Scaling, equation of state, and the instability of the spin-glass phase

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The free energy and equation of state for the Ising-type random-bond spin-glass are generated in $6-\epsilon$ dimensions with the aid of renormalization-group recursion relations. Scaling predictions based on previous renormalization-group work are borne out. The specific heat is found to be smooth with a rounded peak above the transition temperature, and the order-parameter exponent β is greater than 1. We also find, however, that the spin-glass phase is unstable in that certain fluctuations have a negative gap. This instability leads immediately to the nonphysical prediction that the bond average of the square of a spin correlation function is negative. We conclude that there is a serious flaw in current approaches to the spin-glass phase based on the Edwards-Anderson order parameter.

I. INTRODUCTION

The theory of spin-glasses has recently attracted a great deal of attention. The theory which is due to Edwards and Anderson[1] uses as a model for the spin-glass system a set of spin variables on a periodic lattice interacting via exchange bonds that are randomly ferromagnetic and antiferromagnetic. The analysis by Edwards and Anderson of this model contains three crucial elements. The first is the performance of averages over configurations of random bonds so that all spatial inhomogeneities are averaged out. The second is the characterization of the spin-glass phase by the order parameter Q, where we have

$$Q = \langle \langle \underline{S}(x) \rangle \cdot \langle \underline{S}(x) \rangle \rangle_J \; , \tag{1.1}$$

$\underline{S}(x)$ is the spin variable at the lattice site x, and the inner brackets correspond to a spin average for a given configuration of bonds, while the outer brackets $\langle \cdots \rangle_J$ correspond to an average over bonds. The final element in their approach is the use of the replication technique to allow the bond and spin averages to be performed interchangeably. This overcomes the problem posed by the "quenched-in" character of the random exchange which normally requires that all spin averages be taken before the bond averages are performed.

Edwards and Anderson treated this model in the mean-field approximation. They derived the first theoretical prediction of a frozen in magnetization and gave a description of the spin-glass transition.

However, very soon the theory began to run into difficulties. An infinite-range model for which the mean-field approximation of Edwards and Anderson should be exact was formulated by Sherrington and Kirkpatrick.[2,3] Their solution, obtained with the use of the replica method and a saddle-point integration matched Edwards and Anderson's results in all respects. The solution, however, also contained a pathology. The entropy in the Sherrington-Kirkpatrick solution becomes negative at sufficiently low temperatures, and such a result is not possible for the discrete Ising-type model they started with. Further, there is the puzzling prediction of complex thermal exponents for the spin-glass—ferromagnetic—paramagnetic multicritical point in $6-\epsilon$ dimensions, obtained by Chen and Lubensky[4] with the use of the replication technique. Such a prediction is certainly unusual, although it cannot be ruled out on physical grounds, since a complex exponent does not necessarily imply nonreal results for thermodynamic quantities.

It has been generally assumed that these difficulties are due to the use of the replica method. The negative entropy prompted Thouless, Anderson and Palmer[5] to reconsider the Sherrington-Kirkpatrick model to attempt a solution that did not rely on the replica technique. Using a mixture of analytic and numerical methods they obtained results in two temperature regimes, immediate vicinity of the spin-glass transition and very low temperatures. Their results near the transition agreed with those of Sherrington and Kirkpatrick,[2] while at low temperatures they generated a non-negative entropy and results that differed quantitatively and qualitatively from the replica-method solution. On the basis of this Thouless, Anderson and Palmer concluded that the replica technique works near the spin-glass transition but breaks down somehow at low-enough temperatures. The numerical validity of their work on the low-temperature properties has been supported by recent Monte Carlo calculations by Kirkpatrick and Sherrington.[3]

The possibility that the temperature at which the replica method begins to fail is the spin-glass transition has recently been raised by Almeida and Thouless,[6] who investigate the stability of the Sherrington-Kirkpatrick saddle-point solution for the spin-glass order parameter. They find the solution to be unstable in that the quadratic form for the effect on the free energy of fluctuations about that solution is neither positive-definite nor negative-definite to lowest order. In order for the solution to be considered stable (in the light of the current picture of the spin-glass phase) it should be negative definite. That is, the free energy should be a maximum as a function of Q. The fact that there is a possibility of lowering the free energy by a suitable variation in the system's degrees of freedom may be considered evidence that the Sherrington-Kirkpatrick solution is on the wrong energy "sheet". However it is hard to make definite inferences from the results of Almeida and Thouless, since they are for a model with infinite-ranged interactions, and the effect of the instability is $O(1/N)$ where N is the number of spins in the system.

We suggest that the problem with the existing spin-glass theories is of a more fundamental nature than the use of the replica method. We base this on the fact that in all cases where alternate analytical calculations have been performed, they give rise to exactly the same predictions. The complex exponents of Chen and Lubensky are obtained when an alternative formulation of the spin-glass system relying on the first two elements of the Edwards-Anderson approach, but not the replication technique, is applied.[7] Furthermore, it is possible to duplicate exactly the results of Sherrington and Kirkpatrick,[2] including their negative-entropy prediction, with the use of a tree-diagram summation procedure based on the linked-cluster technique of Brout and with no resort to replicas.[8] This last calculation is to be contrasted with that of Thouless, Anderson and Palmer, discussed above, who also perform the tree-diagram sum without the use of replicas. Although their calculation disagrees with the calculation of Sherrington and Kirkpatrick at low temperature, in that region they abandon an analytic approach in favor of numerical techniques. A critical comparison between their numerical solution and the nonreplica analytic solution has not yet been made. Their numerical calculations and those of Kirkpatrick and Sherrington[3] presumably give a correct description of the low-temperature behavior of this model.

The critical point, we believe, of the analytical results is that the predictions in question are not artifacts of the replication technique, but are rather to be associated with the other two elements of the Edwards-Anderson approach, namely, the averaging out of all spatial inhomogeneities and the characterization of the spin-glass phase uniquely in terms of the Edwards-Anderson order parameter, Q.

In this paper we present the results of a renormalization-group calculation in $6-\epsilon$ dimensions for a model based on this order parameter and using the replica technique. What we have done is to carry the calculations of Harris, Chen, and Lubensky[4,9] one step further and to construct explicitly the free energy and equation of state of an Ising spin-glass in $6-\epsilon$ dimensions. In order to make definite predictions about critical behavior, it is necessary to have at hand not only the critical exponents but also the thermodynamic functions. This is especially true when the ϵ expansion predicts power-law singularities weaker than those in mean-field theory.

The thermodynamic functions yield behavior consistent with the exponents predicted by Harris, Chen, and Lubensky on the basis of recursion relations and assumed scaling relations. We obtain a rounded specific-heat peak with a maximum above the transition temperature, and an order parameter exponent β greater than 1. The specific-heat peak is in qualitative accord with experimental observations. Since we are not interested in multicritical behavior of multicomponent spin-glasses we do not have complex exponents to contend with.

These results are overshadowed by the fact that we find an instability of the spin-glass phase which is manifested in the form of fluctuations with a negative gap. The instability, which appears immediately below the spin-glass transition, is found to be directly coupled to the completely unacceptable prediction that the bond average of the square of a correlation function is negative.

What this means about the spin-glass phase is not yet clear. It could be that the spin-glass phase, if one exists, is not characterized by a frozen-in random magnetization. Bray, Moore, and Reed,[10] in particular have performed numerical simulations that they interpret as indicating that the spin-glass phase has no frozen-in order, but rather anomalously long relaxation times. They have argued that the frozen-in magnetization predicted in other approaches is an artifact of the mean-field approximation used, and that in any system with fewer than four spatial dimensions this state will not persist indefinitely. They bolster their arguments by looking at solvable models and by considering analytically the properties of finite random systems.

Alternatively there could be such a phase but with a frozen-in magnetization that requires a more complete description than is provided by the Edwards-Anderson order parameter.

We want to emphasize that our interpretation of the results presented here is that what has broken down is not the replica approach, but rather the description of the bond-averaged spin-glass system in terms of the Edwards-Anderson order parameter.

This means that a reformulation of existing approaches is absolutely essential for further progress in this field.

An outline of the paper is as follows: In Sec. II the replica-based spin-glass Hamiltonian for a spin system with short-range interactions is presented and analyzed in the mean-field approximation. It is then shown that when physically reasonable coefficients are inserted into the Hamiltonian, the mean-field solution in the ordered phase is unstable with respect to fluctuations to lowest order in perturbation theory. This instability has recently been obtained independently by De Dominicis[11] by a diagrammatic expansion, without the use of the replica technique. Further, this instability corresponds, in the finite-range model we consider here, to the instability seen in the infinite-range Sherrington-Kirkpatrick model by Almeida and Thouless.

The terms leading to the instability in this lowest-order calculation are identified and found to be irrelevant in the renormalization-group sense in $6-\epsilon$ dimensions. This suggests the possibility that the instability renormalizes away for $d \lesssim 6$. In fact, we find that destabilizing terms are generated, in higher order, by the renormalization-group calculations.

In addition to Bray *et al.*, the suggestion that the spin-glass phase may not exist below four dimensions has also been made by Fisch and Harris[12] based on high-temperature series expansions. However, it has been generally accepted that in the neighborhood of $d \lesssim 6$, the model considered in this paper gives correct results. The fact that the model is unstable with respect to fluctuations for $d \lesssim 6$ represents a serious problem for this model.

In a very recent paper Bray and Moore[13] have obtained an instability in second-order perturbation theory but find that the instability can be removed for $d > 4$ by breaking the symmetry between different replicas and restoring it at the end by a particular limiting procedure. The significance of this procedure is, however, not very clear.

The results of the renormalization-group calculations are presented in Sec. III for the disordered phase, and in Sec. IV for the ordered phase.

So far we have discussed only the random-bond model of spin-glasses. It has recently been suggested that amorphous magnets with a random uniaxial anisotropy axis also exhibit a spin-glass phase.[14] The calculations presented below apply equally to both models.[4,14]

II. INSTABILITY OF THE MEAN-FIELD SOLUTION

The Landau-Ginzburg-Wilson effective Hamiltonian for an isotropic m-component spin-glass takes the following form in the replica approach:

$$\begin{aligned}\mathcal{H} = {} & \tfrac{1}{4}\int (r+k^2)\sum Q_{ij}^{\alpha\beta}(k)\,Q_{ij}^{\alpha\beta}(-k) - W\int\sum Q_{ij}^{\alpha\beta}(k)\,Q_{jk}^{\beta\gamma}(k')\,Q_{ki}^{\gamma\alpha}(-k-k')\\ & + U\int\sum Q_{ij}^{\alpha\beta}(k)\,Q_{jk}^{\beta\gamma}(k')\,Q_{kl}^{\gamma\delta}(k'')\,Q_{li}^{\delta\alpha}(-k-k'-k'')\\ & + X\int\sum Q_{il}^{\alpha\beta}(k)\,Q_{jl}^{\alpha\beta}(k')\,Q_{ik}^{\alpha\gamma}(k'')\,Q_{jk}^{\alpha\gamma}(-k-k'-k'')\\ & + 2X\int\sum Q_{il}^{\alpha\beta}(k)\,Q_{il}^{\alpha\beta}(k')\,Q_{jk}^{\alpha\gamma}(k'')\,Q_{jk}^{\alpha\gamma}(-k-k'-k'')\\ & + Y\int\sum Q_{ik}^{\alpha\beta}(k)\,Q_{ik}^{\alpha\beta}(k')\,Q_{jl}^{\alpha\beta}(k'')\,Q_{jl}^{\alpha\beta}(-k-k'-k'')\\ & + 2Y\int\sum Q_{il}^{\alpha\beta}(k)\,Q_{ik}^{\alpha\beta}(k')\,Q_{jl}^{\alpha\beta}(k'')\,Q_{jk}^{\alpha\beta}(-k-k'-k'') + O(Q^5)\end{aligned} \tag{2.1}$$

The superscripts (α, β,...) are replica indices while the subscripts (i, j,...) stand for spin components. The tensor components $Q_{ij}^{\alpha\beta}$ are defined by

$$Q_{ij}^{\alpha\beta}(x) = S_i^{\alpha}(x)S_j^{\beta}(x)(1-\delta_{\alpha\beta}) \; . \tag{2.2}$$

In line with the standard application of the replica technique we have

$$\langle Q_{ij}^{\alpha\beta}\rangle = \langle S_i^{\alpha}S_j^{\beta}\rangle = \langle\, \langle S_i\rangle\, \langle S_j\rangle\, \rangle_J \; , \tag{2.3}$$

where in the last term on the right the inner brackets represent a spin average for a given bond (or anisotropy-axis) configuration. A brief review of the replica technique is contained in Appendix A where the derivation of Eq. (2.1) is outlined.

The sums over the indices in Eq. (2.1) are all unrestricted except that $Q_{ij}^{\alpha\beta}=0$ for $\alpha=\beta$.

For simplicity we will in this section confine ourselves to the case of a single-component spin system. The subscripts on the Q's are all the same and can be dropped. The Hamiltonian becomes

$$\mathcal{H}=\tfrac{1}{4}\int (r+k^2)\sum Q^{\alpha\beta}(k)Q^{\alpha\beta}(-k)-w\int\sum Q^{\alpha\beta}(k)\,Q^{\beta\gamma}(k')\,Q^{\gamma\alpha}(-k-k')$$
$$+u\int\sum Q^{\alpha\beta}(k)\,Q^{\beta\gamma}(k')\,Q^{\gamma\delta}(k'')\,Q^{\delta\alpha}(-k-k'-k'')+x\int\sum Q^{\alpha\beta}(k)\,Q^{\alpha\beta}(k')\,Q^{\alpha\gamma}(k'')\,Q^{\alpha\gamma}(-k-k'-k'')$$
$$+y\int\sum Q^{\alpha\beta}(k)\,Q^{\alpha\beta}(k')\,Q^{\alpha\beta}(k'')\,Q^{\alpha\beta}(-k-k'-k'')+O(Q^5)\ . \tag{2.4}$$

The partition function is given by

$$\int e^{-\mathcal{H}\{Q^{\alpha\beta}\}}\delta Q_{\alpha\beta}\ . \tag{2.5}$$

The mean field approximation consists of setting all the $Q^{\alpha\beta}(0)$'s equal and neglecting all $Q^{\alpha\beta}(k\neq 0)$ in the partition function calculation (2.5). We are left with an integral over a single variable Q, where we have

$$Q\equiv Q^{\alpha\beta}(0)\,\delta(k) \tag{2.6}$$

for all (α,β). The partition function then becomes

$$\int e^{-H(Q)}\,dQ\ , \tag{2.7}$$

where

$$\mathcal{H}(Q)=Nn[\tfrac{1}{4}(n-1)rQ^2-w(n-1)\,(n-2)\,Q^3$$
$$+u(n^3-4n^2+6n-3)\,Q^4$$
$$+x(n-1)^2Q^4+y(n-1)\,Q^4]\ , \tag{2.8}$$

and where n is the number of replicas.

For any finite n the integral (2.7), in the limit $N\to\infty$, is given by the integrand with Q replaced by the appropriate extremizing value. The extremum equation is

$$\frac{d\mathcal{H}(Q)}{dQ}=0\ . \tag{2.9}$$

For the subsequent discussion explicit expressions will be needed for the various coefficients in Eq. (2.4) as derived for an Ising spin system. These are

$$r=A(T^2-T_c^2);\ A>0,\ \ w=\tfrac{1}{6}Z^3,$$
$$u=-\tfrac{1}{8}Z^4,\ \ x=\tfrac{1}{4}Z^4,\ \ y=-\tfrac{1}{12}Z^4\ , \tag{2.10}$$

where Z in Eq. (2.10) is the coordination number of the lattice. The important points to note are: (i) r is a monotonically increasing function of temperature, (ii) $w>0$, and (iii) $y<0$.

We now turn to the question of the appropriate solution to the extremum equation (2.9). Standard approaches require that the appropriate extremum be a minimum of $\mathcal{H}(Q)$ so that the integrand in Eq. (2.7) is maximized. In the spin-glass case, for reasons not altogether understood, it is necessary in order to obtain physically sensible results to maximize $\mathcal{H}(Q)$, for $n<1$. This is a feature of all existing spin-glass calculations whether or not the replica technique is being used.[15,16]

The extremum equation is

$$\tfrac{1}{2}r(n-1)Q-3w(n-1)\,(n-2)\,Q^2$$
$$+4\{\tfrac{u}{n}[(n-1)^4+(n-1)]+x(n-1)^2$$
$$+y(n-1)\}\,Q^3=0\ . \tag{2.11}$$

The appropriate solutions are, in the limit $n\to 0$

$$Q=\begin{cases}0, & r>0\\ -\dfrac{r}{12w}+O(r^2), & r<0\end{cases}\ . \tag{2.12}$$

Since $w>0$ when Q is nonzero, it is positive, as it must be [recall Eq. (2.3)]. The contributions of order r^2 in Eq. (2.12) are due to the quartic terms in the Hamiltonian. It has been traditional to neglect them as unimportant in the region of the transition, but we will see that they play an important role there. In fact they are responsible for what we will call the destabilization of the mean-field solution (2.12).

In order to assess the stability of a mean-field result it is traditional to expand the full Hamiltonian with respect to fluctuations about it. What is done is to write

$$Q^{\alpha\beta}(k)=Q\,\delta(k)+q^{\alpha\beta}(k) \tag{2.13}$$

and expand Eq. (2.4) to second order in $q^{\alpha\beta}(k)$. That is, we consider the effect of fluctuations to lowest order in the interaction parameters. The new Hamiltonian consists of the mean-field Hamiltonian (2.8) and a quadratic Hamiltonian $\mathcal{H}_2\{q^{\alpha\beta}\}$ given by

$$\mathcal{H}_2=\tfrac{1}{4}\int(r+k^2)\sum q^{\alpha\beta}(k)\,q^{\alpha\beta}(-k)-3wQ\int\sum_{\beta\neq\gamma}q^{\alpha\beta}(k)\,q^{\alpha\gamma}(-k)$$
$$+2uQ^2[(2n-1)\int\sum q^{\alpha\beta}(k)\,q^{\alpha\beta}(-k)+(2n-2)\int\sum_{\beta\neq\gamma}q^{\alpha\beta}(k)\,q^{\alpha\gamma}(-k)+\int{\sum}'q^{\alpha\beta}(k)\,q^{\gamma\delta}(-k)]$$
$$+2xQ^2[(n+1)\int\sum q^{\alpha\beta}(k)\,q^{\alpha\beta}(-k)+2\int\sum_{\beta\neq\gamma}q^{\alpha\beta}(k)\,q^{\alpha\gamma}(-k)]+6yQ^2\int\sum q^{\alpha\beta}(k)\,q^{\alpha\beta}(-q)\ , \tag{2.14}$$

where $\sum'$ denotes that all the indices are different.

In order to facilitate the task of considering the fluctuation Hamiltonian we introduce three $\frac{1}{2}n(n-1)$ by $\frac{1}{2}n(n-1)$ matrices acting on the $\frac{1}{2}n(n-1)$ vectors $q^{\alpha\beta}$, ($q^{\alpha\beta}=q^{\beta\alpha}$). These matrices are: (i) The identity matrix

$$\underline{I}_{\alpha\beta,\,\gamma\delta}=\begin{cases}1 \text{ if } \alpha=\gamma \text{ and } \beta=\delta \text{ or } \alpha=\delta \text{ and } \beta=\gamma,\\ 0 \text{ otherwise,}\end{cases} \tag{2.15}$$

(ii) The association matrix

$$\underline{R}_{\alpha\beta,\,\gamma\delta}=\begin{cases}1 \text{ if } \alpha=\gamma,\ \beta\neq\delta \text{ or } \alpha=\delta,\ \beta\neq\gamma\\ \text{or } \alpha\neq\gamma,\ \beta=\delta \text{ or } \alpha\neq\delta,\ \beta=\gamma\ ,\\ 0 \text{ otherwise,}\end{cases} \tag{2.16}$$

and (iii) The matrix $\underline{S}$

$$\underline{S}_{\alpha\beta,\,\gamma\delta}=\begin{cases}0 \text{ if } \alpha=\gamma,\ \beta=\delta \text{ or } \alpha=\delta,\ \beta=\gamma\\ 1 \text{ otherwise.}\end{cases} \tag{2.17}$$

The quadratic Hamiltonian then takes the form

$$\begin{aligned}\mathcal{H}_2=\tfrac{1}{2}\int (r+k^2)q(k)\cdot\underline{I}\cdot q(-k)-3wQ\int q(k)\cdot\underline{R}\cdot q(-k)\\ +4uQ^2\int q(k)\cdot[(2n-1)\underline{I}+(n-1)\underline{R}+2(\underline{S}-\underline{R})]\cdot q(-k)\\ +4xQ^2\int q(k)\cdot[(n+1)\underline{I}+\underline{R}]\cdot q(-k)+12yQ^2\int q(k)\cdot\underline{I}\cdot q(-k)\ .\end{aligned} \tag{2.18}$$

To diagonalize the quadratic form (2.18) one must find the eigenvectors and eigenvalues of $\underline{R}$ and $\underline{S}$. These are discussed in Appendix B.[17] Altogether we find three different eigenvalues. In the following, we focus our attention on those eigenvectors which have eigenvalues -2 and -1 when operated on by $\underline{R}$ and $\underline{S}$, respectively. If we take a q proportional to one of these eigenvectors [there are $\frac{1}{2}n(n-3)$ of them], then we can write

$$q(k)=C(k)\underline{v}\ , \tag{2.19}$$

where $\underline{v}$ is the eigenvector. Substituting Eq. (2.19) into Eq. (2.18) we obtain for the contribution of this eigenmode to the fluctuation Hamiltonian

$$\int C(k)\,C(-k)[\tfrac{1}{2}(r+k^2)-6wQ+12uQ^2 +4xQ^2(n-1)+12y\,Q^2]\ . \tag{2.20}$$

Making use of Eq. (2.11) and taking the limit $n\to 0$, the fluctuation Hamiltonian (2.20) reduces to

$$\int C(k)\,C(-k)\,(\tfrac{1}{2}k^2+8yQ^2)\ . \tag{2.21}$$

In order to have an extremum value, the complete quadratic form (2.18) should be positive or negative definite. This is the case for the eigenvectors corresponding to two of the three possible eigenvalues. For the remaining eigenvectors which we consider here, however, the quadratic form is neither positive nor negative definite. The reason is that, according to Eq. (2.10), the term $8yQ^2$ in Eq. (2.21) is negative. That means that for sufficiently small wavenumber k, the coefficient of $C(k)\,C(-k)=|C(k)|^2$ in this equation is also negative, while for larger k the coefficient becomes positive. This is precisely the instability obtained by De Dominicis[11] and for $k=0$ by Almeida and Thouless.[6] Thus the mean field is unstable just below the spin-glass transition to sufficiently long wavelength fluctuations.

It is possible to place a more physical interpretation on the instability. In Sec. IV the instability is discussed in terms of a correlation function where it is shown that the negativity of the coefficient in Eq. (2.21) corresponds to a negative result for a correlation function that must necessarily be positive.

Looking at the Hamiltonian (2.4) formally we can isolate the term leading to the instability. It is the quartic term proportional to $\sum_{\alpha,\beta}(Q^{\alpha\beta})^4$. This can be characterized as a "multiple-loop" term, where a single-loop term of any order has the form

$$Q^{\alpha_1\alpha_2}\,Q^{\alpha_2\alpha_3}\,Q^{\alpha_3\alpha_4}\ldots Q^{\alpha_l\alpha_1}\ ,$$

the summation over each α being unrestricted. A multiple-loop term is generated by requiring that two nonconsecutive α_i's be equal. A graphical classification of terms is contained in Appendix A. The instability results from the graph shown in Fig. 2. In Appendix C it is shown that if the Hamiltonian consists entirely of single-loop terms and the interactions are treated to lowest order, then the lowest-lying terms in the quadratic Hamiltonian correspond to gapless modes. Thus, it is both the existence and the sign of the multiple-loop quartic term that leads to the instability of the mean-field solution.

It is now clear that we have to take a much closer look at the quartic and higher order terms in the spin-glass Hamiltonian. The coefficients of all the terms will alter as we take into more complete account the effects of fluctuations about the mean-field approximation, particularly near the transition to the spin-glass phase. It is necessary to keep very careful track of what happens to the higher order terms as they become renormalized.

A naive prediction is that the destabilizing terms are not important in $6-\epsilon$ dimensions where we will be carrying out our calculations, since fourth-order terms in a Ginzburg-Landau-Wilson Hamiltonian are irrelevant in the renormalization group sense in more than four dimensions. We will find, however, that even though we use an initial Hamiltonian containing only single-loop diagrams, with no quartic interactions of the type shown in Fig. 2, destabilizing terms are generated directly by the renormalization group calculations. Both the instability of the mean-field solution and the unphysical predictions associated with it, emerge as a direct consequence of our calculations.

III. RENORMALIZATION-GROUP CALCULATION FOR THE DISORDERED PHASE

In the renormalization-group calculation we expand about $d=6$. Because they are irrelevant for $d>4$, the quartic terms in Eq. (2.1) can be neglected, and we are left with the standard spin-glass model[9]

$$\mathcal{H}=\frac{1}{4}\int (r+k^2)\sum Q_{ij}^{\alpha\beta}(k)\,Q_{ij}^{\alpha\beta}(-k) - w\int \sum Q_{ij}^{\alpha\beta}(k)\,Q_{jk}^{\beta\gamma}(k')\,Q_{ki}^{\gamma\alpha}(-k-k') \ . \tag{3.1}$$

This model applies both to the random exchange and the random anisotropy spin-glass. In the latter case, $i=j=k=1$. The critical behavior of the random-anisotropy spin-glass is expected to be the same as for an Ising random-exchange spin-glass.[4]

The recursion relations for the disordered phase have been derived by Harris, Lubensky and Chen.[4,9] When written in differential form these are

$$\frac{dr(l)}{dl}=[2-\eta(l)]r(l)-36w^2(l)\times(n-2)m\,\frac{K_6}{[1+r(l)]^2}\ ,$$

$$\frac{dw(l)}{dl}=[\tfrac{1}{2}\epsilon-\tfrac{3}{2}\eta(l)]w(l)+36w^3(l)\times[(n-3)m+1]\,\frac{K_6}{[1+r(l)]^3}\ , \tag{3.2}$$

where

$$\eta(l)=12K_6(n-2)mw^2(l)\ . \tag{3.3}$$

Here $\epsilon=6-d$, and m is the number of spin components. These equations may be integrated up in the usual way.[18] To leading order in ϵ and w^2 we obtain

$$r(l)=t(l)+18K_6(n-2)mw^2(l)\times\left\{1-2t(l)\ln[1+t(l)]-\frac{t^2(l)}{1+t(l)}\right\} \tag{3.4}$$

valid for $r(l)\lesssim 1$, where $t(l)$ and $w(l)$ satisfy the equations

$$\frac{dt(l)}{dl}=[2-\eta(l)]t(l)+72K_6(n-2)mw^2(l)t(l)\ ,$$

$$\frac{dw}{dl}=[\tfrac{1}{2}\epsilon-3\eta(l)]w(l)+36K_6[(n-3)m+1]w^3(l)\ . \tag{3.5}$$

Again to leading order, these equations have the solutions

$$t(l)=te^{2l}W(l)^a\ , \tag{3.6}$$

$$w^2(l)=w^2e^{\epsilon l}W(l)^{-1}\ , \tag{3.7}$$

where

$$W(l)=1-36K_6[(n-4)m+2]\frac{w^2}{\epsilon}(e^{\epsilon l}-1)\ , \tag{3.8}$$

$$a=-\frac{5}{3}\,\frac{(2-n)m}{[(4-n)m-2]}\ , \tag{3.9}$$

and where w and t denote the initial values, $t=t(0)$, $w=w(0)$.

The above solution for $t(l)$ can be used to calculate the free energy in the disordered phase. The free energy per degree of freedom is determined by[18]

$$\frac{F}{nm}=\frac{n-1}{4}K_6\int_0^{l^*}\ln[1+r(l)]e^{-ld}dl\ , \tag{3.10}$$

where l^* is chosen such that $r(l^*)\sim 1$. To leading order we can replace $r(l)$ by $t(l)$ and expand ln $[1+t(l)]$ in powers of $t(l)$. The dominant contribution to the integral will come from the term in which the l dependence of $e^{-ld}t^k(l)$ is weak $\sim e^{\epsilon l}$, so that the integration gives an extra factor of $1/\epsilon$. For $d\sim 6$ that term is $t^3(l)$,

$$\frac{F}{nm}=\frac{n-1}{12}K_6\int_0^{l^*}e^{-ld}t^3(l)\,dl\ . \tag{3.11}$$

This integral is straightforward. Taking the limit $n\to 0$ we obtain

$$\frac{F}{nm}=\frac{t^3}{864w^2(3m+1)}[W(l^*)^b-1]\ , \tag{3.12}$$

where

$$b=-(3m+1)/(2m-1)\ . \tag{3.13}$$

The value of l^* is determined by the condition $t(l^*)=1$. From Eq. (3.6)

$$t(l^*)=1=te^{2l^*}W(l^*)^a\ . \tag{3.14}$$

To leading order, $e^{l^*}=t^{-1/2}$. Then by iteration, we obtain from Eqs. (3.8) and (3.12),

$$\frac{F}{nm}=\frac{t^3}{864w^2(3m+1)} \times\left[\left[1+72K_6(2m-1)\frac{w^2}{\epsilon}(t^{-\epsilon/2}-1)\right]^b-1\right]. \quad (3.15)$$

For $t \approx 0$ the corresponding expression for the specific heat, $C_v=-\partial^2F(t,w)/\partial t^2$, can be written

$$C_v=C_1t-C_2t^{-\alpha}, \quad (3.16)$$

where

$$\alpha=-\left[1+\frac{3m+1}{2m-1}\frac{\epsilon}{2}\right]. \quad (3.17)$$

The coefficients C_1 and C_2 are both positive

$$C_1=\frac{1}{144(3m+1)w^2},\ C_2=C_1\left[72K_6(2m-1)\frac{w^2}{\epsilon}\right]^b. \quad (3.18)$$

This expression for α agrees with the result obtained from scaling.

The critical exponents ν and η for this model have been determined by Harris, Lubensky, and Chen[9]

$$\nu=\frac{1}{2}+\frac{5}{12}\frac{m}{2m-1}\epsilon, \quad \eta=-\frac{1}{3}\frac{m\epsilon}{2m-1}. \quad (3.19)$$

From the scaling relations we have

$$d\nu=2-\alpha, \quad \gamma=(2-\eta)\nu, \quad 2\beta=2-\gamma-\alpha, \quad (3.20)$$

it then follows that:

$$\alpha=-\left[1+\frac{3m+1}{2m-1}\frac{\epsilon}{2}\right], \quad \gamma=1+\frac{m}{2m-1}\epsilon, \quad \beta=1+\frac{m+1}{2m-1}\frac{\epsilon}{4}, \quad (3.21)$$

where β and γ are the exponents of the spin-glass order parameter and of the susceptibility, $\chi=\langle Q_{\alpha\beta}Q_{\alpha\beta}\rangle$.

Because $\alpha<-1$, the leading temperature dependence of C_v when $t\to 0$ is given by the linear term and not by the "singular" term, $t^{-\alpha}$.

The spin-glass susceptibility in the critical region χ is related to the susceptibility far from critically $\chi(l^*)$ by

$$\chi^{-1}=\exp\left[-2l^*+\int_0^{l^*}\eta(l)\,dl\right]\chi^{-1}(l^*), \quad (3.22)$$

where l^* is determined by Eq. (3.14). The trajectory integral can be evaluated with help of Eqs. (3.2) and (3.3), i.e.,

$$\exp\int_0^{l^*}\eta(l)\,dl=W(l^*)^{a/5}. \quad (3.23)$$

We calculate the noncritical susceptibility $\chi^{-1}(l^*)$ to leading order in (w^2,ϵ) using fluctuation corrected Landau theory. From the diagram in Fig. 1(a) we obtain

$$\chi^{-1}(l^*)=r(l^*)-36K_6(n-2)mw^2(l^*) \times\int\frac{k^5\,dk}{r(l^*)+k^2}, \quad (3.24)$$

where we have made use of the propagator

$$G_{ijkl}^{\alpha\beta\gamma\delta}(k)=\langle Q_{ij}^{\alpha\beta}(k)\,Q_{kl}^{\gamma\delta}(-k)\rangle =\frac{1}{r+k^2}(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{il}\delta_{jk}\delta_{\beta\delta}\delta_{\beta\gamma}) \times(1-\delta_{\alpha\beta})(1-\delta_{\gamma\delta}). \quad (3.25)$$

Performing the elementary integrations and eliminating $r(l^*)$, using Eq. (3.4) we find

$$\chi^{-1}(l^*)=t(l^*)-18K_6(n-2)mw^2(l^*) \times t(l^*)[\ln t(l^*)+\tfrac{1}{2}]. \quad (3.26)$$

Then setting $t(l^*)=1$ we obtain simply

$$\chi^{-1}(l^*)=1+O(w^2). \quad (3.27)$$

From Eqs. (3.14), (3.22), and (3.23), the critical susceptibility is then given by

$$\chi^{-1}=tW(l^*)^{6/5a}. \quad (3.28)$$

Iterating this expression we find, in the limit $n\to 0$,

$$\chi^{-1}=t^{\gamma}\text{ with }\gamma=1+\frac{m}{2m-1}\epsilon \quad (3.29)$$

in agreement with the scaling relations discussed above.

FIG. 1. Diagrams contributing to the (a) r_i and (b) $\tilde{h}$ recursion relations.

IV. RENORMALIZATION-GROUP CALCULATION FOR THE ORDERED PHASE

In the ordered phase we consider for simplicity only the case of a single-spin component with the Hamiltonian

$$\mathcal{H}=\frac{1}{4}\int (r+k^2)\sum Q^{\alpha\beta}(k)\,Q^{\alpha\beta}(-k)$$
$$-\int \sum h^{\alpha\beta}(k)\,Q^{\alpha\beta}(-k)$$
$$-w\int \sum Q^{\alpha\beta}(k)\,Q^{\beta\gamma}(k')\,Q^{\gamma\alpha}(-k-k') \qquad (4.1)$$

Here $h^{\alpha\beta}$ is a fictious field which couples linearly to the order parameter. This Hamiltonian describes either an Ising random-exchange spin-glass or a general uniaxial-anisotropy spin-glass. We substitute Eq. (2.13) for $Q^{\alpha\beta}(k)$ and separate $\mathcal{H}$ into its fluctuation part

$$\mathcal{H}=\frac{1}{4}\int (r+k^2)\sum q^{\alpha\beta}(k)\,q^{\alpha\beta}(-k)$$
$$-\int \sum \tilde{h}^{\alpha\beta}(k)\,q^{\alpha\beta}(-k)$$
$$-s\int \sum_{\alpha\neq\gamma} q^{\alpha\beta}(k)\,q^{\gamma\beta}(-k)$$
$$-w\int \sum q^{\alpha\beta}(k)\,q^{\beta\gamma}(k')\,q^{\gamma\alpha}(-k-k')\;, \qquad (4.2)$$

where

$$\tilde{h}^{\alpha\beta}=h^{\alpha\beta}-\tfrac{1}{2}rQ+3(n-2)\,wQ^2\;, \qquad (4.3)$$

$$s=3wQ\;, \qquad (4.4)$$

and its fluctuation-independent (mean-field) part,

$$\mathcal{H}_{\rm mf}=\tfrac{1}{4}rn(n-1)\,Q^2-wn(n-1)\,(n-2)\,Q^3\;, \qquad (4.5)$$

where the spin-glass order parameter Q is now to be determined self-consistently from the condition,

$$\langle q^{\alpha\beta}(k)\rangle=0\;. \qquad (4.6)$$

From Eq. (2.18) the quadratic part of the Hamiltonian (4.2) can be written

$$\mathcal{H}_2=\tfrac{1}{2}\int q\cdot[(r+k^2)\underline{I}-2s\,\underline{R}\,]\cdot q\;, \qquad (4.7)$$

where $\underline{I}$ is the unit matrix and $\underline{R}$ is the association matrix defined by Eq. (2.16), the properties of which is discussed in Appendix B. $\underline{R}$ has eigenvalues, $2(n-2)$, $(n-4)$, and -2. Thus $\mathcal{H}_2$ has three diagonal elements, r_m+k^2, with

$$r_1=r-4(n-2)s\;,$$
$$r_2=r-2(n-4)s\;, \qquad (4.8)$$
$$r_3=r+4s\;.$$

Here r_3 is the mode discussed in Sec. II which is gapless when we substitute for s its mean-field value using cubic terms only, but becomes negative if quartic terms are included. The corresponding degeneracies are

$$1,\quad (n-1),\quad \text{and } n(n-3)/2 \qquad (4.9)$$

respectively. By determining the eigenvectors as well, the propagator in the disordered phase $G^{\alpha\beta\gamma\nu}(k)=\langle q^{\alpha\beta}(k)\,q^{\gamma\delta}(-k)\rangle$, can be determined. However, it turns out to be simpler to determine the propagator diagramatically by treating the s term in perturbation theory and summing the s insertions to all orders.

A. Calculation of the propagator

We define G_a, G_b, and G_c as the components of the propagator with respectively two pairs of indices equal, one pair equal and no indices equal

$$G_a=\langle q^{\alpha\beta}q^{\alpha\beta}\rangle$$
$$G_b=\langle q^{\alpha\beta}q^{\alpha\gamma}\rangle,\quad \beta\neq\gamma \qquad (4.10)$$
$$G_c=\langle q^{\alpha\beta}q^{\gamma\delta}\rangle\ \alpha,\beta\neq\gamma,\delta\;,$$

where G_a, G_b, and G_c are independent of the indices. Then after i insertions the propagator will be of the form $(G_a^{(i)},\,G_b^{(i)},\,G_c^{(i)})$ where the superscript indicates the number of insertions. The following recursion relation is straightforward to construct

$$\begin{pmatrix} G_a^{(i+1)}(k)\\ G_b^{(i+1)}(k)\\ G_c^{(i+1)}(k)\end{pmatrix}=\frac{2s}{r+k^2}\begin{pmatrix}0 & 2(n-2) & 0\\ 1 & (n-2) & (n-3)\\ 0 & 4 & 2(n-4)\end{pmatrix}\begin{pmatrix}G_a^{(i)}\\ G_b^{(i)}\\ G_c^{(i)}\end{pmatrix}. \qquad (4.11)$$

In order to assess the effect of i insertions we diagonalize the matrix in Eq. (4.11). There are three eigenvalues,

$$\lambda_1=2(n-2),\quad \lambda_2=(n-4),\quad \text{and } \lambda_3=-2 \qquad (4.12)$$

with associated eigenvectors

$$\underline{v}_1=\begin{pmatrix}1\\1\\1\end{pmatrix},\quad \underline{v}_2=\begin{pmatrix}1\\(n-4)/2(n-2)\\-2/(n-2)\end{pmatrix},$$
$$\underline{v}_3=\begin{pmatrix}1\\-1/(n-2)\\2/(n-2)\,(n-3)\end{pmatrix}. \qquad (4.13)$$

These eigenvalues are the same as those for the association matrix. The initial propagator $\underline{G}^0$ is of the form

$$\underline{G}^0=\frac{1}{r+k^2}\begin{pmatrix}1\\0\\0\end{pmatrix}$$
$$=\frac{1}{r+k^2}\left[\frac{2}{n(n-1)}\underline{v}_1+\frac{2}{n}\underline{v}_2+\frac{n-3}{n-1}\underline{v}_3\right]. \qquad (4.14)$$

After i insertions we have

$$\underline{G}^{(i)} = \frac{1}{r+k^2}\left(\frac{2s}{r+k^2}\right)^i \left([2(n-2)]^i \frac{2}{n(n-1)}\underline{\nu}_1 + (n-4)^i \frac{2}{n}\underline{\nu}_2 + (-2)^i \frac{n-3}{n-1}\underline{\nu}_3\right) . \tag{4.15}$$

The fully renormalized propagator is obtained by summing over all the insertions

$$\underline{G} = \frac{2}{n(n-1)}\frac{1}{r_1+k^2}\underline{\nu}_1 + \frac{2}{n}\frac{1}{r_2+k^2}\underline{\nu}_2 + \frac{n-3}{n-1}\frac{1}{r_3+k^2}\underline{\nu}_3 . \tag{4.16}$$

By definition the components of $\underline{G}$ are G_a, G_b, and G_c respectively. Supplying all the replica indices, the complete propagator can finally be written

$$\begin{aligned} G^{\alpha\beta\gamma\delta}(k) = {} & G_a(k)\,(1-\delta_{\alpha\beta})\,(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) + G_b(k)\,(1-\delta_{\alpha\beta})\,(1-\delta_{\gamma\delta}) \\ & \times[\delta_{\alpha\gamma}(1-\delta_{\beta\delta})+\delta_{\beta\delta}(1-\delta_{\alpha\gamma})+\delta_{\alpha\delta}(1-\delta_{\beta\gamma})+\delta_{\beta\gamma}(1-\delta_{\alpha\delta})] \\ & + G_c(k)\,(1-\delta_{\alpha\beta})\,(1-\delta_{\gamma\delta})\,(1-\delta_{\alpha\gamma})\,(1-\delta_{\beta\delta})\,(1-\delta_{\alpha\delta})\,(1-\delta_{\beta\gamma}) , \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} G_a(k) &= \frac{2}{n(n-1)}G_1(k) + \frac{2}{n}G_2(k) + \frac{n-3}{n-1}G_3(k) , \\ G_b(k) &= \frac{2}{n(n-1)}G_1(k) + \frac{n-4}{n(n-2)}G_2(k) - \frac{n-3}{(n-1)(n-2)}G_3(k) , \\ G_c(k) &= \frac{2}{n(n-1)}G_1(k) - \frac{4}{n(n-2)}G_2(k) + \frac{2}{(n-1)(n-2)}G_3(k) , \end{aligned} \tag{4.18}$$

and where

$$G_m(k) = \frac{1}{r_m+k^2}, \quad m=1,2,3 , \tag{4.19}$$

with r_m given by Eq. (4.8).

In the disordered phase, $G_m(k)=1/(r+k^2)$ for all m. Then by Eqs. (4.18), $G_a(k)=1/(r+k^2)$ while $G_b(k)=G_c(k)=0$.

In Appendix D these propagators are expressed in terms of random-averaged spin-correlation functions without use of the replica formalism.

B. Recursion relations

To determine the critical behavior in the ordered phase, we need recursion relations for each of the three $r_m(l)$ and for $h^{\alpha\beta}(l)$. For $w(l)$, the solution obtained in the disordered phase will suffice. The s term in Eq. (4.2) need not be further considered, as it has been incorporated into the expressions for $r_m(l)$ and in the propagator given by Eq. (4.17).

To derive the recursion relations for $r_m(l)$ we need only consider the diagram shown in Fig. 1(a) proportional to w^2. Using the propagator in Eq. (4.17) we obtain

$$\begin{aligned} 9K_6 w^2\big(& [(n-2)G_a^2+(n-2)^2G_b^2+(n-2)(n-3)G_c^2]\sum q^{\alpha\beta}q^{\alpha\beta} \\ & +2[(n-2)G_aG_b+(3n-8)G_b^2+(n-2)(n-3)G_bG_c+(n-3)(n-4)G_c^2]\sum_{\beta\neq\gamma} q^{\alpha\beta}q^{\alpha\gamma} \\ & +[2G_aG_c+(n-2)G_b^2+(n-4)(n-5)G_c^2+4(n-4)G_bG_c]\sum{}' q^{\alpha\beta}q^{\gamma\delta}\big) , \end{aligned} \tag{4.20}$$

where we have suppressed all wave-vector dependence. $\sum'$ denotes as before that all the indices are different. As $G_b=G_c=0$, when $Q=0$, we recover trivially the result for the disordered phase $9K_6w^2(n-2)G^2$. We write Eq. (4.20)

$$\begin{aligned} & Aw^2\sum_{\alpha\beta} q^{\alpha\beta}q^{\alpha\beta} + Bw^2\sum_{\beta\neq\gamma} q^{\alpha\beta}q^{\alpha\gamma} \\ & + Cw^2\sum{}' q^{\alpha\beta}q^{\gamma\delta} \end{aligned} \tag{4.21}$$

to obtain the following quadratic "inner shell" Hamiltonian

$$\begin{aligned} \mathcal{H}^{<} = \tfrac{1}{2}q\cdot[& (r+k^2-4Aw^2)\underline{I}-(6wQ+2Bw^2)\underline{R} \\ & -8Cw^2(\underline{S}-\underline{R})]\cdot q , \end{aligned} \tag{4.22}$$

where the matrices $\underline{R}$ and $\underline{S}$ are defined by Eqs. (2.16) and (2.17), respectively. The eigenvalues of $\underline{R}$ and $\underline{S}$ are given in Appendix B. Using those results we obtain for the eigenmodes

$$r_1+k^2-4Aw^2-4Bw^2(n-2)-4Cw^2(n-2)(n-3)\,,$$
$$r_2+k^2-4Aw^2-2Bw^2(n-4)+8Cw^2(n-3)\,,$$
$$r_3+k^2-4Aw^2+4Bw^2-8Cw^2\,. \tag{4.23}$$

Then with the usual rescalings we obtain for the differential recursion relations

$$\frac{dr_1(l)}{dl}=[2-\eta(l)]r_1(l)-4w^2(l)\times[\tilde{A}(l)+(n-2)\tilde{B}(l)+(n-2)(n-3)\tilde{C}(l)]\,,$$
$$\frac{dr_2(l)}{dl}=[2-\eta(l)]r_2(l)-4w^2(l)\times\left[\tilde{A}(l)+\frac{n-4}{2}\tilde{B}(l)-2(n-3)\tilde{C}(l)\right],$$
$$\frac{dr_3(l)}{dl}=[2-\eta(l)]r_3(l)-4w^2(l)\times[\tilde{A}(l)-\tilde{B}(l)+2\tilde{C}(l)]\,, \tag{4.24}$$

where $\eta(l)$ is given by Eq. (3.3). The functions $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are equal to A, B, and C, respectively, except that the propagators $G_{a,b,c}$ are replaced by $g_{a,b,c}$ where

$$g_{a,b,c,}=G_{a,b,c,}(k^2=1)\,. \tag{4.25}$$

Combining these results we can write

$$\frac{dr_m}{dl}(l)=[2-\eta(l)]r_m(l)+w^2\times\sum_{i\leq j}\frac{a_{ij}^{(m)}}{[1+r_i(l)][1+r_j(l)]}\,, \tag{4.26}$$

where $m=1,2,3$ and $a_{ij}^{(m)}$ are rather messy coefficients obtained by substituting for g_a, g_b, and g_c in $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ and for the latter in the recursion relations. Explicit expressions for these coefficients are given in Appendix E.

In view of the initial conditions, given by Eqs. (4.8), we look for solutions of the form

$$t_1(l)=t(l)-2(n-2)6w(l)\,Q(l)\,,$$
$$t_2(l)=t(l)-(n-4)6w(l)\,Q(l)\,, \tag{4.27}$$
$$t_3(l)=t(l)+12w(l)\,Q(l)\,,$$

where $t(l)$ and $w(l)$ are given by Eqs. (3.6) and (3.7) respectively. It is tedious but straightforward to verify that these solutions satisfy the differential recursion relations provided,

$$Q(l)=Q\exp[2-\epsilon/2)l+\int_0^l\eta(l')\,dl']\,. \tag{4.28}$$

The leading correction to order w^2 can then be obtained rather simply with the result

$$r_m(l)=t_m(l)+r_c(l)+r_m'(l) \tag{4.29}$$

where

$$r_c(l)=18K_6(n-2)w^2(l) \tag{4.30}$$

$$r_m'(l)=\frac{w^2(l)}{2}\sum_i a_{ii}^{(m)}\left\{2t_i(l)\ln[1+t_i(l)]+\frac{t_i^2(l)}{1+t_i(l)}\right\}+\frac{w^2(l)}{2}\sum_{i<j}a_{ij}^{(m)}\left[\frac{t_i^2(l)+t_j^2(l)+t_i(l)t_j(l)}{t_i(l)-t_j(l)}\ln\left(\frac{1+t_i(l)}{1+t_j(l)}\right)+\frac{t_i(l)+t_j(l)}{t_i(l)-t_j(l)}\{t_i(l)\ln[1+t_j(l)]-t_j(l)\ln[1+t_i(l)]\}\right]. \tag{4.31}$$

The recursion relation for $\tilde{h}^{\alpha\beta}(l)$ is obtained from the diagram in Fig. 1(b),

$$\frac{d\tilde{h}^{\alpha\beta}}{dl}(l)=[4-\tfrac{1}{2}\epsilon-\tfrac{1}{2}\eta(l)]\,\tilde{h}^{\alpha\beta}(l)+3K_6(n-2)\,w(l)g_b(l)\,, \tag{4.32}$$

where $g_b(l)$ is determined by Eqs. (4.18) and (4.25). There are no g_a or g_c contributions. The solution of this equation is straightforward. We obtain

$$\tilde{h}^{\alpha\beta}(l)=h^{\alpha\beta}(l)-\tfrac{1}{2}t(l)Q(l)+3(n-2)w(l)\,Q^2(l)+\tfrac{3}{2}w(l)(n-2)K_6\left\{\frac{2}{n(n-1)}t_1(l)\{1-t_1(l)\ln[1+t_1(l)]\}+\frac{n-4}{n(n-2)}t_2(l)\{1-t_2(l)\ln[1+t_2(l)]\}-\frac{n-3}{(n-1)(n-2)}t_3(l)\{1-t_3(l)\ln[1+t_3(l)]\}\right\}, \tag{4.33}$$

where

$$h^{\alpha\beta}(l) = h^{\alpha\beta}\exp\left[(4-\tfrac{1}{2}\epsilon)l - \tfrac{1}{2}\int_0^l \eta(l')\,dl'\right] . \tag{4.34}$$

The form of the zeroth-order solution is again suggested by the initial condition Eq. (4.3). The trajectory integral in Eqs. (4.28) and (4.34) is given by Eq. (3.23).

C. Order parameter — specific heat

The fluctuation-corrected noncritical spin-glass order parameter is given implicitly by the condition that, $\langle q^{\alpha\beta}\rangle = 0$. With $l = l^*$ we obtain from the diagram in Fig. 1(b)

$$\tilde{h}^{\alpha\beta}(l^*) + 3w(l^*)\,(n-2)K_6 \int_0^1 k^5\,dk\, G_b(r_m(l^*),k) = 0 , \tag{4.35}$$

with $G_b(k)$ given by Eq. (4.18). Performing the integrals and substituting Eq. (4.33) for $\tilde{h}^{\alpha\beta}(l)$ we find

$$h^{\alpha\beta}(l^*) - \tfrac{1}{2}t(l^*)Q(l^*) + 3(n-2)w(l^*)\,Q^2(l^*) - \tfrac{3}{2}w(l^*)\,(n-2)K_6$$
$$\times\left[\frac{2}{n(n-1)}t_1^2(l^*)\ln t_1(l^*) + \frac{n-4}{n(n-2)}t_2^2(l^*)\ln t_2(l^*) - \frac{n-3}{(n-1)(n-2)}t_3^2(l^*)\ln t_3(l^*)\right] = 0 . \tag{4.36}$$

We note from Eqs. (4.27) that $t_1(l) = t_2(l)$ for $n = 0$. In the ordered phase it is convenient to choose l^* such that

$$t_1(l^*) = t_2(l^*) = 1 . \tag{4.37}$$

Then for $n \to 0$,

$$h^{\alpha\beta}(l^*) - \tfrac{1}{2}t_3(l^*)\,Q(l^*) + \tfrac{9}{2}K_6 w^2(l^*) \times t_3^2(l^*)\ln t_3(l^*) = 0 , \tag{4.38}$$

or to leading order

$$h^{\alpha\beta}(l^*)/Q(l^*) = \tfrac{1}{2}t_3(l^*) . \tag{4.39}$$

Thus in the ordered phase, where we have $Q \neq 0$, it follows that in the limit $h^{\alpha\beta} \to 0$:

$$t_3(l^*) = t(l^*) + 12w(l^*)\,Q(l^*) = 0 \tag{4.40}$$

or

$$Q = (|t|/12w)\,W(l^*)^{-1} , \tag{4.41}$$

where $W(l)$ is given by Eq. (3.8). For $m = 1$, $n = 0$, we have

$$W(l^*) = 1 + 72K_6\frac{w^2}{\epsilon}(e^{\epsilon l^*} - 1) . \tag{4.42}$$

From the condition $t_3(l^*) = 0$ and Eqs. (4.27) it further follows:

$$t_1(l^*) = -t(l^*) , \tag{4.43}$$

such that, by Eq. (4.37), l^* is determined by

$$e^{2l^*} = \frac{1}{|t|}W^{5/3}(l^*) . \tag{4.44}$$

By iteration, the leading temperature dependence of the order parameter as $|t| \to 0$ is then given by

$$Q \sim |t|^\beta \text{ where } \beta = 1 + \tfrac{1}{2}\epsilon \tag{4.45}$$

in agreement with the scaling relations (for $m = 1$) discussed in Sec. III.

From Eqs. (4.5) and (4.7)–(4.9), the free energy (per degree of freedom) can be written

$$\frac{F}{n} = \frac{1}{4}rQ^2(n-1) - wQ^3(n-1)(n-2)$$
$$+\frac{1}{2}\frac{K_6}{n}\int_0^{l^*} dl\, e^{-ld}\left[\ln[1+r_1(l)] + (n-1)\ln[1+r_2(l)] + \frac{n(n-3)}{2}\ln[1+r_3(l)]\right] . \tag{4.46}$$

The mean-field contribution is given by the first two terms while the effect of the fluctuations is described by the trajectory integral. The leading contribution to the latter comes from the terms

$$\frac{1}{6}\frac{K_6}{n}\int_0^{l^*} dl\, e^{-ld}\left[t_1^3(l) + (n-1)t_2^3(l) + \frac{n(n-3)}{2}t_3^3(l)\right] \tag{4.47}$$

or making use of Eqs. (4.27), from

$$(n-1)K_6\int_0^{l^*} dl\, e^{-ld}\left[\frac{t^3(l)}{12}+18(n-2)w^2(l)\,Q^2(l)\,t(l)-36(n-2)^2w^3(l)\,Q^3(l)\right]. \tag{4.48}$$

These integrals are evaluated with the help of the differential equations for $t(l)$ and $w(l)$, Eqs. (3.5). The l dependence of $Q(l)$ is given by Eq. (4.28). The first term in Eq. (4.48) is identical to expression (3.11) for the free energy in the disordered phase. The mean-field terms in Eq. (4.46) are cancelled by the contributions from the lower limit of the second and third terms in the trajectory integral, and we are left simply with

$$\frac{F}{n}=\frac{|t|^3}{3456w^2}\,[W(l^*)^{-4}+1]\ , \tag{4.49}$$

with l^* determined by Eq. (4.44). The specific heat obtained from Eq. (4.49) can be combined with the expression for the disordered phase to give

$$C_v=C_1t-C_2|t|^{-\alpha};\quad T\gtrless T_c\ , \tag{4.50}$$

where C_1, C_2, and α are given by Eqs. (3.17) and (3.18), with $m=1$.

This result should be contrasted with the mean-field result, which predicts a cusp in the specific heat. Also, from scaling arguments one only obtains the $t^{1+2\epsilon}$ term, which predicts a peak (with zero slope) at T_c. From Eq. (4.50) it is clear that any peak will occur for $T>T_c$, as the linear term dominates the behavior near T_c. This qualitative result is in agreement with experiments on dilute spin-glasses.[19]

D. Spin-glass susceptibilities

The contribution of the diagram in Fig. 1(a) to the fluctuation corrected susceptibilities at $l=l^*$ and to the recursion relations for $r_m(l)$ differ only by the range of integration in k space. Thus from Eq. (4.26) we obtain

$$\chi_m^{-1}(l^*)=r_m(l^*)+w^2(l^*)\times\sum_{i\le j}a_{ij}^{(m)}\int_0^1\frac{k^5\,dk}{[r_i(l^*)+k^2][r_j(l^*)+k^2]}\ , \tag{4.51}$$

where the coefficients $a_{ij}^{(m)}$ are listed in Appendix E. Substituting for $r_m(l^*)$ the expression (4.29) obtained by integrating up the differential recursion relations then gives

$$\chi_m^{-1}(l^*)=t_m(l^*)+w^2(l^*)\sum a_{ii}^{(m)}t_i(l^*)\,[\ln t_i(l^*)+\tfrac{1}{2}]+\tfrac{1}{2}w^2(l^*)\sum_{i<j}\frac{a_{ij}^{(m)}}{t_i(l^*)-t_j(l^*)}\,[t_i^2(l^*)\ln t_i(l^*)-t_j^2(l^*)\ln t_j(l^*)]\ . \tag{4.52}$$

In the limit $n=0, t_1(l)=t_2(l)$. Further making use of Eqs. (4.37) and (4.40), we set $t_3(l^*)=0$ and $t_1(l^*)=1$. The susceptibilities then reduce to

$$\chi_{1,2}^{-1}(l^*)=1+O(w^2)\ , \tag{4.53}$$

$$\chi_3^{-1}(l^*)=-72K_6w^2(l^*)\ , \tag{4.54}$$

where we have made use of the expressions for $a_{ij}^{(m)}$ given in Appendix E. From Eqs. (3.22) and (3.23) we have

$$\chi_m^{-1}=e^{-2l^*}W(l^*)^{-1/3}\chi_m^{-1}(l^*)\ . \tag{4.55}$$

Since $\chi_{1,2}^{-1}(l^*)=1+O(w^2)$ we thus obtain

$$\chi_{1,2}^{-1}=\frac{1}{|t|}\,W(l^*)^2\sim|t|^{-\gamma}\ , \tag{4.56}$$

where $\gamma=1+\epsilon$, as in the disordered phase.

We note that the susceptibility $\chi_3^{-1}(l^*)$, given by Eq. (4.54), is negative. Because χ_3 is by definition positive, as shown in Appendix D, this represents a major inconsistency in the theory. In the mean-field approximation $\chi_3^{-1}=0$ suggesting that there may be a gapless mode, analogous to spin waves in a ferromagnet for which the transverse susceptibility, $\chi_T^{-1}=0$. However, as soon as fluctuations are taken into account χ_3^{-1} becomes negative.

The remarkable thing is that the instability of the spin-glass phase shows up only in χ_3^{-1}. For all other physical properties explicit scaling functions can be calculated. Critical exponents can be extracted in agreement with the usual hyperscaling relations. The specific-heat peaks at a temperature above T_c in agreement with the experiments. In spite of all these apparent successes the instability in χ_3 makes it clear that the current picture of the spin-glass phase must be revised. The fact that the spin-glass solutions considered so far, maximize the free energy may be another manifestation of the breakdown of current spin-glass theories. We can offer no concrete suggestions except perhaps that the complete elimination of the disorder by the bond averaging may be the source of the problem and, perhaps, it will be necessary to keep some aspect of the randomness in the calculation until the end. Spatial inhomogeneities and paramagnetic clusters might well be crucial.

APPENDIX A

In this Appendix we present a brief review of the replica method. As applied to spin-glasses, the replica method begins with the identity

$$\lim_{n\to 0}\frac{1}{n}\langle Z^n\rangle_J - 1 = \langle \ln Z\rangle_J \ , \tag{A.1}$$

where Z is the partition function of a system of classical spins interacting through exchange interactions of random sign and strength. The brackets $\langle\ldots\rangle_J$ stands for an average over configurations of bond strengths. The exchange Hamiltonian is

$$\mathcal{H}\{\vec\sigma(x)\} = \sum_{(x,x')} J(x,x')\vec\sigma(x)\cdot\vec\sigma(x') \ . \tag{A.2}$$

The $\vec\sigma$'s are classical spin variables on a regular, ordered lattice, so that the arguments x are discrete variables. The $J(x,x')$, couple $\vec\sigma$'s on nearest-neighbor sites.

The probability that a given $J(x,x')$ will have a value between J and $J+dJ$ is

$$P(J) = \left[\frac{1}{2\pi\bar J^2}\right]^{1/2} e^{-J^2/2\bar J^2} \ . \tag{A.3}$$

The expectation value of any bond strength is thus zero and

$$\langle J(x,x')^2\rangle_J = \left[\frac{1}{2\pi\bar J^2}\right]^{1/2}\int_{-\infty}^{\infty} e^{-J^2/2\bar J^2}J^2\,dJ = \bar J^2 \ . \tag{A.4}$$

The average of any quantity $f\{J(x,x')\}$ over the ensemble of bond strength is

$$\langle f\{J(x,x')\}\rangle_J \equiv \int\cdots\int f\{J(x,x')\} \times \prod_{(x,x')} P(J(x,x'))\,dJ(x,x') \ . \tag{A.5}$$

Equation (A.5) defines the bracket in Eq. (A.1). The partition function in Eq. (A.1) is

$$Z \equiv \sum_s e^{-\mathcal{H}\{\sigma(x)\}/kT} \ , \tag{A.6}$$

where $\sum_s$ is the sum over spin configurations.

Since the bonds are quenched rather than annealed the appropriate thermodynamic quantity to average over the bond ensemble is the free energy $F\{J(x,x')\}$, where

$$F\{J(x,x')\} = -kT\ln Z \ . \tag{A.7}$$

All the thermodynamics thus appear to be contained in the right-hand side of Eq. (A.1).

To evaluate $\langle Z^n\rangle_J$ we set up n replicas of the system

$$Z^n = \prod_{\alpha=1}^{n}\sum_s \exp\left[\frac{1}{kT}\sum_{(x,x')} J(x,x')\vec\sigma^\alpha(x)\cdot\vec\sigma^\alpha(x')\right] . \tag{A.8}$$

Then we have

$$\langle Z^n\rangle_J = \int\cdots\int \sum_s \exp\left[\frac{1}{kT}\sum_{(x,x')}\sum_{\alpha=1}^{n} J(x,x')\vec\sigma^\alpha(x)\cdot\vec\sigma^\alpha(x')\right] \times \prod_{(x,x')} P(J(x,x'))\,dJ(x,x') \ . \tag{A.9}$$

The averages over the $J(x,x')$'s can be performed to yield

$$\langle Z^n\rangle_J = \sum_s \exp\left[\sum_{(x,x')}\frac{1}{2}\left[\frac{\bar J}{kT}\right]^2\left[\sum_{\alpha=1}^{n}[\vec\sigma^\alpha(x)\cdot\vec\sigma^\alpha(x')]^2 + \sum_{\alpha\neq\beta}\vec\sigma^\alpha(x)\cdot\vec\sigma^\alpha(x')\,\vec\sigma^\beta(x)\cdot\vec\sigma^\beta(x')\right]\right] . \tag{A.10}$$

The terms $[\sigma^\alpha(x)\cdot\sigma^\alpha(x)']^2$ are disregarded. They reduce to a constant for a single-component system. Otherwise, they give rise to an order parameter $Q_{ij}^{\alpha\alpha}$, diagonal in the replica indices, which can be shown to have a lower transition temperature than the off-diagonal elements $Q_{ij}^{\alpha\beta}$, $\alpha\neq\beta$.

The remaining terms in Eq. (A.10) are decoupled with the use of the identity

$$\exp 2\left(\frac{\tilde{J}}{kT}\right)^2 \sigma_i^\alpha(x)\sigma_j^\beta(x)\sigma_i^\alpha(x')\sigma_j^\beta(x')$$

$$= \left(\frac{(kT)^2}{4\pi\tilde{J}^2}\right)^{1/2} \int_{-\infty}^{\infty} dq_{ij}^{\alpha\beta}(x,x') \exp\left[-q_{ij}^{\alpha\beta}(x,x')^2\left(\frac{kT}{2\tilde{J}}\right)^2 + q_{ij}^{\alpha\beta}(x,x')[\sigma_i^\alpha(x)\sigma_j^\beta(x)+\sigma_i^\alpha(x')\sigma_j^\beta(x')] - \left(\frac{\tilde{J}}{kT}\right)^2[\sigma_i^\alpha(x)^2\sigma_j^\beta(x)^2+\sigma_i^\alpha(x')^2\sigma_j^\beta(x')^2]\right] \quad (A.11)$$

Introducing $q_{ij}^{\alpha\beta}(x,x')$'s at every nearest-neighbor bond and for every $\sigma_i^\alpha\sigma_j^\beta$ pair, with $i=j$ included, but not $\alpha=\beta$, we can decouple the spin degrees of freedom on different sites. If we define

$$Q_{ij}^{\alpha\beta}(x) \equiv \sum_{x'} q_{ij}^{\alpha\beta}(x,x') \quad (A.12)$$

we are left with the following sum at a given site:

$$\sum_s \exp\left[\frac{1}{2}\sum_{\alpha,\beta,i,j} \sigma_i^\alpha(x)\sigma_j^\beta(x)Q_{ij}^{\alpha\beta}(x)\right] . \quad (A.13)$$

By introducing a weighting function

$$W(\vec{\sigma}^\alpha) = \delta(|\vec{\sigma}^\alpha|^2 - 1) ,$$

the sum over spin configurations in Eq. (A.13) can be replaced by an integral over $\vec{\sigma}^\alpha$,

$$\int \cdots \int \prod_\alpha W(\vec{\sigma}^\alpha)\, d\vec{\sigma}^\alpha(x) \times \exp\left[\frac{1}{2}\sum_{\substack{\alpha\neq\beta\\ i,j}} Q_{ij}^{\alpha\beta}(x)\sigma_i^\alpha(x)\sigma_j^\beta(x) + \sum h_i^\alpha\sigma_i^\alpha(x)\right] , \quad (A.14)$$

where we have in addition introduced a set of linear fields.

If we expand the exponential in this sum with respect to the $Q\sigma\sigma$ term, sum and re-exponentiate, we obtain an expression that can be shown with the use of standard methods to be expressible in terms of a linked cluster sum of the type shown in Fig. 2. Each edge in a given linked graph corresponds to a factor of $Q_{ij}^{\alpha\beta}(x)$. Each vertex corresponds to the cumulant average $\langle\alpha_i^\alpha\sigma_j^\alpha\cdots\rangle_c$, where a σ_i^α is associated with a vertex if the $\binom{\alpha}{i}$ end of a $Q_{ij}^{\alpha\beta}$ line is incident there. The cumulant average is given by

$$\langle\sigma_i^\alpha\sigma_j^\alpha\ldots\rangle = \frac{\partial}{\partial h_i^\alpha}\frac{\partial}{\partial h_j^\alpha}\cdots \times \ln\left[\int_{-\infty}^{\infty} W(\vec{\sigma}^\alpha)\, d\vec{\sigma}^\alpha \times \exp\sum_i h_i^\alpha\sigma_i^\alpha\right]_{h_i^\alpha=0} . \quad (A.15)$$

In the case of single component spins the term in brackets, which we will call $M_0(h)$, is

$$M_0(h) = 2\cosh h . \quad (A.16)$$

Each graph must be divided by a symmetry factor corresponding to the number of ways the graph can be mapped into itself by flipping and/or rotating it and by permuting identical edges that connect the same pair of vertices.

Because of the fact that the vertices are cumulants, sums over replica indices are unrestricted in each graph subject to the requirement that the edges must join vertices corresponding to different replicas. The graphs (a)–(e) correspond to the terms in Eqs. (2.1) and (2.4). The instability discussed in Sec. II results from graph (e).

Graphs (a),(b),(c) and (f) are single loop graphs. It is interesting to note that they are the only graphs we would have in a single-component system, if the spin-weighting function were Gaussian, since if we have

$$W(\sigma) = e^{-\sigma^2/2\tilde{\sigma}^2} , \quad (A.17)$$

then

$$M_0(h) = \int e^{h\sigma}e^{-\sigma^2/2\tilde{\sigma}^2}\, d\sigma = (2\pi\tilde{\sigma}^2)^{1/2}e^{h\tilde{\sigma}^2/2} \quad (A.18)$$

and

$$\ln M_0(h) = \tfrac{1}{2}\ln(2\pi\tilde{\sigma}^2) + h^2\tfrac{1}{2}\tilde{\sigma}^2 . \quad (A.19)$$

The only nonvanishing cumulant average is $\langle\sigma^2\rangle_c = \tilde{\sigma}^2$.

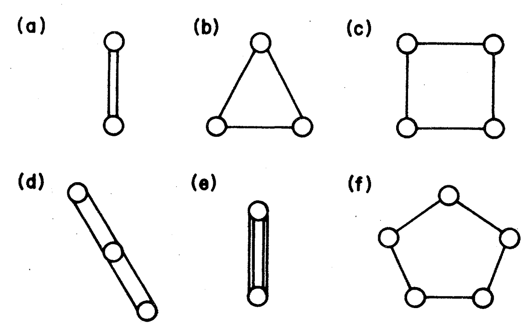

FIG. 2. Low-order diagrams contributing to a linked-cluster expansion of the free energy.

From the discussion in Appendix C it follows that for a Gaussian spin-weighting function there is no instability to lowest order. Instead a gapless mode is obtained. However, in a renormalization group calculation, an instability would presumably again develop as described in Sec. IV.

APPENDIX B

In this Appendix we discuss the properties of the matrices $\underline{R}$ and $\underline{S}$ introduced in Sec. II.

We will begin with the simplest matrix $\underline{S}$. $\underline{S}$ is the m by m matrix all the diagonal elements of which are zero, all the other elements being equal to one. A relationship that is straightforward to verify is

$$\underline{S}^2 = (m-1)\,\underline{I} + (m-2)\,\underline{S}\ , \tag{B.1}$$

where $\underline{I}$ is the m by m identity matrix. If we multiply an eigenvector of $\underline{S}$ by both sides of Eq. (B.1) we obtain

$$\lambda^2 = (m-1) + (m-2)\lambda\ , \tag{B.2}$$

λ being the eigenvalue. Solving for λ we obtain two solutions, $\lambda = -1$ and $\lambda = m-1$. The multiplicities of these two eigenvalues, M_{-1} and M_{m-1}, are obtained by noting that the sum of them is m, and that because $\underline{S}$ is traceless, and the trace is an invariant

$$M_{-1}(-1) + M_{m-1}(m-1) = 0\ . \tag{B.3}$$

These two conditions yield

$$M_{-1} = m-1,\ M_{m-1} = 1\ . \tag{B.4}$$

Eigenvectors corresponding to these two eigenvalues are straightforward to construct. The un-normalized eigenvector $\underline{v}_{m-1}$ is the column vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \tag{B.5}$$

and an eigenvector $\underline{v}_{-1}$ is

$$\begin{pmatrix} 1 \\ -\dfrac{1}{m-1} \\ -\dfrac{1}{m-1} \\ -\dfrac{1}{m-1} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}. \tag{B.6}$$

Other eigenvectors can be obtained by putting 1 elsewhere in the column and $1/(m-1)$ everywhere except at that location.

The association matrix $\underline{R}$ is familiar to graph theorists. It operates on vectors $\underline{v}^{\alpha\beta}$ spanning an $\frac{1}{2}n(n-1)$ dimensional space, the unordered pair (α,β), $\alpha \neq \beta$, corresponding to the index of a given axis in that space. The components of a vector $\underline{v}^{\alpha\beta}$ are most conveniently displayed on an n by n matrix as shown below

$$\begin{pmatrix} 0 & v^{12} & v^{13} & v^{14} & v^{15} & \cdots \\ & 0 & v^{23} & v^{24} & v^{15} & \cdots \\ & \cdot & 0 & v^{34} & v^{35} & \cdots \\ & & \cdot & 0 & v^{45} & \cdots \\ & & \cdot & & 0 & \cdots \\ & & & & \vdots & \end{pmatrix}.$$

Since the pair of indices α,β is not ordered, we need fill the matrix only above the diagonal.

For every coordinate $v^{\alpha\beta}$ we can define two sets of coordinates, the set of first associates and the set of second associates. A coordinate $v^{\alpha\delta}$ belongs to the set of first associates if one of the pair $(\gamma\delta)$ equals α or β and the other does not. Thus v^{23} is a first associate of v^{12} while v^{34} is not. A coordinate $v^{\gamma\delta}$ is a second associate of $v^{\alpha\beta}$ if neither γ nor δ equals α or β. Any coordinate will be either a first or second associate of any other coordinate. Since the relationship of being a first or second associate is reciprocal we can speak of a pair of elements as being first or second associates.

A given elements has $2n-4$ first associates and $\frac{1}{2}(n-2)(n-3)$ second associates. The association matrix $\underline{R}$ as defined by Eq. (2.14) is an $\frac{1}{2}n(n-1)$ by $\frac{1}{2}n(n-1)$ matrix which connects only those elements that are first associates.

In order to find the eigenvalues and the eigenvectors of $\underline{R}$ we note, first, that $\underline{R}$ commutes with $\underline{S}$, where now $m = \frac{1}{2}n(n-1)$ and $\underline{S}$ operates on the vector space spanned by the $v^{\alpha\beta}$'s. This means that $\underline{R}$ and $\underline{S}$ are simultaneously diagonalizable.

To find the eigenvalues of $\underline{R}$ we note that the following relationship may be verified by inspection:

$$\underline{R}^2 = 2(n-2)\,\underline{I} + (n-2)\,\underline{R} + 4(\underline{S} - \underline{R}) \tag{B.7}$$

We can use Eq. (B.7) to find the eigenvalues of $\underline{R}$. We start by multiplying an eigenvector of $\underline{R}$ that has the eigenvalue -1 when multiplied by $\underline{S}$. We obtain

$$\lambda^2 = 2(n-2) + (n-2)\lambda - 4 - 4\lambda\ . \tag{B.8}$$

This equation has two solutions,

$$\lambda = n-4 \tag{B.9}$$

and

$$\lambda = -2\ . \tag{B.10}$$

The other eigenvalue is that of the eigenvector of $\underline{S}$

with the single eigenvalue $m-1=\frac{1}{2}n(n-1)-1$. This eigenvector is of the form (B.5). Multiplying it by $\underline{R}$ yields the eigenvalue $2(n-2)$, corresponding to the number of second associates of a given element.

The multiplicities, M_λ, of the eigenvalues follows from the fact that their total number is $\frac{1}{2}n(n-1)$ and that the matrix $\underline{R}$ is traceless, so that the sum of its eigenvalues equals zero,

$$1+M_{-2}+M_{n-4}=n(n-1)/2 \tag{B.11}$$

$$2(n-2)+M_{n-4}(n-4)+M_{-2}(-2)=0\ , \tag{B.12}$$

we find immediately for M_{n-4} and M_{-2}

$$M_{n-4}=n-1$$
$$M_{-2}=n(n-3)/2\ . \tag{B.13}$$

The eigenvectors of $\underline{R}$ can be constructed as follows. We choose a principal component, $\nu^{\alpha\beta}$ and give it the value 1, give all first associate elements the value V_1 and all second associate elements the value V_2. The values corresponding to the three eigenvalues of $\underline{R}$ are as follows,

$$\lambda_1=2(n-2);\ \begin{cases} V_1=1 \\ V_2=1\end{cases}, \tag{B.14}$$

$$\lambda_2=n-4;\ \begin{cases} V_1=\dfrac{n-4}{2(n-2)} \\ V_2=\dfrac{-2}{n-2}\end{cases}, \tag{B.15}$$

$$\lambda_3=-2;\ \begin{cases} V_1=\dfrac{-1}{n-2} \\ V_2=\dfrac{2}{(n-2)(n-3)}\end{cases}. \tag{B.16}$$

If we denote the eigenvector by the index $(\alpha\beta)$ of the principal component then we call the first eigenvector $\underline{\nu}_1^{\alpha\beta}$, the second kind $\underline{\nu}_2^{\alpha\beta}$ and the third $\underline{\nu}_3^{\alpha\beta}$. All of the $\underline{\nu}_1^{\alpha\beta}$'s are clearly the same, while it appears that we can form $\frac{1}{2}n(n-1)$ different $\underline{\nu}_2^{\alpha\beta}$'s and $\frac{1}{2}n(n-1)$ $\underline{\nu}_3^{\alpha\beta}$'s. However we know from the multiplicity of their respective eigenvalues that there can be at most $n-1$ independent $\underline{\nu}_2^{\alpha\beta}$'s and $\frac{1}{2}[(n-2)(n-3)]$ independent $\underline{\nu}_3^{\alpha\beta}$'s. In the case of the $\underline{\nu}_2^{\alpha\beta}$'s it is possible to verify that all other $\underline{\nu}_2^{\alpha\beta}$'s can be expressed in terms of the $n-1$ $\underline{\nu}_2^{1\beta}$'s (or the $n-1$ $\underline{\nu}_2^{2\beta}$'s etc.) as follows:

$$\underline{\nu}_2^{\alpha\beta}=\frac{n-4}{n-2}(\underline{\nu}_2^{1\alpha}+\underline{\nu}_2^{1\beta})-\frac{2}{n-2}\sum_{\gamma\neq\alpha,\beta}\underline{\nu}_2^{1\gamma} \tag{B.17}$$

for $\alpha,\beta\neq 1$. In the case of the $\underline{\nu}_3^{\alpha\beta}$'s there are n linear relationships between the eigenvectors, which reduce the number of independent eigenvectors to the requisite $\frac{1}{2}n(n-3)$. The relationships are of the following form:

$$\sum_{\beta\neq\alpha}\underline{\nu}_3^{\alpha\beta}=0\ , \tag{B.18}$$

where the sum is over β only. Equation (B.18) holds for each of the n α's.

APPENDIX C

In Sec. II we noted that the instability of the mean-field solution for the spin-glass was due to a multiple-loop quartic term. In this Appendix we show that the modes giving rise to the instability are strictly gapless if the Hamiltonian consists, instead, entirely of single-loop terms. That is, we consider an expansion to all orders in $Q_{ij}^{\alpha\beta}$, which yields an infinite number of interaction parameters. The stability of the mean-field solution is then tested to lowest order in these parameters.

To begin, we will need an expression for $\underline{S}^i$ where $\underline{S}$ is an n by n version of the matrix discussed in Appendix B. Using Eq. (B.1) we can verify that

$$\underline{S}^i=a_i\underline{I}+b_i\underline{S}\ , \tag{C.1}$$

where we have

$$a_i=1/n[(n-1)^i+(n-1)(-1)^i] \tag{C.2}$$

and

$$b_i=1/n[(n-1)^i-(-1)^i]\ . \tag{C.3}$$

In the limit $n\to 0$

$$a_i\to(-1)^i(1-i)\ , \tag{C.4}$$

$$b_i\to(-1)^{i+1}i\ . \tag{C.5}$$

This means that

$$\mathrm{Tr}(\underline{S}^i)=na_i=[(n-1)^i+(n-1)(-1)^i] \tag{C.6}$$

so that

$$\lim_{n\to 0}(1/n)\,\mathrm{Tr}(\underline{S}^i)=(-1)^i(1-i)\ . \tag{C.7}$$

Some results that will be useful later are

$$C_k\equiv\sum_{i=0}^{k}a_ib_{k-i}$$
$$=\frac{1}{n^2}\left((k+1)(n-1)^k-(n-1)(k+1)(-1)^k\right.$$
$$\left.+\frac{n-2}{n}[(n-1)^{k+1}-(-1)^{k+1}]\right)\ , \tag{C.8}$$

$$D_k \equiv \sum_{i=0}^{k} a_i a_{k-i}$$

$$= \frac{1}{n^2}\left\{(k+1)(n-1)^k + (n-1)^2(k+1)(-1)^k + \frac{2(n-1)}{n}[(n-1)^{k+1} - (-1)^{k+1}]\right\}, \quad \text{(C.9)}$$

$$E_k \equiv \sum_{i=0}^{k} b_i b_{k-i}$$

$$= \frac{1}{n^2}\left\{(k+1)(n-1)^k + (k+1)(-1)^k - 2/n[(n-1)^{k+1} - (-1)^{k+1}]\right\}. \quad \text{(C.10)}$$

The single-loop Hamiltonian is of the form

$$\int \sum_{\alpha\beta} u_2(k)\, Q^{\alpha\beta}(k)\, Q^{\alpha\beta}(-k) + u_3 \int \sum_{\alpha\beta\gamma} Q^{\alpha\beta}(k)\, Q^{\beta\gamma}(k')\, Q^{\gamma\alpha}(-k-k') + u_4 \int \sum_{\alpha\beta\gamma\delta} Q^{\alpha\beta}(k)\, Q^{\beta\gamma}(k')\, Q^{\gamma\delta}(k'') \times Q^{\delta\alpha}(-k-k'-k'') + \ldots \quad \text{(C.11)}$$

The only restriction on the sums over replica indices is that there are no diagonal elements, $Q^{\alpha\alpha}(k)$. In the mean-field approximation we neglect all $Q(k)$'s with $k \neq 0$ and set all $Q^{\alpha\beta}(0)$'s equal. Then the Hamiltonian becomes

$$\sum_{i=2}^{\infty} u_i Q^i \operatorname{Tr} \underline{S}^i , \quad \text{(C.12)}$$

where

$$u_2 \equiv u_2(0) . \quad \text{(C.13)}$$

Using Eq. (C.7) we have for the Hamiltonian (C.12) in the limit $n \to 0$

$$n \sum_{i=2}^{\infty} u_i Q^i (-1)^i (1-i) . \quad \text{(C.14)}$$

The extremum equation is

$$n \sum_{i=2}^{\infty} u_i Q^{i-1} i (-1)^i (1-i) = 0 . \quad \text{(C.15)}$$

When there is a $Q \neq 0$ solution of Eq. (C.15) we can write

$$n \sum_{i=2}^{\infty} u_i Q^{i-2} i (-1)^i (i-1) = 0 . \quad \text{(C.16)}$$

Now, we turn to the fluctuation Hamiltonian. First, we write $Q^{\alpha\beta}(k) = Q\,\delta(k) + q^{\alpha\beta}(k)$. Then, substituting this into Eq. (C.11) we extract all terms of zeroth and second order in $q^{\alpha\beta}$. Because of Eq. (C.15) there are no linear terms. The zeroth-order term is just the mean-field Hamiltonian (C.12), while the second-order term is rather tedious to extract. We will content ourselves with describing in detail how one contribution is obtained and leave to the reader the task of verifying our final expression.

There will be five kinds of quadratic terms: $q^{\alpha\beta}q^{\beta\alpha}$, $q^{\alpha\beta}q^{\alpha\beta}$, $q^{\alpha\beta}q^{\beta\gamma}$, $q^{\alpha\beta}q^{\alpha\gamma}$, and $q^{\alpha\beta}q^{\gamma\delta}$. The first and second kind of quadratic terms are actually identical, since $q^{\alpha\beta} = q^{\beta\alpha}$, but are generated in different ways. Similarly the third and fourth terms must also be extracted by looking at different terms in the q expansion of the Hamiltonian (C.11), even though they are identical. We will concentrate on the first two terms.

If we look at a given order term in Eq. (C.11) a second order in $q^{\alpha\beta}$ part of it will look like

$$u_i Q^{i-2} \sum q^{\alpha\beta}(k) S^j_{\beta\gamma} q^{\gamma\delta}(-k) S^{i-j-2}_{\delta\alpha} . \quad \text{(C.17)}$$

If we pick out the $q^{\alpha\beta}q^{\beta\alpha}$ part of Eq. (C.17) we have for its coefficient

$$u_i Q^{i-2} S^j_{\beta\beta} S^{i-j-2}_{\alpha\alpha} = u_i Q^{i-2} a_j a_{i-j-2} . \quad \text{(C.18)}$$

The coefficient of the $q^{\alpha\beta}q^{\alpha\beta}$ part of Eq. (C.17) is

$$u_i Q^{i-2} b_j b_{i-j-2} . \quad \text{(C.19)}$$

Summing over all contributions to the quadratic Hamiltonian from the ith-order term in Eq. (C.11) we have for the coefficient of $q^{\alpha\beta}q^{\beta\alpha}$ and $q^{\alpha\beta}q^{\alpha\beta}$

$$\frac{i}{2} u_i Q^{i-2} \sum_{j=0}^{i-2} (a_j a_{i-j-2} + b_j b_{i-j-2}) = \frac{i}{2} u_i Q^{i-2} (D_{i-2} + E_{i-2}) , \quad \text{(C.20)}$$

with D_i and E_i as defined in Eqs. (C.9) and (C.10). We can write

$$\sum_{\alpha\beta} q^{\alpha\beta} q^{\alpha\beta} = 2\, \underline{q} \cdot \underline{I} \cdot \underline{q} . \quad \text{(C.21)}$$

Thus we have from one ith-order term in (C.11) the quadratic contribution

$$i u_i Q^{i-2} (D_{i-2} + E_{i-2})\, \underline{q}(k) \cdot \underline{I} \cdot \underline{q}(-k) . \quad \text{(C.22)}$$

Adding all the different kinds of quadratic contributions from the ith-order term and then summing to all orders we have

$$\sum_{i=2}^{\infty} i u_i(k)\, Q^{i-2} [\underline{q}(k) \cdot \underline{I} \cdot \underline{q}(-k)\, (D_{i-2} + E_{i-2}) + \underline{q}(k) \cdot \underline{R} \cdot \underline{q}(-k)\, (C_{i-2} + E_{i-2}) + 2\, \underline{q}(k) \cdot (\underline{S} - \underline{R}) \cdot \underline{q}(k) E_{i-2}] . \quad \text{(C.23)}$$

In Eq. (C.23), we have

$$u_i(k)=\begin{cases}u_2(k), & i=2\\ u_i, & i>2\end{cases} \tag{C.24}$$

and $\underline{S}$ is the $\frac{1}{2}n(n-1)$ by $\frac{1}{2}n(n-1)$ matrix considered in Appendix B rather than the n by n matrix discussed heretofore in this section. To diagonalize Eq. (C.23) we take $\underline{q}(k)$ to be proportional to an eigenvector of $\underline{S}$ and $\underline{R}$. The eigenvector of interest is one of the eigenvectors with eigenvalues -1 and -2, respectively. This is the eigenvector that gave rise to the instability in Sec. II. Then, substituting into Eq. (C.23)

$$\underline{q}(k)=C(k)\,\underline{\nu}\ , \tag{C.25}$$

where $\underline{\nu}$ is one of those eigenvectors, we have

$$\sum_{i=2}^{\infty} iu_i(k)\,Q^{i-2}C(k)\,C(-k)\times[(D_{i-2}+E_{i-2})-2(C_{i-2}+E_{i-2})+2E_{i-2}]\ . \tag{C.26}$$

Using Eqs. (C.8)–(C.10) we obtain

$$\sum_{i=2}^{\infty} iu_i(k)\,Q^{i-2}C(k)\,C(-k)\,(-1)^{i-2}(i-1)\ . \tag{C.27}$$

If Q is nonzero, we can add on the left-hand side of Eq. (C.16) to obtain for our final result

$$2[u_2(k)-u_2(0)]\,C(k)\,C(-k)\ . \tag{C.28}$$

If $u_2(k)=u_2(0)+k^2$, we have for this contribution to the fluctuation Hamiltonian, the quadratic Hamiltonian of a gapless mode.

APPENDIX D

In Sec. IV we introduced the spin-glass susceptibilities $\chi_m=G_m$, $m=1,2,3$ in terms of thermal averages of pairs of operators $q^{\alpha\beta}$. In this Appendix we express these susceptibilities in terms of random-averaged spin-correlation functions, without use of the replica formalism.

In Eq. (4.18) $G_{a,b,c}$ are expressed in terms of $G_{1,2,3}$. We invert this set of equations to obtain

$$G_1=G_a+2(n-2)G_b+\tfrac{1}{2}(n-2)(n-3)G_c\ ,$$
$$G_2=G_a+(n-4)G_b-(n-3)G_c\ ,$$
$$G_3=G_a+G_c-2G_b\ . \tag{D.1}$$

In the limit $n\to 0$, G_1 and G_2 become equal, and there are only two different propagators

$$G_1=G_2=G_a-4G_b+3G_c\ ,$$
$$G_3=G_a-2G_b+G_c\ . \tag{D.2}$$

By using techniques similar to those discussed in Appendix B of Ref. 3, we can show that

$$G_a=\langle q_k^{\alpha\beta}q_l^{\alpha\beta}\rangle=\sum_{ij}(\eta^{1/2})_{ki}(\eta^{1/2})_{lj}\,\langle\langle S_iS_j\rangle\,\langle S_iS_j\rangle\rangle_J\ ,\quad k\neq l,$$
$$G_b=\langle q_k^{\alpha\beta}q_l^{\alpha\gamma}\rangle=\sum_{ij}(\eta^{1/2})_{ki}(\eta^{1/2})_{lj}\,\langle\langle S_iS_j\rangle\,\langle S_i\rangle\,\langle S_j\rangle\rangle_J\ ,\quad k\neq l,$$
$$G_c=\langle q_k^{\alpha\beta}q_l^{\gamma\delta}\rangle=\sum_{ij}(\eta^{1/2})_{ki}(\eta^{1/2})_{lj}\,\langle\langle S_i\rangle^2\langle S_j\rangle^2\rangle_J,\quad k\neq l\ , \tag{D.3}$$

where for nearest-neighbor interactions

$$\eta_{ij}=\frac{\sigma^2}{2(kT)^2}\gamma_{ij}\ , \tag{D.4}$$

where σ is the variance of $p(J_{ij})$ and

$$\gamma_{ij}=\begin{cases}1, & i,\,j \text{ nearest neighbors}\\ 0 & \text{otherwise}\end{cases}\ .$$

The subscripts i,j,k,l now refer to lattice sites. From Eqs. (D.2) we can express the diagonal propagators, G_m, in terms of these correlation functions

$$G_{1,2}^{kl}=\sum_{ij}(\eta^{1/2})_{ki}(\eta^{1/2})_{lj}\times(\langle\langle S_iS_j\rangle\,\langle S_iS_j\rangle\rangle_J-4\langle\langle S_iS_j\rangle\,\langle S_i\rangle\,\langle S_j\rangle\rangle_J+3\langle\langle S_i\rangle^2\langle S_j\rangle^2\rangle_J),\quad \text{all } k,l\ , \tag{D.5}$$

while

$$G_3^{k,l}=\sum_{ij}(\eta^{1/2})_{ki}(\eta^{1/2})_{lj}\,\langle(\langle S_iS_j\rangle-\langle S_j\rangle)^2\rangle_J,\quad \text{all } k,l\ . \tag{D.6}$$

We note that G_3 is necessarily positive.

APPENDIX E

In this Appendix we list the coefficients $a_{ij}^{(m)}$ which enters in the recursion relations for $r_m(l)$ and in the fluctuation-corrected susceptibilities. Let

$$K=\frac{-36K_6}{n^2(n-1)^2(n-2)^2}\ .$$

Then we have

$$a_{11}^{(1)}=K(4n^6-36n^5+128n^4-224n^3+192n^2-64n)\ ,\quad a_{12}^{(1)}=K(n^5-4n^3)\ ,$$
$$a_{22}^{(1)}=K(n^7-14n^6+77n^5-212n^4+308n^3-224n^2+64n)\ ,\quad a_{13}^{(1)}=0\ ,\quad a_{23}^{(1)}=0\ ,$$
$$a_{33}^{(1)}=K(2n^6-16n^5+48n^4-92n^3+24n^2)\ , \tag{E.1}$$

while

$$a_{11}^{(2)}=0\ ,\quad a_{12}^{(2)}=K(2n^6-26n^5+128n^4-296n^3+320n^2-128n)\ ,a_{22}^{(2)}=(18n^5-132n^4+338n^3-352n^2+128n)$$
$$a_{13}^{(2)}=0\ ,\quad a_{23}^{(2)}=K(n^7-12n^6+51n^5-88n^4+48n^3)\ ,a_{33}^{(2)}=K(2n^6-18n^5+54n^4-62n^3+24n^2)\ , \tag{E.2}$$

and

$$a_{11}^{(3)}=a_{12}^{(3)}=0\ ,\quad a_{22}^{(3)}=K(n^4-10n^3+33n^2-40n+16)n^2\ ,a_{13}^{(3)}=4Kn(2n^3-10n^2+16n-8)\ ,$$
$$a_{23}^{(3)}=4Kn(2n^4-14n^3+30n^2-26n+8)\ ,a_{33}^{(3)}=Kn^2(n^5-9n^4+27n^3-23n^2-12n+16)\ . \tag{E.3}$$

These coefficients contain, in general, terms proportional to $1/n$, which diverge as $n\rightarrow 0$. However, in the expressions in the text, these coefficients occur only in combinations such as

$$a_{11}^{(m)}+a_{12}^{(m)}+a_{22}^{(m)},\ a_{12}^{(m)}+a_{23}^{(m)},\ \text{and}\ (2a_{11}^{(m)}+a_{22}^{(m)}+\tfrac{3}{2}a_{12}^{(m)})n\ , \tag{E.4}$$

which all approach finite constant values in the limit $n\rightarrow 0$.

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