Collective modes and nonequilibrium effects in current-carrying superconductors

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Dynamical equations for the order parameter and the electric potential in a short-mean-free-path superconductor near the critical temperature are derived from microscopic theory. The effects of supercurrents and various pair-breaking mechanisms on collective modes and relaxation processes are discussed.

I. INTRODUCTION

In this paper we study the kinetics of short-mean-free-path superconductors in situations that are not necessarily close to equilibrium. The main aim of the work is to use kinetic equations based on microscopic theory to understand the spectrum of collective modes in the presence of superflow, magnetic fields, and various scattering mechanisms. The paper extends previous work[1,2] by each of the authors. In Ref. 1, the basic scheme used here—a dynamical equation for the order parameter coupled to a Boltzmann equation for the quasiparticle distribution function—has been discussed and used to derive physical results. This theory is based on the work of Eilenberger,[3] Eliashberg,[4] and Usadel.[5] In Ref. 2, it was pointed out that as the critical current is approached in a superconducting filament, certain collective modes become progressively less stable. In this work we show, among other things, how the methods of Ref. 1 lead to predictions about current-induced mode softening in various regimes of frequency and scattering strength. One result of the study is that the equations used in Ref. 2 are inconsistent, because of a too naive treatment of phonon scattering. Consequently, the quantitative results presented in that paper are modified.

In outline, the plan of this paper is as follows. In Sec. II, we review and slightly generalize the method of Ref. 1, leading to coupled kinetic equations for the order parameter and quasiparticle distribution function. We allow for the presence of a supercurrent and discuss various pair-breaking mechanisms. In Sec. III we show how the quasiparticle distribution function can be eliminated from the problem for linear deviations from true equilibrium, or from a metastable state of steady supercurrent. In the low-frequency limit we find that the quasiparticles can be eliminated even when the order parameter varies in space and time, provided that these variations are sufficiently gentle. As a result we obtain a time-dependent Ginzburg-Landau equation which is valid for arbitrary pair-breaking. In Sec. IV, we give a review of solutions corresponding to deviations from a true equilibrium state in which no supercurrent is flowing. In Sec. V the problem of how collective modes are modified by supercurrent flows is examined, and several results are collected which should be amenable to experimental test. The main result of Ref. 2, a mode softening as the current approaches the critical value, is found again. Also the decay of an electric field in superconductors is investigated. Finally, in Sec. VI the time-dependent Ginzburg-Landau equation derived in Sec. III is used to estimate the rate of current-reducing fluctuations, leading to modifications of results based on the simple phenomenological time-dependent Ginzburg-Landau equation. An Appendix contains mathematical details about some integrals which occur in the reduction of the general theory to usable form.

II. KINETIC EQUATIONS

A. Fundamental equations

In this section, we will review the derivation of the time-dependent Ginzburg-Landau equation and the Boltzmann equation for short-mean-free-path superconductors close to the transition temperature. We will carefully take into account spatial variations of the equilibrium order parameter both in absolute magnitude and in phase. Various pair-breaking mechanisms will be discussed, but we will not rederive the electron-phonon collision operator which appears in the Boltzmann equation. For details we refer to Eliashberg[4] and Ref. 1.

We start from the fundamental equation derived by Eilenberger[3] and by Larkin and Ovchinnikov,[6] in a form obtained by Usadel,[5] which applies to the dirty limit $1/\tau_{\rm imp} \gg T$, i.e.,

$$\left\{\left[\omega\tau_3 + i\hat{U} + i\hat{\Sigma}_{\rm ph} + \frac{1}{2\tau_s}\tau_3\hat{G}\tau_3, \hat{G}\right]\right\}_{\omega_\nu,\omega_\nu'} = D\,\{[\nabla - ieA\,\tau_3, \hat{G}[\nabla - ieA\,\tau_3, \hat{G}]]\}_{\omega_\nu,\omega_\nu'} \quad (1)$$

The notation follows Ref. 1. The Green's function $\hat{G}$ is an angular average

$$\hat{G}(\omega_\nu,\omega_\nu';r) = \int \frac{d\Omega_p}{4\pi}\,\hat{G}_{\vec{p}_F}(\omega_\nu,\omega_\nu';r) \quad (2)$$

of the contracted Green's function

$$\hat{G}_{\vec{p}_F}(\omega_\nu,\omega_\nu';r) = \frac{i}{\pi}\int d\xi_p\,\hat{G}(\omega_\nu,\omega_\nu';r,\vec{p}) \ , \quad (3)$$

where the argument of the integral in Eq. (3) is obtained after partial Fourier transformation with respect to the relative space coordinates. The frequencies ω_ν, ω_ν' are Matsubara frequencies; the difference $\omega_0 = \omega_\nu - \omega_\nu'$ represents the external frequency. $\hat{U}$ denotes a perturbation of the system, for instance in the case of an electric potential it is given by $e\Phi$. The phonon self-energy is $\hat{\Sigma}_{\rm ph}$, and τ_s is the spin-flip time. Finally, D is the diffusion coefficient $D = \frac{1}{3}v_F^2\tau_{\rm imp}$. We use the short-hand notation

$$\{AB\}_{\omega_\nu,\omega_\nu'} = T\sum_{\omega_\nu''} A(\omega_\nu,\omega_\nu'')B(\omega_\nu'',\omega_\nu') \ .$$

The contracted Green's functions obey the normalization condition[1,3,6]

$$\{\hat{G}(r)\hat{G}(r)\}_{\omega_\nu,\omega_\nu'} = (1/T)\delta_{\omega_\nu,\omega_\nu'} \ . \quad (4)$$

B. Equilibrium solutions

In stable and metastable, i.e., supercurrent-carrying, equilibrium the solution of Eq. (1) can be written

$$\hat{G}^{(\rm eq)}(\omega_\nu) = \alpha(\omega_\nu)\tau_3 + \beta(\omega_\nu)\tau_\theta \ ;$$

$$\hat{\Sigma}_{\rm ph}^{(\rm eq)}(\omega_\nu) = -\frac{i}{2\tau_E}\,{\rm sgn}\,\omega_\nu\tau_3 - i\Delta_0\tau_\theta \ , \quad (5)$$

where we have $\tau_\theta = e^{-i\theta(r)\tau_3}\tau_1$. (Renormalization effects proportional to the electron-phonon coupling parameter have been neglected.) The absolute value and phase of the order parameter are $\Delta_0(r)$ and $\theta(r)$; τ_E is the inelastic electron-phonon scattering time. The normalization condition Eq. (4) yields $\alpha^2 + \beta^2 = 1$, which is consistent with the $\underline{1}$ component of Eq. (1). From the $\tau_{\theta+\pi/2}$ component we find

$$\tilde{\omega}_\nu\beta - \Delta_0\alpha = -\left[\frac{1}{\tau_s} + \frac{D}{2}Q^2\right]\alpha\beta + \frac{D}{2}[(\nabla^2\beta)\alpha - (\nabla^2\alpha)\beta] \ , \quad (6)$$

where

$$\tilde{\omega}_\nu = \omega_\nu + (\tfrac{1}{2}\tau_E)\,{\rm sgn}\,\omega_\nu \quad (7)$$

and

$$Q = \nabla\theta + 2eA \ . \quad (8)$$

If the terms on the right-hand side of Eq. (6) can be neglected, the solution is simply

$$\alpha(\omega_\nu) = \tilde{\omega}_\nu/(\tilde{\omega}_\nu^2 + \Delta_0^2)^{1/2} \ ,$$
$$\beta(\omega_\nu) = \Delta_0/(\tilde{\omega}_\nu^2 + \Delta_0^2)^{1/2} \ . \quad (9a)$$

In the limit $|\tilde{\omega}_\nu| \gg \Delta_0$, these expressions reduce to

$$\alpha(\omega_\nu) = {\rm sgn}\,\omega_\nu, \quad \beta(\omega_\nu) = \Delta_0/|\tilde{\omega}_\nu| \ . \quad (9b)$$

For arbitrary right-hand sides of Eq. (6) we find the same limiting forms Eq. (9b) if we redefine $\tilde{\omega}_\nu$

$$\tilde{\omega}_\nu = \omega_\nu + \Gamma\,{\rm sgn}\,\omega_\nu \ , \quad (10)$$

as long as the conditions $|\tilde{\omega}_\nu| \gg \Delta_0$ and $T \gg \Gamma$ are satisfied. The pair-breaking parameter Γ, which is given by

$$\Gamma = \frac{1}{2\tau_E} + \frac{1}{\tau_s} + \frac{D}{2}Q^2 - \frac{D}{2}\frac{\nabla^2\Delta_0}{\Delta_0} \ , \quad (11)$$

includes the various pair-breaking mechanisms in superconductors[7]: electron-phonon scattering, paramagnetic impurities, supercurrent and magnetic field, and spatial variation of the magnitude of the order parameter. (Gap anisotropy is not considered here.)

From the remaining components of Eq. (1) we find $2(\nabla\beta)Q + \beta(\nabla Q) = 0$ which [see Eq. (28)] just yields $j_s = $ const. One the other hand, if we insert $\beta(\omega_\nu)$ given by Eqs. (9) and (11) into the self-consistency equation $\Delta_0 = \pi\lambda T\sum_{\omega_\nu}\beta(\omega_\nu)$, and use a proper cutoff for the summation, we find for $\Gamma \ll T$ the Ginzburg-Landau equation for Δ_0:

$$[\alpha - \beta\Delta_0^2 - \xi^2(0)(Q^2 - \nabla^2)]\Delta_0 = 0 \ , \quad (12)$$

where $\alpha = (T_c - T)/T_c$ and $\beta = 7\zeta(3)/8\pi^2T^2$, and in the dirty limit $\xi^2(0) = (\pi/8T)D$.

C. Nonequilibrium equations

Nonequilibrium situations may be characterized by deviations of the order parameter from the equilibrium value in magnitude (longitudinal) and in phase (transverse), i.e.,

$$\delta\hat{\Delta} = \delta\Delta^L\tau_\theta + \delta\Delta^T\tau_{\theta+\pi/2} \quad (13)$$

Both $\delta\Delta^L$ and $\delta\Delta^T$ are real (this notation differs from Ref. 1). The corresponding change in the Green's function is denoted by

$$\delta\hat{G}(\omega_\nu,\omega_\nu';r) = a^L\tau_3 + b^L\tau_\theta + a^T\underline{1} + b^T\tau_{\theta+\pi/2} \ . \quad (14)$$

The linearized form of the fundamental Eq. (1) and the self-consistency relation

$$\delta\Delta = \pi\lambda T \sum_{\omega_\nu} (b^L \tau_\theta + b^T \tau_{\theta+\pi/2}) \tag{15}$$

will result in a complete set of equations for these deviations.

The frequency $\omega_0 = \omega_\nu - \omega_\nu'$ represents the external frequency. We have to perform the change from the Matsubara frequency to real frequency $\omega_0 \to -i\omega + 0$. This is easily possible in those terms where ω_ν and ω_ν' have the same sign. However, if we have $\omega_\nu > 0$ and $\omega_\nu' < 0$ this process needs particular care. Correspondingly we split the sum in Eq. (15) into two regular (r) contributions ($\omega_\nu < 0$ and $\omega_\nu > \omega_0$) and into an anomalous (a) contribution ($0 < \omega_\nu < \omega_0$). The regular contributions yield the linearized form of the simple time-dependent Ginzburg-Landau equation.[8] The result is well known and we will not repeat the derivation here

$$\frac{\pi}{8T}\,\delta\dot{\Delta}^L = [\alpha - 3\beta\Delta_0^2 - \xi^2(0)(Q^2 - \nabla^2)]\delta\Delta^L - \xi^2(0)[(\nabla Q + 2Q\nabla)\delta\Delta^T + 4Qe\,\delta A\,\Delta_0] + \pi T \sum_{\omega_\nu}^{(a)} b^L$$

$$\frac{\pi}{8T}\,\delta\dot{\Delta}^T = [\alpha - \beta\Delta_0^2 - \xi^2(0)(Q^2 - \nabla^2)]\delta\Delta^T + \xi^2(0)[(\nabla Q + 2Q\nabla)\delta\Delta^L + 4e\,\delta A\,(\nabla\Delta_0)] + \pi T \sum_{\omega_\nu}^{(a)} b^T \ . \tag{16}$$

The remaining contributions from the anomalous terms in Eq. (16) will turn out to be important. In order to determine them we follow the work of Eliashberg.[4]

As becomes obvious by looking at a contribution like

$$\delta\hat{G}^{(1)}(\omega_\nu, \omega_\nu') \sim \hat{G}^{(\mathrm{eq})}(\omega_\nu)\,\hat{U}(\omega_\nu, \omega_\nu')\,\hat{G}^{(\mathrm{eq})}(\omega_\nu') \ ,$$

the quantity $\delta\hat{G}(\omega_\nu, \omega_\nu')$ as a function of a continuous complex-variable z ($\omega_\nu \to z$, $\omega_\nu' = \omega_\nu - \omega_0$) has at least (depending on $\hat{U}$) two cuts in the complex plane, one along the imaginary axis and one for $\mathrm{Re}\,z = \omega_0$. If we change from a frequency summation to an energy integral we obtain

$$T \sum_{\omega_\nu} \delta\hat{G}(\omega_\nu, \omega_\nu') = \int_{-\infty}^{+\infty} \frac{dE}{4\pi i}\,\delta\hat{G}_E \ , \tag{17}$$

where

$$\delta\hat{G}_E = \delta\hat{G}_E^R \tanh\frac{E - \omega/2}{2T} - \delta\hat{G}_E^A \tanh\frac{E + \omega/2}{2T} + \delta\hat{G}_E^{(a)} \ . \tag{18}$$

From the summation over the regions where both ω_ν and ω_ν' are positive (negative), we again obtain regular contributions involving the retarded (advanced) Green's function. This is the result of analytic continuation $\omega_\nu \to -iE - \frac{1}{2}i\omega + (-)0$, when we have both $\mathrm{Re}\,\omega_\nu, \mathrm{Re}\,\omega_\nu' > (<) 0$. On the other hand from the summation over the interior region we obtain the contribution involving a different –, the anomalous Green's function $\delta\hat{G}_E^{(a)}$. Equations (17) and (18) define this quantity.

When performing the analytic continuation in the anomalous region we will encounter combinations of equilibrium Green's functions, as in the following example:

$$\tfrac{1}{2}[\alpha(\omega_\nu) - \alpha(\omega_\nu')] \to \tfrac{1}{2}(\alpha_{E+\omega/2}^R - \alpha_{E-\omega/2}^A) = N_1(E) \ . \tag{19a}$$

This combination is the reduced density of states in a superconductor. In the limit where we have $\Gamma = \omega = 0$, it reduces to the BCS[9] normalized density of states. The effect of a finite value of $1/\tau_E$ is to smear out the singularities. (We will discuss this further in Sec. III.) Similarly we define

$$N_2(E) = \tfrac{1}{2}(\beta_{E+\omega/2}^R + \beta_{E-\omega/2}^A) \ ,$$

$$R_2(E) = \frac{1}{2i}(\beta_{E+\omega/2}^R - \beta_{E-\omega/2}^A) \ . \tag{19b}$$

For $\Gamma = \omega = 0$ we have $R_2(E) = (\Delta_0/E)N_1(E)$. N_1 and N_2 are even functions of the energy, while R_2 is an odd function.

Considering the definitions of the contracted Green's function and Eqs. (17) and (18) we recognize that $-\frac{1}{4}N(0)(\delta\hat{G}_E)_{11}$ is the change in the quasiparticle density per spin direction and per unit energy. Since, at $\omega = 0$, the contribution

$$\tfrac{1}{2}N(0)(a_E^R - a_E^A)[-\tfrac{1}{2}\tanh(E/2T)]$$

is due to the change $N(0)\delta N_1(E)$ in the density of states [Eq. (18)] we conclude that the anomalous part of $(\delta\hat{G}_E)_{11}$ is proportional to a change δf_E in the quasiparticle distribution function, and we have

$$\delta f_E^{L,T} = -a_E^{L,T(a)}/4N_1(E) \ . \tag{20}$$

In the anomalous region the normalization [Eq. (4)]

provides the following relations between $a^{(a)}$ and $b^{(a)}$:

$$b_E^{L(a)}=iR_2(e)a_E^{L(a)}/N_1(E)\ ,$$
$$b_E^{T(a)}=-iN_2(E)a_E^{T(a)}/N_1(E)\ . \tag{21}$$

If we set $\hat{\Sigma}_{\rm ph}^{(\rm eq)}=-i\Delta_0\tau_\theta$ and

$$\delta\hat{\Sigma}_{\rm ph}=-i(\delta\Delta^L\tau_\theta+\delta\Delta^T\tau_{\theta+\pi/2})$$

(neglecting the inelastic scattering contributions for the moment) it is straightforward to perform the analytic continuation of Eq. (1) in the anomalous region. We finally insert the electron-phonon collision operator derived in Ref. 1 and obtain the following Boltzmann equation for the quasiparticle distribution function $\delta f_E^{L(T)}$:

$$N_1\delta\dot{f}^L-K^L(\delta f^L)-D\nabla(M^L\nabla\delta f^L)+2D\left[\nabla(QN_2R_2\delta f^T)+QN_2R_2\frac{e\delta\dot{A}}{4T\cosh^2(E/2T)}\right]=\frac{R_2}{4T\cosh^2(E/2T)}|\dot{\Delta}|$$

$$N_1\delta\dot{f}^T+2|\Delta|N_2\delta f^T-K^T(\delta f^T)-D\nabla\left[M^T\left(\nabla\delta f_+^T\frac{e\delta\dot{A}}{4T\cosh^2(E/2T)}\right)\right]+2D\nabla(QN_2R_2\delta f^L)$$
$$=\frac{N_2}{4T\cosh^2(E/2T)}|\Delta|\dot{\theta}+\frac{N_1}{4T\cosh^2(E/2T)}e\dot{\Phi}\ , \tag{22}$$

where

$$M^L(E)=N_1^2(E)-R_2^2(E)\ ,$$
$$M^T(E)=N_1^2(E)+N_2^2(E)\ . \tag{23}$$

Consideration of the inhomogeneous terms in the system of Eq. (22) shows that δf_E^T is an even function of energy, describing changes in the particle number, while δf_E^L is an odd function describing changes in the energy distribution. The presence of a phase gradient of the equilibrium order parameter or a vector potential, results in a coupling of longitudinal and transverse distribution functions. The collision operators K^L and K^T are defined in Ref. 1. We merely mention here, that they can be written as the sum of a "scattering-out" term

$$-(1/\tau_E)N_1(E)\delta f_E^{L(T)} \tag{24}$$

and a "scattering-in" term which is an integral operator. Near T_c, the inelastic electron-phonon scattering rate $1/\tau_E$ is given by the normal-state value and is the same for both the (L) and (T) operator. The transverse collision operator, furthermore, satisfies particle number conservation

$$\int_{-\infty}^{+\infty}dE\,K^T(\delta f^T)=0\ . \tag{25}$$

Note, however, that due to the term $2|\Delta|N_2\delta f_E^T$, which could be considered part of the collision operator, a conversion betwen normal and supercurrent takes place, and the quasiparticle number is not conserved in general processes. Furthermore, energy can be transferred from the electrons to the phonons, which in turn are assumed to thermalize immediately with the surroundings, and consequently the electronic energy is not conserved.

Using the normalization relations (21), it is now straightforward to perform the remaining summation in Eq. (16) and find

$$\frac{\pi}{8T}\left(\frac{\partial}{\partial t}+2ie\Psi\right)\Delta$$
$$=[\alpha-\beta|\Delta|^2+\xi^2(0)(\nabla-2ieA)^2]\Delta\ , \tag{26}$$

where the effective potential is given by

$$2e\Psi=\frac{8T}{\pi|\Delta|}\int_{-\infty}^{+\infty}dE\,[N_2(E)\delta f_E^T-iR_2(E)\delta f_E^L]\ . \tag{27}$$

Strictly speaking we have derived here only the linearized form of Eq. (26). However, the generalization is obvious. The question of whether it is sufficient to treat the quasiparticle distribution function in linear approximation can be answered positively, even when the order parameter changes drastically, as long as there is a strong relaxation mechanism, e.g., electron-phonon scattering and as long as the changes evolve sufficiently slowly in time. The limit of negligible electron-phonon scattering with diffusion as the only relaxation mechanism has recently been studied by Larkin and Ovchinnikov.[10] They also derived a nonlinear Boltzmann equation.

We complete the set of kinetic equations by deriving expressions for the current density and the charge density. The expression for the current in the dirty limit is given by[6]

$$j = i\frac{\pi\sigma_0}{2e}\operatorname{Tr}\tau_3 T\sum_{\omega_\nu}\{\hat{G}(\nabla - ieA\tau_3)\hat{G} + ieA\tau_3\}_{\omega_\nu,\omega_\nu'} . \tag{28}$$

From this we obtain the linearized form of $j = j_s + j_n$, where we can consider j_s as the supercurrent given by

$$j_s = -(\pi\sigma_0/4eT)|\Delta|^2 Q , \tag{29}$$

while j_n is the normal current

$$j_n = -\frac{\sigma_0}{e}\int_{-\infty}^{+\infty} dE\left[M^T\left(\nabla\delta f_E^T + \frac{e\delta\dot{A}}{4T\cosh^2(E/2T)}\right) - 2QN_2R_2\delta f_E^L\right] . \tag{30}$$

Finally, the expression for the charge density

$$\rho = 2eN(0)\left(-\frac{i\pi}{2}\operatorname{Tr}T\sum_{\omega_\nu}\hat{G}(\omega_\nu,\omega_\nu') - e\Phi\right) \tag{31}$$

can be transformed into

$$\rho = 2eN(0)\left(\int_{-\infty}^{+\infty} dE\, N_1(E)\delta f_E^T - e\Phi\right) . \tag{32}$$

Note that the set of kinetic equations implies the continuity equation

$$\dot{\rho} + \operatorname{div} j = 0 . \tag{33}$$

III. REDUCTION OF THE KINETIC EQUATION

Under certain conditions, for example in the presence of sufficiently strong relaxation mechanisms, or if we consider the effect of small external perturbation, we can assume that the deviation of the quasiparticle distribution function from a local equilibrium value is small. The equilibrium distributions may correspond to stationary stable or metastable, homogeneous or gently inhomogeneous states, and even to states varying slowly in time. Under these conditions we can eliminate the quasiparticle distribution function from the set of coupled kinetic equations. In this section we will first demonstrate this for the case where the superconductor is in a homogeneous current carrying state and where no magnetic field is applied. In this case we can choose a gauge in which the vector potential vanishes ($A = 0$) and the equilibrium order parameter is of the form $\Delta(x) = \Delta_0 e^{-iqx}$. Furthermore, only linear deviations of the order parameter are considered. In the case of low frequencies and long wavelengths we will be able to generalize the results to the nonlinear regime. But we also will discuss high-frequency effects like the Carlson-Goldmann mode.[11]

In all but the collisionless high-frequency modes, inelastic electron-phonon scattering is of central importance in carrying a nonequilibrium distribution to equilibrium. A simple relaxation approximation, in which the collision operators $K(\delta f)$ are replaced by the scattering-out term Eq. (24), violates the particle number conservation Eq. (25), and hence the continuity equation. On the other hand, the full collision operators are too complicated to allow an exact analytic solution. Since a mere numerical solution is likely to be untransparent, we replace the exact transverse collision operator by a reduced operator[12] satisfying Eq. (25), i.e.,

$$K^T(\delta f^T) \to -\frac{1}{\tau_E}N_1(E)\delta f_E^T + \frac{N_1(E)}{4T\cosh^2(E/2T)}\frac{1}{\tau_E}\times\int_{-\infty}^{+\infty} N_1(E')\delta f_{E'}^T\, dE' , \tag{34}$$

where we assume τ_E to be energy independent. This reduced operator has the same lowest eigenvalue 0 and eigenfunction $\delta f_E^T = \delta\mu/4T\cosh^2(E/2T)$, describing a shift in the chemical potential, as the exact operator. The scattering-in term of the (L) collision operator only leads to corrections of order $(\Delta_0/T)^2$, and hence is neglected.

The kinetic equations have to be completed by Poisson's equation $-\nabla^2\Phi = 4\pi\rho$. As long as we consider wavelengths which are large compared to the Thomas-Fermi screening length $\lambda_{TF} \approx 10^{-8}$ cm, we can put $\rho = 0$ in Eq. (32). This relation then provides a convenient method to simplify Eq. (34) further. We also can assume that the order parameter only varies on a spatial scale which is larger or of the order of magnitude of the Ginzburg-Landau coherence length $\xi(T) = (1/\alpha^{1/2})\xi(0)$. Therefore we have $Dq^2 \lesssim O(\Delta_0^2/T) << \Delta_0$. If we Fourier transform with respect to time $(-i\omega)$ and space (ik), and use the notation

$$1/\tau \equiv -i\omega + 1/\tau_E , \tag{35}$$

we find

$$Q^L\delta f_E^L = \frac{-i\omega}{4T\cosh^2(E/2T)}R_2\delta\Delta^L - \frac{2iDkq(-i\omega)}{4T\cosh^2(E/2T)}\frac{R_2N_2^2}{Q^T}\delta\Delta^T - \frac{2iDkq(1/\tau)}{4T\cosh^2(E/2T)}\frac{R_2N_2N_1}{Q^T}e\Phi \tag{36}$$

$$Q^T\delta f_E^T = \frac{-i\omega}{4T\cosh^2(E/2T)}N_2\delta\Delta^T + \frac{1/\tau}{4T\cosh^2(E/2T)}N_1e\Phi - \frac{2iDkq(-i\omega)}{4T\cosh^2(E/2T)}\frac{R_2^2N_2}{Q^L}\delta\Delta^L ,$$

where

$$Q^L = (1/\tau)N_1 + Dk^2 M^L$$

and

$$Q^T = (1/\tau)N_1 + 2\Delta_0 N_2 + Dk^2 M^T \ . \tag{37}$$

Here, we neglected terms of order $Dk^2 Dq^2 N_2^2 R_2^2 / Q^L Q^T$, which are small except for energies very close to the value of the energy gap, if this exists, and yield only negligible contributions upon integration. When evaluating $2e\Psi^{L(T)}$ and $\delta\rho$ we encounter the following integrals:

$$X_{ij} = \int_{-\infty}^{+\infty} dE \, \frac{1}{\cosh^2(E/2T)} \, \frac{N_i N_j}{Q^T} \ , \quad i,j = 1,2 \ ,$$

$$X_L = \int_{-\infty}^{+\infty} dE \, \frac{1}{\cosh^2(E/2T)} \, \frac{R_2^2}{Q^L} \ , \tag{38}$$

$$Y_{ij} = \int_{-\infty}^{+\infty} dE \, \frac{1}{\cosh^2(E/2T)} \, \frac{R_2^2 N_i N_j}{Q^L Q^T} \ .$$

It is convenient to split off a term from X_{11} as follows:

$$X_{11} = \frac{4T}{1/\tau + Dk^2} + \tilde{X}_{11} \ .$$

With these definitions we find easily from the linearized Ginzburg-Landau equation and the equations for $\delta\rho$, the following equations:

$$\left[\frac{\pi}{8T}\left(-i\omega - \frac{2i\omega}{\pi} X_L\right) + \xi^2(0)k^2 + 2\beta\Delta_0^2\right]\delta\Delta^L - \frac{4i\xi^2(0)kq}{\pi}\frac{1}{\tau} Y_{12} e\Phi + 2i\xi^2(0)kq\left(1 + \frac{2i\omega}{\pi} Y_{22}\right)\delta\Delta^T = 0 \tag{39a}$$

$$\left[\frac{\pi}{8T}\left(-i\omega + \frac{2i\omega}{\pi} X_{22}\right) + \xi^2(0)k^2\right]\delta\Delta^T - \frac{1}{4T} X_{12}\frac{1}{\tau} e\Phi - 2i\xi^2(0)kq\left(1 + \frac{2i\omega}{\pi} Y_{22}\right)\delta\Delta^L = 0 \tag{39b}$$

$$-\frac{i\omega}{4T} X_{12}\delta\Delta^T + \left(\frac{1}{\tau}\frac{1}{4T}\tilde{X}_{11} - \frac{\xi^2(0)k^2}{(\pi/8T)(1/\tau) + \xi^2(0)k^2}\right) e\Phi - \frac{4}{\pi}\xi^2(0)kq\,\omega\, Y_{12}\delta\Delta^L = 0 \ . \tag{39c}$$

Before we can proceed to calculate the various integrals, we have to discuss the form of the spectral quantities. According to Eq. (19a) they are defined by the analytically continued forms $\alpha^{R(A)}$ and $\beta^{R(A)}$. In the limit where no other pair-breaking effect but inelastic electron-phonon scattering is present, the relevant quantities are

$$\alpha^{R(A)}_{E\pm\omega/2} = \frac{-iE \pm \gamma}{[(-iE \pm \gamma)^2 + \Delta_0^2]^{1/2}} \ ,$$

$$\beta^{R(A)}_{E\pm\omega/2} = \frac{\Delta_0}{[(-iE \pm \gamma)^2 + \Delta_0^2]^{1/2}} \ , \tag{40}$$

where we have $\gamma = -i\omega/2 + 1/2\tau_E \equiv 1/2\tau$. The square roots appearing in the retarded (advanced) quantities have a cut in the complex plane extending from $-\Delta_0 \mp i\gamma$ to $\Delta_0 \mp i\gamma$. The sign is chosen such that for large energies E the square root goes to $\mp iE$. On the other hand, in the limits of large energies $|E| \gg \Delta_0$ or of strong pair breaking $\Gamma \gg \Delta_0$, the approximation Eq. (9b) is valid and the spectral quantities assume the simple forms

$$N_1(E) = 1 \ , N_2(E) = \frac{\Delta\gamma}{E^2 + \gamma^2} \ ,$$

$$R_2(E) = \frac{\Delta E}{E^2 + \gamma^2} \ , \tag{41}$$

where γ is in general defined

$$\gamma = -i\omega/2 + \Gamma \tag{42}$$

Although we have expressions for the spectral quantities only in the two limiting cases, this will turn out to be sufficient to determine the required integrals. Furthermore, in the following, we will restrict ourselves to the limit

$$\Delta_0^2/T \ll |1/\tau| = |1/\tau_E - i\omega| \ , \tag{43}$$

which allows us to neglect the diffusive terms $Dk^2 M^{L(T)}$ in $Q^{L(T)}$ as long as we have $k \lesssim 1/\xi(T)$ and allows us to simplify Eq. (39). In the Appendix we will calculate the integrals and also discuss the corrections due to the diffusive terms. The resulting values of the integrals X are

$$\tilde{X}_{11} = -\pi\Delta_0\tau \frac{(2\gamma\tau)^{1/2}}{(1 + \gamma/2\Delta_0^2\tau)^{1/2}} \ ,$$

$$X_{12} = \frac{\pi}{2}\frac{(2\gamma\tau)^{1/2}}{(1 + \gamma/2\Delta_0^2\tau)^{1/2}} \ ,$$

$$X_{22} = \frac{\pi}{2} - \frac{\pi}{4\Delta_0\tau}\frac{(2\gamma\tau)^{1/2}}{(1 + \gamma/2\Delta_0^2\tau)^{1/2}} \ , \tag{44}$$

$$X_L = \pi\Delta_0\tau\{[1 + (\gamma/\Delta_0)^2]^{1/2} - \gamma/\Delta_0\} \ .$$

Note, that these forms are obtained both in the case $\gamma = 1/2\tau$, using Eq. (40), as well as for arbitrary pair-breaking effects in the limit $|\gamma| \gg \Delta_0$, using Eq. (41). The integrals Y_{ij} in the limit of small

$|\gamma| << \Delta_0$ are more complicated. In the case where we have $\gamma = 1/2\tau$ we find

$$Y_{22} \approx \frac{\pi}{4}\tau \frac{1}{1+(1/2\Delta_0\tau)^2} \quad . \tag{45}$$

The corrections, some elliptic integrals, are of order $|(1/\Delta_0\tau)\ln|\Delta_0\tau||$. Of Y_{12} it is sufficient to know that we have $Y_{12} \lesssim O(|\tau \ln|\Delta_0\tau||)$. On the other hand, the limit $|\gamma| >> \Delta_0$ can easily be evaluated. We find

$$Y_{22} = \begin{cases} \dfrac{\pi}{16}\dfrac{\Delta_0^4\tau^2}{\gamma^3} & \text{for } \Delta_0^2 << \left|\dfrac{\gamma}{\tau}\right| , \\[2ex] \dfrac{\pi}{16}\dfrac{\Delta_0^2\tau}{\gamma^2} & \text{for } \Delta_0^2 >> \left|\dfrac{\gamma}{\tau}\right| , \end{cases} \tag{46a}$$

and

$$Y_{12} = \begin{cases} \dfrac{\pi}{8}\dfrac{\Delta_0^3\tau^2}{\gamma^2} & \text{for } \Delta_0^2 << \left|\dfrac{\gamma}{\tau}\right| , \\[2ex] \dfrac{\pi}{4}\dfrac{\Delta_0\tau}{\gamma} & \text{for } \Delta_0^2 >> \left|\dfrac{\gamma}{\tau}\right| . \end{cases} \tag{46b}$$

Since we have $|\omega\tau| \lesssim 1$ and $|\omega/\gamma| \lesssim 1$, we see that the only case where the corrections due to Y_{22} are significant, is the limit where Eq. (45) applies. While $\delta\Delta^L$ and $\delta\Delta^T$ may be of the same order of magnitude, the electric potential $e\Psi$ has to be compared to $(\Delta_0/T)\delta\Delta^T$. This becomes obvious by looking at the approximate equation $\delta j = \delta j_s - \sigma_0\nabla\Phi$. We, therefore, can in all cases neglect the $e\Phi$ term in Eq. (39a). After insertion of $\delta\Delta^L$ from Eq. (39b) into Eq. (39c), we see that in the latter equation the $\delta\Delta^L$ term can be neglected as long as we have $\omega << \Delta_0$. If we define

$$\eta = (2\gamma\tau)^{1/2}(1+\gamma/2\Delta_0^2\tau)^{1/2} \quad ,$$
$$\zeta = [1+(\gamma/\Delta_0)^2]^{1/2} - \frac{\gamma}{\Delta_0} + \frac{1}{2\Delta_0\tau} \tag{47}$$

we obtain from Eq. (39)

$$\left[-i\omega\frac{\pi}{8T}2\Delta_0\tau\zeta - \xi^2(0)\nabla^2 + 2\beta\Delta_0^2\right]\delta\Delta^L + \left[1+\frac{2i\omega}{\pi}Y_{22}\right]2\xi^2(0)q\nabla\delta\Delta^T = 0 \quad , \tag{48a}$$

$$\left[-i\omega\frac{\pi}{8T}\frac{\eta}{2\Delta_0\tau} - \xi^2(0)\nabla^2\right]\delta\Delta^T - \frac{\eta}{\tau}\frac{\pi}{8T}e\Phi - \left[1+\frac{2i\omega}{\pi}Y_{22}\right]2\xi^2(0)q\nabla\delta\Delta^L = 0 \quad , \tag{48b}$$

$$-\frac{i\omega}{\tau}\frac{\pi}{8T}\eta\delta\Delta^T - \left[\frac{\pi\Delta_0}{4T}\eta\frac{1}{\tau} - \frac{8T}{\pi}\xi^2(0)\nabla^2\right]e\Phi = 0 \quad . \tag{48c}$$

In deriving a similar set of equations,[2] one of the authors made an approximation equivalent to keeping only the scattering-out term (24) in K^T. The consequent violation of the conservation law makes the equations of Ref. 2 incorrect. The main physical point made in that work, the existence of a soft mode at the critical moment, is present in the system (48), and is discussed further below.

We note that if the frequency is small, i.e., $\omega << 1/\tau_E$ (which means $\omega Y_{22} << 1$), the coupling of the quasiparticles to the order parameter affects only the coefficients of the time derivatives and of the electric potential in the time-dependent Ginzburg-Landau equation. In this limit we can generalize the elimination of the quasiparticle distribution, as discussed above, to the case where the order parameter Δ_0 varies in space and time, provided that the spatial scale of this variation is large or comparable to $\xi(T)$, and that the time scale is much larger than the inelastic electron-phonon scattering time. In this case we are not even restricted to a linear form of the Ginzburg-Landau equation, whereas the conditions Eq. (43) and $\omega << 1/\tau_E$ guarantee that the quasiparticle distribution is close to a local equilibrium characterized by the local $\Delta(x,t)$ and $\Phi(x,t)$, and a linear form of the Boltzmann equation is sufficient. Finally, since none of the above mentioned arguments depend on the magnitude of q for $q \lesssim q_c$, we can also include a static magnetic-field $H \lesssim H_c$. We thus obtain, as long as the conditions

$$\omega << 1/\tau_E, \quad |\Delta|^2/T << 1/\tau_E \tag{49}$$

are satisfied, the time-dependent Ginzburg-Landau equation

$$\frac{\pi}{8T}\left[\frac{\eta}{2|\Delta|\tau_E}\left[\frac{\partial}{\partial t}+2ie\Phi\right] + \left[2|\Delta|\tau_E\zeta - \frac{\eta}{2|\Delta|\tau_E}\frac{|\dot{\Delta}|}{|\Delta|}\right]\right]\Delta = [\alpha - \beta|\Delta|^2 + \xi^2(0)(\nabla - 2ieA)^2]\Delta \quad , \tag{50}$$

which we combine with the equation for the current j

$$j=(\pi\sigma_0/4eT)\,\mathrm{Im}\Delta^*(\nabla-2ieA)\Delta-\sigma_0\nabla\Phi \quad (51)$$

Note, that the linear form of Eq. (51) follows from Eq. (48c) $\delta\rho=0$, combined with the Ginzburg-Landau equation (48b). In the limit where no other pair-breaking mechanism but inelastic electron-phonon scattering is present ($\gamma=1/2\tau=1/2\tau_E$) Eq. (50) has been recently derived by Kramer and Watts-Tobin.[13] A similar equation with different coefficients describing short filaments, where spatial derivatives are large $Dk^2 \gg 1/\tau_E$, has been presented by Golub.[14] Finally, in the limit of gapless superconductors, $\Gamma/2|\Delta|^2\tau_E \gg 1$, Eq. (50) reduces to the simple time-dependent Ginzburg-Landau equation derived for this limit before.[8]

IV. SUPERCONDUCTORS WITHOUT SUPERCURRENTS

In order to convey some feeling for the various terms in the time-dependent Ginzburg-Landau equation, we start with the discussion of the case where in equilibrium no (or only a small) supercurrent is flowing. Also the equilibrium vector potential is assumed to be small. Most of the results which we will derive have been presented before.[1,15–17]

In the considered limit we find two independent modes involving only changes of the order parameter either in magnitude or in phase. From Eq. (48a) we obtain immediately the time characterizing the decay of a spatially homogeneous nonequilibrium value of $|\Delta|$ to equilibrium[1,12]

$$\tau_R^L=\frac{\pi^3T}{7\zeta(3)\Delta_0}\,\tau_E\left[[1+(\Gamma/\Delta_0)^2]^{1/2}-\frac{\Gamma}{\Delta_0}+\frac{1}{2\Delta_0\tau_E}\right] . \quad (52a)$$

This result is valid for arbitrary pair-breaking $\Gamma \ll T$ as long as we have $1/\tau_R^L \ll 1/\tau_E$. (This condition allowed us to replace $1/\tau$ by $1/\tau_E$, and γ by Γ.) On the other hand, in the limit of negligible electron-phonon scattering but strong pair breaking we have $\Gamma/\tau_E \ll \Delta_0^2$, but $\Gamma \gg \Delta_0$ and find

$$\tau_R^L=\Gamma/\Delta_0^2 \quad (52b)$$

Among others, Schuller and Gray[18] have investigated this relaxation process experimentally. The temperature dependence of their result (applying to $\Gamma \ll \Delta_0$) is in good agreement with Eq. (52).

When examining the transverse equations, we start with an even simpler ideal system: an uncharged superconductor. In this case the time derivative of the electric potential $-i\omega e\Phi$ drops out of Eq. (48b). [Note, however, that $1/\tau_E e\Phi$ was introduced formally replacing the scattering-in term of the reduced collision operator, and has to be eliminated by a correspondingly modified form of Eq. (48c).] At low-frequencies $\omega \ll 1/\tau_E$ there exists a solution[17] with

$$\omega^2=c^2k^2(1-i\omega\tau_R^T) \ , \quad (53)$$

where

$$c=2\Delta_0\xi^2(0) \quad (54)$$

and

$$\tau_R^T=\frac{4T}{\pi\Delta_0}\left(\frac{\tau_E}{2\Gamma}\right)^{1/2}\left(1+\frac{\Gamma}{2\Delta_0^2\tau_E}\right)^{1/2} . \quad (55)$$

The time τ_R^T is the relaxation time characteristic of transverse processes.[1] These include relaxation of a difference between the electrochemical potential of the quasiparticles and the Cooper pairs, which was first investigated by Clarke and Tinkham.[19] In the hydrodynamic limit where we have $\omega\tau_R^T \ll 1$, we thus find a propagating mode with a phase velocity given by Eq. (54) which can be identified to be the velocity of hydrodynamic fourth sound $c^2=c_4^2=(n_s/n)c_1^2$. For high-frequencies $\omega \gg 1/\tau_E$, we can neglect the electron-phonon scattering and find, if we also neglect other pair-breaking effects, a propagating mode $\omega=ck$, where the phase velocity is

$$c=(2\Delta_0 D)^{1/2} \quad (56)$$

The presence of the charge in a real superconductor has the effect that in the low-frequency region there exists only a decaying solution

$$\omega=-i\,\frac{\pi\Delta_0^2}{2T}=-i\,\frac{n_s}{n}\,\frac{1}{\tau_{\mathrm{imp}}} . \quad (57)$$

The fourth sound mode is overdamped, since the constraint of approximate charge neutrality requires that the normal component performs a counteroscillation to compensate the charge wave of the superfluid component. The ohmic losses of the normal component result in a damping given by Eq. (57). At high frequencies we find the same damping mechanism. In this frequency regime, however, propagation is still possible. Neglecting pair-breaking effects we obtain the dispersion relation[15]

$$\omega=-\frac{i}{2}\,\frac{n_s}{n}\,\frac{1}{\tau_{\mathrm{imp}}}\pm\left[c^2k^2-\left(\frac{1}{2}\,\frac{n_s}{n}\,\frac{1}{\tau_{\mathrm{imp}}}\right)^2\right]^{1/2} , \quad (58)$$

where the bare phase-velocity c is the same as in the uncharged case Eq. (56). This result is in good quantitative agreement with the propagating mode found by Carlson and Goldman[11] in most of the range of parameters where the experiments have been performed. The agreement can even be improved[17] if the integrals X_{ij} are calculated including the diffusive terms neglected in Eq. (44) (see the Appendix). This leads to corrections in the resonant frequencies of order $(\Delta/T)\ln(\Delta/\omega)$, and also the damping is increased by an amount of order Dk^2. Pair-breaking effects, due to paramagnetic impurities for example, lead to an additional damping of the mode given by $-i\Gamma/2$ (for small Γ).[16] We, therefore, conclude that as soon as we have $\Gamma \gtrsim \mathrm{Re}\omega$, the mode becomes overdamped.

Finally, we find that under stationary conditions the spatial scale for variations of the electric potential is given by

$$\nabla^2\Phi = (1/D\tau_R^T)\Phi \tag{59}$$

This is of importance in the problem of normal metal-superconductor boundary, where a normal current is converted into a supercurrent.[1] Actually in Eq. (59) a complication results from the fact that within a region of order $\xi(T)$, from the boundary $\Delta_0(x)$, and $\Gamma(x)$, and thus τ_R^T depend on the space coordinate. In many cases, however, we have $(D\tau_R^T)^{1/2} >> \xi(T)$ and hence this region is of minor importance. The resulting characteristic penetration length $(D\tau_R^T)^{1/2}$ for the electric potential, yields a measure for an extra resistance in the superconducting boundary region. An extension of this result to general situations involving finite supercurrents will be given in Sec. V.

V. SUPERCONDUCTORS IN THE CURRENT-CARRYING STATE

In a superconductor carrying a homogeneous current, the two linear modes discussed in Sec. IV are coupled, as follows from the set of Eqs. (48). A nontrivial solution exists only when the corresponding secular determinant vanishes, which yields

$$k^2\left[\left(-i\omega 2\Delta_0\tau\zeta + Dk^2 + \frac{16T}{\pi}\beta\Delta_0^2\right)\times\left(-i\omega\frac{\eta}{2\Delta_0\tau} + \frac{\pi\Delta_0}{4T}\frac{\eta}{\tau} + Dk^2\right) - 4Dq^2\left(1 + i\frac{2\omega}{\pi}Y_{22}\right)^2\left(\frac{\pi\Delta_0}{4T}\frac{\eta}{\tau} + Dk^2\right)\right] = 0 \ . \tag{60}$$

There always exists a solution for $k=0$, which is simply the gauge transformation. From the remaining equation we again find in the high-frequency region $\omega >> \Gamma$ a propagating mode with a modified dispersion relation. (In the considered limit $Y_{22} = +i\pi/4\omega$.) The velocity of sound is reduced to a value given by

$$c^2 = 2\Delta_0 D\left(1 - \frac{Dq^2}{2\Delta_0}\right) , \tag{61}$$

while the damping is increased due to the pair-breaking effect of the supercurrent as follows from Eq. (11):

$$\mathrm{Im}\omega \approx -\left(\frac{\pi\Delta_0^2}{4T} + \frac{1}{4}Dq^2\right) . \tag{62}$$

This effect should be observable in the order-parameter structure factor, which can be investigated in an experiment of the type performed by Carlson and Goldman,[11] if an additional supercurrent in the plane of the junction is imposed.[20]

By adding a normalized driving-force term $(\frac{1}{2}N_0)$ to the right-hand sides of Eqs. (48a) and (48b), we obtain the pairfield susceptibility for arbitrary values of $\omega\tau_E$, i.e., $\chi = \chi^L + \chi^T$,

$$\chi^L = \frac{8T}{2\pi N_0}\frac{1}{Dk^2}\frac{Dk^2(T) - 2iDkq\left(1 + \frac{2i\omega}{\pi}Y_{22}\right)(R)}{(L)(T) - 4Dq^2\left(1 + \frac{2i\omega}{\pi}Y_{22}\right)^2(R)} , \tag{63}$$

$$\chi^T = \frac{8T}{2\pi N_0}\frac{1}{Dk^2}\frac{\left[(L) + 2iDkq\left(1 + \frac{2i\omega}{\pi}Y_{22}\right)\right](R)}{(L)(T) - 4Dq^2\left(1 + \frac{2i\omega}{\pi}Y_{22}\right)^2(R)} ,$$

where

$$(L) = -i\omega 2\Delta_0\tau\zeta + Dk^2 + \frac{16T}{\pi}\beta\Delta_0^2 ,$$

$$(T) = -\frac{i\omega}{2\Delta_0\tau}\eta + \frac{\pi\Delta_0}{4T}\frac{\eta}{\tau} + Dk^2 ,$$

$$(R) = \frac{\pi\Delta_0}{4T}\frac{\eta}{\tau} + Dk^2 .$$

From the pairfield susceptibility, the structure factor $S(k,\omega)$ is obtained by means of the fluctuation dissipation theorem

$$S(k,\omega) = (2T/\omega)\,\mathrm{Im}\chi(k,\omega) .$$

In the low-frequency region $\omega << 1/\tau_E$, Eq. (60) can be solved directly, i.e.,

$$\omega_\pm = -\frac{i}{2}\left[\frac{2\Delta_0\tau_E}{\eta}\left(\frac{1}{\tau_R^T}+Dk^2\right)+\frac{1}{2\Delta_0\tau_E\zeta}\left(\frac{16T}{\pi}\beta\Delta_0^2+Dk^2\right)\right]$$
$$\pm\frac{i}{2}\left[[\cdots]^2-\frac{4}{\eta\zeta}\left(\frac{1}{\tau_R^T}+Dk^2\right)\left(\frac{16T}{\pi}\beta\Delta_0^2+Dk^2-4Dq^2\right)\right]^{1/2} . \qquad (64)$$

In the limit where no supercurrent is flowing ($q=0$), the solution ω_+ reduces to the longitudinal relaxation-rate Eq. (52a), while the solution ω_- is connected with the transverse mode in this limit. In a homogeneous current-carrying state, the order parameter satisfies $\Delta_0^2=\Delta_{00}^2(1-q^2/3q_c^2)$, where $\Delta_{00}^2=\alpha/\beta$ and $q_c=1/3^{1/2}\xi(T)$ is the phase gradient corresponding to the critical supercurrent j_c. We thus find that at j_c, the frequency ω_+ for fluctuations with zero wave vector, k goes to zero. This mode softening shows that at the critical current the superconductor becomes unstable against the formation of long-wavelength fluctuations, which has the result that no current larger than j_c can be stable.[2]

In the most interesting case of a superconductor with energy gap $\Delta_0 >> \Gamma$, the form (64) can be expanded for all positive Dk^2 and arbitrary supercurrents, and we find

$$\omega_+ = -i\frac{1}{2\Delta_0\tau_E}\left(\frac{16T}{\pi}\beta\Delta_0^2+Dk^2-4Dq^2\right) ,$$
$$\omega_- = -i\,\frac{\pi\Delta_0^2}{2T}(1+Dk^2\tau_R^T) . \qquad (65)$$

Comparison with Eq. (52a) shows that the rate of relaxation of the magnitude of the order parameter is slowing down as the supercurrent is increased. This effect should be directly observable in the type of experiment as that of Schuller and Gray[18] or in the order-parameter structure factor as a sharpening of the central peak. The relaxation rate for fluctuations with finite wave vector k also becomes smaller as q goes to q_c however, it remains finite. The second solution ω_- is little affected by the supercurrent in the considered parameter range.

While the softening of the homogeneous mode shows that supercurrents above j_c are unstable, it does not describe the experimental situations at or below j_c. If large supercurrents are imposed on a superconductor, localized phase slip centers appear,[21] where a part of the current is carried by a normal current, which is driven by a gradient of the voltage. A complete analysis of localized solutions is not possible within a linearized theory. However, by an investigation in terms of localized solutions of the form $\sim\exp(-\kappa|x|)$, we can obtain valuable information. Since solutions of this form have an unphysical kink at the origin, they are not acceptable in an infinite domain. But if localized solutions of the nonlinear equations exist, their asymptotic behavior should be described by the exponential solution. Also, solutions of this type are acceptable if we consider boundary problems.

We therefore, extend our analysis to negative values of $k^2=-\kappa^2$, and find three different nondecaying solutions. The first $\omega=0$ solution, which exists for arbitrary currents $j\leq j_c$, has a spatial scale given by

$$\kappa_{sp}^2=2\left(1-\frac{q^2}{q_c^2}\right)/\xi^2(T) . \qquad (66)$$

In the limit $q=q_c$, this is the softening homogeneous solution discussed above. The solution agrees with the asymptotic form of the saddlepoint solution,[22] which is a bounded stationary solution of the nonlinear Ginzburg-Landau equation. Both involve no voltage, the order parameter is locally depressed but the current is kept constant by a corresponding increase of the phase gradient. This agreement of the $\exp(-\kappa|x|)$ solution with an exact solution, suggests that sensible information is also contained in the next two solutions.

The second $\omega=0$ solution of the exponential type has a spatial scale given by

$$\kappa_\Phi^2=1/D\tau_R^T . \qquad (67)$$

This solution involves a voltage. The amplitudes of the corresponding eigensolution satisfy the relation

$$\frac{\pi\Delta_0}{4T}\left(\frac{16T}{\pi}\beta\Delta_0^2-\frac{1}{\tau_R^T}-4Dq^2\right)\delta\Delta^L=2Dq(\nabla e\Phi) ,$$

which means, that the magnitude of the order-parameter $|\Delta|$ follows the local value of the supercurrent, while the amplitude of $\delta\Delta^T$ guarantees that the total current is conserved. At a certain current j^*, where we have $\kappa_\Phi=\kappa_{sp}$, and the two discussed so far coincide, an instability is indicated. For $\Delta_{00}>>\Gamma\approx\frac{1}{2}\tau_E$, this current follows from

$$\left(\frac{q^*}{q_c}\right)^2 = 1 - \frac{\pi^4}{56\zeta(3)}(2/3)^{1/2}\frac{1}{\Delta_{00}\tau_E} \quad . \tag{68}$$

Below j^* the order-parameter $|\Delta|$ is increased in the region where the normal current is parallel to the supercurrent (positive $q\nabla e\Phi$), corresponding to the resulting local decrease of the supercurrent below its asymptotic value. Above j^*, however, $|\Delta|$, is reduced in this region.

In a boundary problem, the last solution describes simultaneous injection of a supercurrent and a normal current. In the limit $q=0$ we recover the normal metal-superconductor boundary problem discussed in Sec. IV. The characteristic decay length $(D\tau_R^T)^{1/2}$, and thus the extra resistance of the boundary, depend only weakly on the magnitude of the supercurrent, in as far as τ_R^T depends on $\Delta(q)$ and on the pair-breaking effect of the supercurrent.

In this connection it is important to notice that Eq. (59) is also valid for stationary situations both in the nonlinear case and in the presence of a supercurrent or a magnetic field. The spatial dependence of $\tau_R^T(x)$, again requires in principle a self-consistent solution of Eq. (59), combined with the Ginzburg-Landau equation for $|\Delta|$ and the equation $J=\text{const}$. However, this dependence can be neglected in most of the range where a normal current flows, if the length $(D\tau_R^T)^{1/2}$ is large compared to the spatial scale of the order parameter. This result is of some importance in the discussion of phase slip centers. Although the present understanding is that phase slip centers evolve in time, the time-averaged potential still seems to be described by Eq. (59). Experiments by Dolan and Jackel[23] show that the spatial dependence of the voltage is indeed given by $\exp(-\kappa_\Phi|x|)$, and they also verify the temperature dependence of τ_R^T. Furthermore, in recent experiments Kadin *et al.*[24] showed that the resistance of a phase slip center depends on the pair-breaking effect of a magnetic field, in a way consistent with the expression Eq. (55) for the relaxation-time τ_R^T. A solution of the nonlinear equations, which involves a voltage and can be constructed by methods proposed by Bezuglyi *et al.*[25] is presently being investigated by the authors.

Finally, we find a nondecaying but oscillating solution of the linear equations (48) of the form $\sim\exp(-\eta|x|)$ at a value of κ, where the square bracket in Eq. (64) vanishes if at the same time the argument of the square root is negative. This requires

$$\kappa_{\rm osc}^2 = \frac{1}{D}\left(\frac{1}{\tau_R^T} + \frac{\eta}{(2\Delta_0\tau_E)^2\zeta}\,\frac{16T}{\pi}\beta\Delta_0^2\right)\Bigg/\left(1+\frac{\eta}{(2\Delta_0\tau_E)^2\zeta}\right) \tag{69}$$

and since in a superconductor with gap $\Delta_0 > \Gamma$ one has $1/\tau_R^T - D\kappa_{\rm osc}^2 < 0$, the supercurrent has to be large

$$q^2 \geq q^{**2} = \frac{1}{4D}\left(\frac{16T}{\pi}\beta\Delta_0^2 - \frac{1}{\tau_R^T}\right)\Bigg/\left(1+\frac{\eta}{(2\Delta_0\tau_E)^2\zeta}\right) \quad . \tag{70}$$

The resonant frequency is then given by

$$\omega = \frac{1}{2\Delta_0\tau_E\zeta}\left\{\left(\frac{16T}{\pi}\beta\Delta_0^2 - \frac{1}{\tau_R^T}\right)\Bigg/\left(1+\frac{\eta}{(2\Delta_0\tau_E)^2\zeta}\right) \times 4D(q^2-q^{**2})\right\}^{1/2} \quad . \tag{71}$$

In the limit $\Delta_0 >> \Gamma \approx 1/2\tau_E$ these expressions reduce to

$$\kappa_{\rm osc} \approx 1/(D\tau_R^T)^{1/2}\ , \quad q^{**} \approx q^*$$

from Eq. (68) and

$$\omega \approx \frac{1}{\tau_E}\,\frac{7\zeta(3)\Delta_{00}(2/3)^{1/2}}{\pi^3 T}\left(\frac{q^2-q^{*2}}{q_c^2}\right)^{1/2} \quad . \tag{72}$$

The eigensolution corresponding to this frequency involves again a finite voltage. We find

$$\frac{\pi\Delta_0}{4T}2q\,\delta\dot{\Delta}^L = \frac{16T}{\pi}\,\frac{\beta\Delta_0^2}{2\Delta_0\tau_E}(\nabla e\Phi) \quad , \tag{73}$$

which shows that $\delta\Delta^L$ and j_n oscillate with a phase difference of $\frac{1}{2}\pi$. The ratio of the amplitudes can be obtained by inserting the frequency. At the onset current $q=q^{**}$, the voltage goes to zero.

VI. TIME SCALE OF INTRINSIC RESISTIVE FLUCTUATIONS

The time-dependent Ginzburg-Landau Eq. (50) differs from the simple form in the property that the time scales for the fluctuations of the magnitude and of the phase of the order parameter are no longer equal. This in turn will modify the time scale for intrinsic resistive fluctuations. It was shown by Langer and Ambegaokar[22] that the dissipation rate of a supercurrent in a thin wire is essentially determined by an exponential activation-factor $e^{-\Delta F/T}$, where ΔF is the difference in free energy between minimum and saddle-point solutions. It can be calculated from the time-independent Ginzburg-Landau theory. In addition a prefactor $\Omega(T)$, which describes the time scale of the fluctuations of the order parameter in the order-parameter space, was calculated by McCumber and Halperin.[26] In this section we will illustrate how the modification of the time-dependent Ginzburg-Landau equation changes their result.

By linearizing Eq. (50) in the limit of low-frequencies $\omega << 1/\tau_E$ we obtain

$$\tau \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \delta\Delta^L \\ \delta\Delta^T \end{pmatrix} = -M_1 \begin{pmatrix} \delta\Delta^L \\ \delta\Delta^T \end{pmatrix} , \qquad (74)$$

where

$$M_1 = -\begin{pmatrix} 1-3\psi_0^2 - \left(\theta_0'^2 - \frac{\partial^2}{\partial x^2}\right) & -\left(\theta_0'' + 2\theta_0' \frac{\partial}{\partial x}\right) \\ \left(\theta_0'' + 2\theta_0' \frac{\partial}{\partial x}\right) & 1-\psi_0^2 - \left(\theta_0'^2 - \frac{\partial^2}{\partial x^2}\right) \end{pmatrix} . \qquad (75)$$

We measured x in units of $\xi(T)$, ψ_0 is the magnitude of the order parameter Δ_0 normalized to $\Delta_{00} = (\alpha/\beta)^{1/2}$, and $\tau = (\pi/8T)(1/\alpha)$. These equations are a generalization of Eq. (48) in as far as they describe deviations of the order parameter from general space-dependent stationary solutions

$$\psi^{eq}(x) = \psi_0(x) e^{-i\theta_0(x)} ,$$

including in particular the saddle-point solution,[22] which is given by following expressions:

$$\psi_0^2(x) = [(1-3K^2)\tanh^2 y + 2K^2] , \qquad (76)$$

where

$$y = x[(1-3K^2)/2]^{1/2} , \quad K(1-K^2) = J$$

and the current J is normalized such that we have $0 \leq |J| \leq (\frac{4}{27})^{1/2}$. We do not specify the phase, but merely mention

$$J = -\psi_0^2(x)\theta_0' . \qquad (77)$$

In Eq. (74) we neglected the electric potential. It was shown by McCumber and Halperin that this is allowed near the minimum and the saddle point. The operator M_1 differs from their operator only by a unitary transformation. On the left-hand side, however, the factors $\nu_1 = 2\Delta_0\tau_E\zeta$ and $\nu_2 = (1/2\Delta_0\tau_E)\eta$ change the physical picture. If these factors are unity, the time-dependent Ginzburg-Landau equation describes an isotropic viscous motion of the order parameter in the order-parameter space under the influence of a potential given by the Ginzburg-Landau free energy. With ν_1 and ν_2 different from unity, the situation rather corresponds to a viscous motion with an anisotropic viscosity. However, by changing to the new variables

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} (\nu_1)^{1/2} & 0 \\ 0 & (\nu_2)^{1/2} \end{pmatrix} \begin{pmatrix} \delta\Delta^L \\ \delta\Delta^T \end{pmatrix} , \qquad (78)$$

we obtain

$$\tau \frac{\partial}{\partial t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -M_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \qquad (79)$$

where

$$M_2 = -\begin{pmatrix} \frac{1}{(\nu_1)^{1/2}} \left[1-3\psi_0^2 - \left(\theta_0'^2 - \frac{\partial^2}{\partial x^2}\right)\right] \frac{1}{(\nu_1)^{1/2}} & -\frac{1}{(\nu_1)^{1/2}} \left(\theta_0'' + 2\theta_0' \frac{\partial}{\partial x}\right) \frac{1}{(\nu_2)^{1/2}} \\ \frac{1}{(\nu_2)^{1/2}} \left(\theta_0'' + 2\theta_0' \frac{\partial}{\partial x}\right) \frac{1}{(\nu_1)^{1/2}} & \frac{1}{(\nu_2)^{1/2}} \left[1-\psi_0^2 - \left(\theta_0'^2 - \frac{\partial^2}{\partial x^2}\right)\right] \frac{1}{(\nu_2)^{1/2}} \end{pmatrix} , \qquad (80)$$

is a Hermitian operator, which allows the interpretation and procedure of calculating the transition rates as used by McCumber and Halperin.

The following results are obtained immediately:

(i) Both for expansion around a minimum (m) as well as a saddle point(s) there exists one solution with eigenvalue zero, corresponding to a constant phase shift.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{m_1 \text{ or } s_2} = \text{const} \begin{pmatrix} 0 \\ (\nu_2)^{1/2}\psi_0(x) \end{pmatrix} \qquad (81)$$

(ii) There exists a second eigenvalue zero for expansion around the saddle-point solution. In the limit where we have $\nu_1 = \nu_2 = 1$, this solution describes a translation of the saddle-point solution in x space

$$\begin{pmatrix} \delta\Delta^L \\ \delta\Delta^T \end{pmatrix}_{s_3} = \delta x \begin{pmatrix} \psi_0' \\ \psi_0\theta_0' \end{pmatrix} = \text{const} \frac{1}{\psi_0(x)} \begin{pmatrix} \frac{(1-3K^2)^{3/2}}{2^{1/2}} \frac{\tanh y}{\cosh^2 y} \\ -K(1-K^2) \end{pmatrix} . \qquad (82)$$

This connection allows one to relate the integration over the phase space of the corresponding fluctations to the physical length of the system. The corresponding solution in the general case $(\nu_1 \neq \nu_2 \neq 1)$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{s_3} = \text{const} \frac{1}{\psi_0(x)} \times \begin{pmatrix} (\nu_1)^{1/2} \frac{(1-3K^2)^{3/2}}{2^{1/2}} \frac{\tanh y}{\cosh^2 y} \\ -(\nu_2)^{1/2} K(1-K^2) \end{pmatrix} , \qquad (83)$$

still has the eigenvalue zero. In principle it should be possible to relate Eq. (83) to a solution of an operator, the linear form of which is given by Eq. (80). However, only in the limit of vanishing or maximum current can this be easily done.

To proceed further we make two rather restrictive simplifications. First, although ν_1 and ν_2 are in general functions of the space variable and are different for minimum and saddle-point solution, we neglect this dependence and treat ν_1 and ν_2 as constant parameters. In spite of this approximation, we will be able to see how the results change qualitatively. Second, even with this simplification we are able to find the eigenvalues of M_2 only in the limit $J \to 0$. In this limit M_2 becomes diagonal, the eigenvalues are those obtained by McCumber and Halperin multiplied by $1/\nu_1$ for magnitude fluctuations, and multiplied by $1/\nu_2$ for phase fluctuations.

Again there exists a negative eigenvalue at the saddle point, the eigensolution for $J \to 0$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{s_1} = \text{const} \begin{bmatrix} 0 \\ 1/\cosh(x/2^{1/2}) \end{bmatrix} , \tag{84}$$

and thus the eigenvalue is

$$\epsilon_{s_1}(\nu_1, \nu_2 \neq 1) = \frac{1}{\nu_2} \epsilon_{s_1}(\nu_1 = \nu_2 = 1) . \tag{85}$$

Furthermore, the ratio of the products of all nonzero eigenvalues at the minimum, and the saddle point enters the expression for the transition rate

$$\left[\prod_{i;\epsilon_{m_i} \neq 0} \epsilon_{m_i} \Big/ \prod_{j;\epsilon_{s_j} \neq 0} \epsilon_{s_j} \right]^{1/2} .$$

If all eigenvalues were different from zero, the factor ν_1 and ν_2 in numerator and denominator would just cancel. However, since the zero eigenvalue fluctuation corrresponding to the shift of the saddle point involves magnitude fluctuations, there is one factor $1/\nu_1$ missing in the denominator.

The remaining coefficients are unmodified. We thus find that the transition rate [see Eq. (2.36) of Ref. 26] obtains a factor

$$\begin{aligned} \Omega(T;\nu_1, \nu_2 \neq 1) &= \frac{1}{\nu_2(\nu_1)^{1/2}} \Omega(T;\nu_1 = \nu_2 = 1) \\ &= \frac{(2\Delta_0 \tau_E)^{1/2}}{\eta(\zeta)^{1/2}} \Omega(T) , \end{aligned} \tag{86}$$

where $\Omega(T)$ has been calculated by McCumber and Halperin. As compared to their result we find an increase of the transition rate and consequently of the resistance by a factor $(2\Delta_0\tau_E)^{1/2}/\eta(\zeta)^{1/2}$.

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APPENDIX

In the following we will illustrate the calculation of the integrals X_{ij}, X_L, and Y_{ij} for temperatures close to T_c, considering different physical limits. As pointed out above we know analytic expressions for the spectral quantities $N_i(E)$ and $R_2(E)$ only if we have either $\Gamma = \frac{1}{2}\tau_E$, i.e., no pair-breaking mechanism but inelastic electron-phonon scattering is present, or if we have $|E| + \Gamma >> \Delta_0$, i.e., the pair breaking is strong or the energy is large. Nevertheless, we will be able to give general results for these integrals.

We first assume that the inequality (43) is satisfied which, as we will show later, allows us to neglect the diffusive term in the denominator of Q^T and Q^L. As an example we investigate X_{12} in the limit $\Gamma = \frac{1}{2}\tau_E$ (i.e., γ reduces to $\gamma = \frac{1}{2}\tau = \frac{1}{2}\tau_E - \frac{1}{2}i\omega$),

$$X_{12} \approx \int_{-\infty}^{\infty} \frac{dE}{\cosh^2(E/2T)} \frac{N_1(E)N_2(E)}{(1/\tau)N_1(E) + 2\Delta_0 N_2(E)} , \tag{A1}$$

where N_1 and N_2 follow from Eq. (40). After some elementary transformations we obtain

$$\begin{aligned} X_{12} = +\frac{i}{4} \int_{-\infty}^{+\infty} dE \, \frac{1}{E} \{ &[(-iE - 1/2\tau)^2 + \Delta_0^2]^{-1/2} \\ &- [(-iE - 1/2\tau)^2 + \Delta_0^2]^{-1/2} \} \end{aligned} \tag{A2}$$

The convergence of the argument of the integral allowed us to put $\cosh^2(E/2T) = 1$ to lowest order in Δ/T. Since $1/\tau$ has a positive real part, the second square root is analytic in the upper-half plane. We close the contour of integration in the upper-half plane and contract it to follow the cut of the first square root, which is real and positive below the cut and negative above. After the substitution $E - i/2\tau \to x$ we obtain

$$X_{12} = \frac{i}{2} \int_{-\Delta_0}^{+\Delta_0} dx \, \frac{1}{(\Delta_0^2 - x^2)^{1/2}} \frac{1}{x + i/2\tau} \tag{A3}$$

and thus we have

$$X_{12} = \frac{1}{2}\pi[1 + (1/2\Delta_0\tau)^2]^{-1/2} . \tag{A4}$$

On the other hand, in the limit $\gamma >> \Delta_0$, we can use the spectral quantities in the form given in Eq. (41). As long as we have γ, $\Delta(\gamma\tau)^{1/2} << T$ we can again put $\cosh^2(E/2T)=1$ and easily obtain

$$X_{12}=\frac{\pi}{2}\frac{(2\gamma\tau)^{1/2}}{(1+\gamma/2\Delta_0^2\tau)^{1/2}} \quad . \tag{A5}$$

The latter result includes Eq. (A4). Obviously the detailed form of the spectral quantities for $|E|\lesssim\Delta_0$, which differs in the two treatments above, is of minor importance in the calculation of the integral. We therefore are allowed to use Eq. (A5) for general values of Γ.

Similarly we calculate the remaining integrals. The results are listed in Eqs. (44)–(46).

There are cases where the inequality Eq. (43) is not strictly satisfied. An example is provided by the resonance frequency of the Carlson Goldman mode[11] in a certain range of parameters. In the experiments of Ref. 11, which were performed in aluminum, the inelastic electron-phonon scattering as well as other pair-breaking mechanisms were small. In Ref. 17 the integrals X_{ij},X_L have been calculated exactly including the diffusive terms in the limit $\Gamma=0$ corresponding to the experimental situation. The results are combinations of elliptic integrals depending on ω and Dk^2. By expanding these results or by directly expanding the integrals, we find that to lowest order the corrections due to the diffusive terms are of order $(Dk^2/\Delta_0)\ln(\Delta_0/\omega)$. Although Dk^2/Δ_0 is small in the experiments, the correction of the resonant frequency is of order $(\Delta_0/T)\ln(\Delta_0/\omega)$, since when calculating the resonant frequency as shown in Sec. IV, the leading terms in ω/Δ_0 cancel. This accounts for the deviation between experimental results and Eq. (58) in the case of small frequencies. Also in the absence of any inelastic scattering and pairbreaking, it is only the diffusive term in X_L which is responsible for the central peak in the order parameter structure factor.[11,17]

On the other hand, in the presence of inelastic electron-phonon scattering or pair breaking the corrections to the integrals are of order $(Dk^2/\Delta_0)\ln|\Delta_0/\gamma|$ for $|\gamma|<\Delta_0$, and are of order Dk^2/Δ_0 for $|\gamma|>>\Delta_0$. Thus, as long as condition (43) is satisfied the diffusive terms can be neglected.

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