). After summation over i in the cell α we have

$$h_2' = 4f_1K_2(g_1+g_2)h_2 \ , \quad h_3' = f_1K_2(g_1+7g_2)h_2 \ ,$$
$$h_4' = 4f_1K_2g_2h_2 \ , \tag{A4}$$

where a and b are defined in Eq. (3.18).

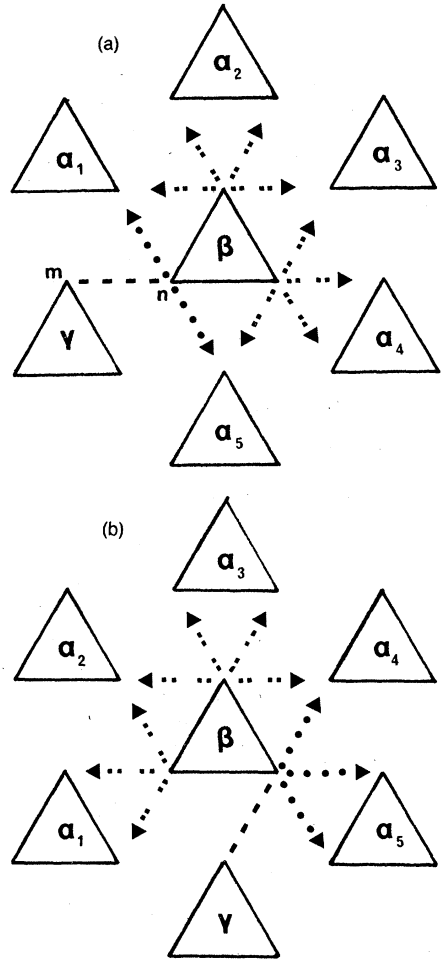

FIG. 3. Contributions to Eq. (A5) from two different orientations of the cells β and γ to which σ_n and σ_m belong, respectively. α_r are the possible locations of the cell α to which σ_i and σ_j belong. The locations of σ_j are marked by arrows. The dashed line contributes f_1. The dotted lines contribute $g_1(1+\bar{f}_2)$ and the double-broken lines contribute $g_2(1+\bar{f}_2)$.

(b) A typical term in the sum is

$$\langle\langle(h_2\sigma_j)K_2^2(\sigma_k\sigma_l - \langle\sigma_k\sigma_l\rangle^\alpha)(\sigma_n\sigma_m - \langle\sigma_n\sigma_m\rangle^\alpha)\rangle\rangle^\alpha \ . \tag{A5}$$

A nonzero contribution is obtained only if each of the brackets in Eq. (A5) has a spin in a cell to which at least one other spin, from other bracket, belongs. Since this term is of order $O(V^2h_2)$, the only contributions are from $h_2 = O(1)$ which connects σ_i and σ_j in the same cell α. The arrangement which gives nonzero contribution is

$$\sigma_k \subset \alpha \ , \quad \sigma_l, \sigma_n \subset \beta \ , \quad \sigma_m \subset \gamma \ ,$$

and

$$\alpha \neq \beta \neq \gamma \ .$$

In Figs. 3(a) and 3(b) we plot two different orientations of the cells β and γ. We preferred, for convenience to keep the cell γ constant and to mark by arrows the possible locations of σ_j. The contributions from these terms are

$$\begin{aligned} h_2' &= 8f_1K_2^2(1+\bar{f}_2)(g_1+g_2)h_2 \ , \\ h_3' &= 2f_1K_2^2(1+\bar{f}_2)(g_1+7g_2)h_2 \ , \\ h_4' &= 8f_1K_2^2(1+\bar{f}_2)g_2h_2 \ . \end{aligned} \tag{A6}$$

Collecting together Eq. (A3)–(A5) and Eq. (3.17), one obtains $\underline{\Omega}$ [Eq. (3.18)].

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