Tomonaga fermions and the Dirac equation in three dimensions

A. Luther
Nordita, Copenhagen, Denmark
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A boson representation for fermion operators in three-dimensional quantum systems is constructed. It solves the bosonization problem for the electron gas, posed by Tomonaga, and has a natural extension to the case of Dirac fermions. Two bosons are required for the two fermions of the Dirac equation. Several applications of these results are suggested.

I. INTRODUCTION

Fascination with the construction of fermion states from a superposition of boson states dates back to the early years of quantum mechanics. The pioneering attempts by Bloch,[1] which were later extended and given an intuitive foundation by Tomonaga,[2] called attention to the conceptual simplicity that results if all excitations can be described within a boson language. More recently, these ideas and their logical continuations have had a profound impact on problems in two dimensions.

In many model field theories in two dimensions, the distinction between fermion and boson representations becomes largely a matter of choice. No fundamental difference between them exists. Thus the Hamiltonian of the Thirring model,[3] or Luttinger model[4] can be expressed in terms of fermion fields and currents, or, equivalently, in canonical boson fields.[5,6] These equivalences have also been exploited in connection with the sine-Gordon equation and massive Thirring model,[7] field-theory equivalence of the eight vertex model,[8] and SU(2) generalizations of these models.[9]

Beyond finding interesting relations between models, however, the bosonization of Fermi fields has been important as a tool in finding solutions[7-9] for these problems in two dimensions. In higher dimensions, it is generally believed that bosonization of fermion fields is substantially different or perhaps not possible at all. In answer to this question, I report the discovery of a boson operator satisfying the Dirac equation and obeying anticommutation relations. These properties are used to demonstrate that the n-point functions are given correctly in this new representation.

The ideas behind this representation are illustrated in the Luttinger model of interacting fermions in two space-time dimensions. For simplicity, consider the free-particle case (extensions to include interactions are straightforward[10]). The Hamiltonian is given by[5]

$$\mathcal{H} = -iv_F \int_0^L dx \left(\psi_1^\dagger(x)\frac{\partial}{\partial x}\psi_1(x) - \psi_2^\dagger(x)\frac{\partial}{\partial x}\psi_2(x)\right)$$
$$= 2\pi v_F\, L^{-1}\sum_{k>0}\left[\rho_1(k)\rho_1(-k) + \rho_2(-k)\rho_2(k)\right], \qquad (1.1)$$

where v_F is the Fermi velocity, L is the length of the Fermi system, $\psi_1(x)$ represents fermions with positive group velocity, and $\psi_2(x)$ represents the negative. The product of density operators $\rho_1(x) = :\psi_1^\dagger(x)\psi_1(x):$ and $\rho_2(x) = :\psi_2^\dagger(x)\psi_2(x):$, suitably normal ordered, describes the free-particle kinetic energy in boson operators.[5]

The Fermi field $\psi_j(x)$ is defined in the canonical representation, with $j=1,2$, by

$$\psi_j(x) = L^{-1/2}\sum_k e^{ikx}a_{j,k}\,,$$
$$(a_{j,k}, a^\dagger_{j',k'}) = \delta_{k,k'}\delta_{j,j'}\,, \qquad (1.2)$$

where k represents a discrete wave number, $2\pi L^{-1}$ times an integer. The density operators satisfy boson operator algebra

$$\rho_j(x) = L^{-1}\sum_p \rho_j(p)e^{-ipx}$$
$$[\rho_j(-p), \rho_j(p')] = -pL(2\pi)^{-1}(-)^j\delta_{p,p'}\delta_{j,j'}\,, \qquad (1.3)$$

and the boson operator that represents the Fermi operator of Eq. (1.2) is given by[10]

$$\psi_j(x) = (2\pi\alpha)^{-1/2}\exp[\phi_j(x) - i(-)^j k_F x]$$
$$\phi_j(x) = (-)^{j+1}2\pi L^{-1}\sum_{k\neq 0} k^{-1}\rho_j(k)e^{-ikx-\alpha|k|/2}\,, \qquad (1.4)$$

where α is a short-distance cutoff parameter, which is taken to zero after calculations, and k_F is the Fermi momentum. In this representation, the correlation functions are given by

$$\langle\psi_j^\dagger(x,t)\psi_j(x_j't')\rangle = (2\pi\alpha)^{-1}\exp\langle\phi_j^2 - \phi_j(x_0t)\phi_j(x_j't')\rangle\,, \qquad (1.5)$$
$$\langle\phi_j^2 - \phi_j(x,t)\phi_j(x_j't')\rangle\rangle = \ln\left[\frac{i\alpha}{(-)^j(x'-x)-(t-t')+i\alpha}\right],$$

where the overall phase factor $ik_F(x-x')$ has been suppressed. The first equation follows from the Baker-Hausdorf formula, the second from using Eq. (1.4) in Eq. (1.1). It can be verified that this boson representation also satisfies the equal-time anticommutation relations, since $x-x'\to x'-x$ results in the appearance of $\ln(-1)=i\pi$ in the exponent. These operators have been extensively used in problems of interacting fermions and their generalizations to models with higher symmetry. While there are other choices for this operator representation,[11] the particular choice here makes explicit the relation to density operators and short-distance cutoff α.

The logarithm of Eq. (1.5) is essential to construct a boson operator that represents the Fermi field. The representation chosen below for four space-time dimensions also makes use of a logarithmic correlation function in an exponent, which appears to be the key to constructing anticommutator algebra.

This paper begins with the Tomonaga problem in three dimensions, that is, the boson representation for fermions in a Fermi sea. This problem provides a bridge between one dimension, where everything is known, to the three-dimensional case of interest. After solving this problem, the Tomonaga problem, the extension for the Dirac equation will be clear. Finally, the paper closes with a discussion of possible applications of this operator representation in statistical mechanics and field theory.

II. TOMONAGA FERMIONS ON THE FERMI SEA

To begin the discussion of Tomonaga fermions, consider the filled Fermi sea in three dimensions at zero temperature, characterized by a Fermi momentum k_F ($\hbar=1$) and Fermi velocity v_F. The Hamiltonian for the problem is

$$\mathcal{H}=v_F\sum_{\vec{k},\sigma}(|\vec{k}|-k_F)a^\dagger_{\vec{k},\sigma}a_{\vec{k}\sigma}\,, \tag{2.1}$$

where the operators $a_{\vec{k},\sigma}$ satisfy the fermion anticommutation relations

$$(a_{\vec{k},\sigma},a^\dagger_{\vec{k}',\sigma'})_+=\delta_{\vec{k},\vec{k}'}\delta_{\sigma,\sigma'}\,,$$
$$(a_{\vec{k},\sigma},a_{\vec{k}',\sigma'})_+=0\,. \tag{2.2}$$

Field operators are used to define the n-particle correlation functions, and the goal of this section is to construct a boson representation for these field operators. In the conventional picture, the operator is defined by

$$\psi_\sigma(\vec{x},t)=V^{-1/2}\sum_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}a_{\vec{k},\sigma}(t)\,, \tag{2.3}$$

with V as the volume of a box with periodic boundary conditions. The correlation function of interest is given by

$$\langle 0|\psi^\dagger_\sigma(\vec{x},t)\psi_{\sigma'}(\vec{x}',t')|0\rangle$$
$$=\delta_{\sigma,\sigma'}V^{-1}\sum_{\vec{k}}n_{\vec{k},\sigma}e^{-i\vec{k}\cdot\vec{R}+ik\tau v_F-iv_Fk_F\tau}\,, \tag{2.4}$$

where $\vec{R}=\vec{x}-\vec{x}'$, $\tau=t-t'$,

and $|0\rangle$ is the ground state. This correlation function takes a simpler form in spherical coordinates. Defining

$$k_F^2(8\pi^3)^{-1}C(\vec{R},\tau)=\langle 0|\psi^\dagger_\sigma(\vec{x},t)\psi_\sigma(\vec{x}',t')|0\rangle\,,$$

the result can be written

$$C(\vec{R},\tau)=k_F^{-2}\int_0^{k_F}k^2dk\int d\Omega\,e^{-i\vec{k}\cdot\vec{R}+i(k-k_F)v_F\tau}\,. \tag{2.5}$$

For large space-time separations, the integral can be expanded about the upper limit, giving the result

$$C(\vec{R},\tau)=i\int\frac{d\Omega\exp(-i\vec{k}_F\cdot\vec{R})}{\hat{k}_F\cdot\vec{R}-v_F\tau+i\alpha} \tag{2.6}$$

where $\hat{k}_F$ is the unit vector at the angle Ω on the Fermi surface. The quantity α appears because, in evaluating the integral in Eq. (2.5), an exponential cutoff has been used, rather than the $\vec{k}=0$ cutoff in Eq. (2.5). The remainder term, indicated by $O(k_F^{-1})$, has contributions of order α^{-1} appearing as well. However, these are not important, for we will be interested in the limit $\alpha^{-1}=k_F\to\infty$.

This limit is perfectly analogous to the continuum limit of field theories on a lattice, for k_F is equivalent to the inverse lattice constant, and separations are measured in units of the lattice constant. To be absolutely precise it should be realized that the spectrum of Eq. (2.1) is not that of Eq. (2.6), since the latter includes states to minus infinity, rather than to $E_{\vec{k}=0}$, as in Eq. (2.1). The difference should be understood with the Luttinger model as comparison: consider the fermion spectrum for fixed angle Ω, with the momentum label taken to run from $-\infty$ to $+\infty$. For any angle Ω, introduce two fields, one describing states moving to the "right," the other to the "left." The Hamiltonian for this situation is

$$\mathcal{H}_F=v_F\int d\Omega\sum_k(k-k_F)a^\dagger_{1k\Omega}a_{1k\Omega}-(k+k_F)a^\dagger_{2k\Omega}a_{2k\Omega}\,, \tag{2.7}$$

where k is now a one-dimensional label, the radial momentum variable, and the Fermi operators satisfy the anticommutation relations

$$(a_{jk\Omega}, a^\dagger_{j'k'\Omega'})_+ = \delta_{k,k'}\delta_{\Omega,\Omega'}\delta_{j,j'},$$
$$(a_{jk\Omega}, a_{j'k'\Omega'})_+ = 0, \tag{2.8}$$

the angle Ω runs over the entire unit sphere, and $\delta_{\Omega,\Omega'}$ is the Kronecker symbol. This model differs from the original model of Eq. (2.1) through the addition of negative-energy states far below the Fermi energy. Since we are interested in the contributions from the region near the Fermi energy, these additional states will turn out to be unimportant. In addition, there is an apparent difference in spin degeneracy. This is, however, only notational. Since Ω runs over the entire unit sphere, and there is a twofold degeneracy (the "one" and "two" Fermi fields), it is apparent that the "one" fermion at angle Ω is equivalent to spin "plus," while the "two" fermion at angle $\Omega+\pi$ is equivalent to spin "minus" at angle Ω.

The corresponding fermion field operators, $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$, are given by the same definition as Eq. (2.3), and can be used to calculate the fermion correlation functions in the asymptotic region. The result is found to be

$$8\pi^3 k_F^{-2}\langle\psi_1^\dagger(\vec{x},t)\psi_1(\vec{x}',t')\rangle = i\int \frac{d\Omega\,\exp(-i\vec{k}_F\cdot\vec{R})}{\hat{k}_F\cdot\vec{R}-u_F\tau+i\alpha} + O(k_F^{-1}), \tag{2.9}$$

the same as Eq. (2.6). The remainder, $O(k_F^{-1})$, in Eq. (2.9) is not equal, in general, to that of Eq. (2.6), and the equivalence between the two pictures is only to the leading term.

The fermion-boson duality now becomes straightforward to define. The definition of the Fermi field, specified in Eq. (2.8), is, in fact, a projection from the three-dimensional world onto the radial equation of Eq. (2.7). Since the radial equation involves only a single quantum variable k, it is possible immediately to write down the corresponding boson representation. Furthermore, it is also clear that the Hamiltonian of Eq. (2.7) is, in each angle, a radial one-dimensional equation. It can therefore be written in a simple form involving only boson operators,

$$\mathcal{H}_B = 2\pi v_F L^{-1}\sum_{k>0}\int d\Omega[\rho_1(k,\Omega)\rho_1(-k,\Omega) + \rho_2(-k,\Omega)\rho_2(k,\Omega)], \tag{2.10}$$

where the operators $\rho_j(k,\Omega)$ are defined to satisfy the commutation relations

$$[\rho_j(k,\Omega),\rho_{j'}(-k',\Omega')] = (-)^j\delta_{j,j'}\delta_{\Omega,\Omega'}kL(2\pi)^{-1}\delta_{k,k'}, \tag{2.11}$$

and the summation over k is one dimensional. Using these boson operators, it is helpful to define a further set of operators, which are analogous to the radial "phase" operators of the one-dimensional problem, Eq. (1.4). These are given by

$$\Phi_j(\Omega,\hat{k}_F\cdot\vec{x}) = (-)^{j+1}(2\pi L^{-1})\sum_{k\neq 0} k^{-1}\rho_j(k,\Omega) \times e^{-ik(\hat{k}_F\cdot\vec{x})-|k|\alpha/2}. \tag{2.12}$$

These steps contain the essence of the simplification of the three-dimensional problem, and the operators of Eq. (2.12) are of great importance. The proof that Eq. (2.10) represents the fermion problem of Eq. (2.7) is reduced to the original proof for the Luttinger model. The definition of the operators ρ_1 and ρ_2 at each angle Ω is a natural generalization and can obviously be related to results known in the random-phase approximation (RPA) picture. However, it should be emphasized that these operators are not identical to the RPA density operators, for these create single fermion particle-hole states at a fixed angle. The relation to the RPA bosonization will be discussed later, after the complete bosonization of Fermi operators has been formulated.

The bosonization of the Fermi field operator can be split into several steps. The first step invents the operator that gives the correct single fermion correlation function. The second is the construction of the operator anticommutation relations, while the third deals with the proof that multifermion correlation functions are given correctly.

Proceeding with the first step, consider the operators ψ_1 and ψ_2, defined in each angle Ω:

$$\psi_j(\Omega,\hat{k}_F\cdot\vec{x}) = k_F(8\pi^3\alpha)^{-1/2} \times \exp[\phi_j(\Omega,\hat{k}_F\cdot\vec{x}) - i(-)^j k_F x]. \tag{2.13}$$

It is necessary to specify the angular label Ω more precisely. Consider placing a mesh on the unit sphere, discretizing the angular variables, with a uniform distribution of N points. To each point a label Ω is attached. It is not important in what sequence these points are labeled, and, for definiteness, a numbering starting at the south pole spiralling longitude by longitude up to the north pole can be used. The angular integrations are then defined to be

$$\int d\Omega = \lim_{N\to\infty}\frac{1}{N}\sum_{\Omega}, \tag{2.14}$$

where the sum is over the mesh.

Fermion fields $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$ are defined from the discrete operators, in Eq. (2.13), by the relations

$$\psi_j(\vec{x}) = N^{-1/2}\sum_{\Omega}\psi_j(\Omega,\hat{k}_F\cdot\vec{x}), \tag{2.15}$$

where the sum is over the mesh points, and the eventual limit $N\to\infty$ is understood. Calculation of the single fermion functions follows from the Baker-Hausdorf formula, for, using Eqs. (2.13) and (2.10), the problem is reduced to a free-boson problem. The equations that summarize this calculation are

$$\langle\psi_j^\dagger(\vec{x},t)\psi_j(\vec{x}',t')\rangle=k_F^2(8\pi^3\alpha)^{-1}N^{-1}\sum_{\Omega,\Omega'}\langle\exp[-\phi_j(\Omega,\hat{k}_F\cdot\vec{x},t)]\exp[\phi_j(\Omega',\hat{k}_F'\cdot\vec{x}',t')]\rangle, \tag{2.16}$$

where the brackets indicate expectation value in Eq. (2.10), and the average of the exponential operators on the right-hand side of Eq. (2.16) is given by

$$\langle\exp[-\phi_j(\Omega,\hat{k}_F\cdot\vec{x},t)]\exp[\phi_j(\Omega',\hat{k}_F'\cdot\vec{x}',t')]\rangle=\exp\langle\phi_j^2-\phi_j(\Omega,\hat{k}_F\cdot\vec{x},t)\phi_j(\Omega',\hat{k}_F'\cdot\vec{x}',t')\rangle, \tag{2.17}$$

where the ϕ_j^2 expectation value depends on none of its arguments Ω, $\vec{x}$, or t. The time dependence of these boson operators is given by Eq. (2.10) [or Eq. (2.7), since they are equivalent] and is a simple

$$\rho_1(\Omega,k,t)=\rho_1(\Omega,k)e^{-ikt}.$$

The complete expectation value in Eq. (2.17) is evaluated by standard procedures and is

$$\langle\phi_j^2-\phi_j(\Omega,\hat{k}_F\cdot\vec{x},t)\phi_j(\Omega',\hat{k}_F'\cdot\vec{x}',t')\rangle=2\pi L^{-1}\sum_{k>0}k^{-1}e^{-\alpha k}(\delta_{\Omega,\Omega'}e^{ik(\hat{k}_F\cdot\vec{R}-\tau v_F)}-1)$$
$$=\begin{cases}\ln\left(\dfrac{i\alpha}{\pm\hat{k}_F\cdot\vec{R}-\tau v_F+i\alpha}\right), & \Omega=\Omega'\\ -\infty, & \Omega\neq\Omega',\end{cases} \tag{2.18}$$

where the upper (lower) sign is for $j=1$ ($j=2$) with $\vec{R}=x-x'$, $\tau=t-t'$. The divergence for $\Omega\neq\Omega'$ follows from the infrared singularity at $k=0$, in the infinite volume ($L^3\to\infty$) limit. This property is extremely crucial for the bosonization and is already familiar in the one-dimensional case.

Collecting factors, this integral leads to the following result for the Fermi correlation functions

$$C(\vec{R},\tau)=N^{-1}\sum_{\Omega,\Omega'}\delta_{\Omega,\Omega'}\frac{ie^{-i\vec{k}_F\cdot\vec{R}}}{\hat{k}_F\cdot\vec{R}-\tau v_F+i\alpha}, \tag{2.19}$$

which agrees with Eq. (2.6) in the $N\to\infty$ limit. A similar result follows for the $\langle\psi_2^\dagger\psi_2\rangle$ expectation value. This completes the boson representation of the fermion correlation functions.

There remains the problem of locality, the anticommutation property, which in the original Fermi representation states that

$$[\psi_j(\vec{x}),\psi_{j'}^\dagger(\vec{x}')]_+=\delta^{(3)}(\vec{x}-\vec{x}')\delta_{j,j'},$$
$$[\psi_j(\vec{x}),\psi_j(\vec{x}')]_+=0. \tag{2.20}$$

It turns out, as might be expected on intuitive grounds, that the boson representation discussed here will not reproduce the $\delta^{(3)}(x-x)$, since that clearly involves excitations well away from the Fermi level. [In Sec. III, it will be shown how to construct the $\delta^{(3)}(x-x')$ for the boson representation of Dirac fermions.] However, the anticommutators are important to reproduce even in the Tomonaga case, so that the multi-Fermi correlation functions will be given correctly. Obviously, the boson representation of Eq. (2.13) is inadequate, because the operators commute.

It is well known in one-dimensional systems that the conversion of commutation to anticommutation relations can be effected with the Jordan-Wigner transformation. A useful extension of this to higher dimensions is needed here.

This extension is given by the definition

$$\psi_j(\vec{x})=N^{-1/2}\sum_{\Omega}\psi_j(\Omega,\hat{k}_F\cdot\vec{x})O_\Omega, \tag{2.21}$$

where O_Ω is an ordering operator, defined by $O_\Omega=e^{iJ_\Omega}$, with

$$J_\Omega=\frac{\pi}{2}\sum_{\Omega'=1}^{\Omega-1}[\rho_1(\Omega)+\rho_2(\Omega)]. \tag{2.22}$$

The operator $\rho_j(\Omega)$ is the limit $k\to0$ of the operator $\rho_j(\Omega,k)$, and this satisfies the commutation relation

$$[\rho_1(\Omega'),\phi_1(\Omega,\hat{k}_F\cdot\vec{x})]=\lim_{k'\to 0}2\pi L^{-1}\times\sum_{k\neq 0}[\rho_1(k',\Omega'),\rho_1(k,\Omega)]k^{-1}\times e^{-ik(\hat{k}_F\cdot\vec{x})}=\delta_{\Omega,\Omega'}\ , \quad (2.23)$$

and this result is independent of the sign of k' in the limit $k'\to 0$. The operator O_Ω has the following important properties:

$$[O_\Omega, O_{\Omega'}]=[O_\Omega,\mathcal{H}]=0$$

$$O_\Omega O_\Omega^\dagger=1$$

$$O_\Omega\psi_j(\Omega,\hat{k}_F\cdot\vec{x})O_{\Omega'}^\dagger=\begin{cases}-\psi_j(\Omega,\hat{k}_F\cdot\vec{x})\ , & \Omega'>\Omega\\ +\psi_j(\Omega,\hat{k}_F\cdot\vec{x})\ , & \Omega'\leq\Omega\end{cases} \quad (2.24)$$

As an example of the anticommutation relations, consider the operator relations

$$[\psi_1(\Omega,\hat{k}_F\cdot\vec{x})O_\Omega,\psi_1(\Omega',\hat{k}'_F\cdot\vec{x}')O_{\Omega'}]_+=0\ ,$$

which vanishes (for fixed finite cutoff), since the ordering operator O_Ω inserts a minus sign for $\Omega>\Omega'$ when pulled to the right or for $\Omega<\Omega'$, $O_{\Omega'}$ inserts the minus sign when it is pulled to the right. For $\Omega=\Omega'$, the operators anticommute directly, as seen by exponentiating the product. (This case, $\Omega=\Omega'$, is of course identical to the one-dimensional problem.) This completes the construction of the anticommuting operators using the generalized Jordan-Wigner transformation.

Moving on, consider the calculation of n-point correlation functions using this operator construction, as an example of these anticommutation properties. An (unordered) four-fermion function representing either a ψ_1 or a ψ_2 field is given by

$$\langle\psi^\dagger(1)\psi^\dagger(2)\psi(3)\psi(4)\rangle=k_F^4N^{-2}(8\pi^3\alpha)^{-2}\sum_{\Omega_1\ldots\Omega_4}\langle O_{\Omega_1}^\dagger\psi^\dagger(\Omega_1,1)O_{\Omega_2}^\dagger\psi^\dagger(\Omega_2,2)\psi(\Omega_3,3)O_{\Omega_3}\psi(\Omega_4,4)O_{\Omega_4}\rangle\ , \quad (2.25)$$

where the numerals stand for the space-time point of the form $2\equiv\vec{x}_2, t_2$, and the parenthesis arguments on the right of Eq. (2.25) represent $(O_{\Omega_2},2)\equiv(\Omega_2,\hat{k}_2\cdot\vec{x}_2,t_2)$ in a natural way. To calculate the expectation value, it is necessary to work the operator O_{Ω_1} to the right, where it operates on the vacuum and gives unity. In the process, the sign changes give the proper minus signs corresponding to the different permutations.

The most direct way to establish this result is to recognize that only certain combinations of the angles contribute in the summation. For example, in Eq. (2.25), only the combination $\Omega_1=\Omega_2$, $\Omega_3=\Omega_4$ will contribute. Thus calculation of an n-point function separates into two steps: (i) finding the angular configuration, and (ii) evaluating the sign of the configuration. The first part can be carried out dropping all ordering operators O_Ω completely (since they give only c-number contributions). Then the correlation function is greatly simplified, and for the example of Eq. (2.25), repeated use of the Baker-Hausdorf formula gives

$$\langle\psi^\dagger(\Omega_1,1)\psi(\Omega_2,2)\psi^\dagger(\Omega_3,3)\psi(\Omega_4,4)\rangle$$
$$=\exp\{-[2g(0)-\delta_{\Omega_1,\Omega_2}g(\Omega_1,\vec{R}_{12},\tau_{12})+\delta_{\Omega_1,\Omega_3}g(\Omega_1,\vec{R}_{13},\tau_{13})-\delta_{\Omega_1,\Omega_4}g(\Omega_1,\vec{R}_{14},\tau_{14})$$
$$-\delta_{\Omega_2,\Omega_3}g(\Omega_2,\vec{R}_{23},\tau_{23})+\delta_{\Omega_2,\Omega_4}g(\Omega_2,\vec{R}_{24},\tau_{24})-\delta_{\Omega_3,\Omega_4}g(\Omega_3,\vec{R}_{34},\tau_{34})]\}\ , \quad (2.26)$$

where

$$g(\Omega,\vec{R},\tau)=\int_0^\infty\frac{dk}{k}\exp[ik(\hat{k}_F\cdot\vec{R}-\tau+i\alpha)]$$

is equal to minus infinity due to the infrared singularity. Unless this singularity can be canceled by combining with the $g(0)$ (the result for $\vec{R}=\tau=0$) the entire correlation function is $\exp(-\infty)$ and vanishes. The only combinations of the angles which result in this cancellation are $\Omega_1=\Omega_3$ with $\Omega_2=\Omega_4$, and $\Omega_1=\Omega_2$ with $\Omega_3=\Omega_4$. (The configuration with all equal will contribute an extra factor N^{-1} compared to the above two and is thus negligible in the $N\to\infty$ limit.)

The second part consists of evaluating the signs of these two configurations. Using the commutation properties of O_Ω, in (2.23), it is easy to pull the operators with identical angle labels together, collecting minus signs along the way, finally making use of $O_\Omega O_\Omega^\dagger=1$. The result is a minus sign for the $\Omega_1=\Omega_3$, $\Omega_2=\Omega_4$ configuration and a plus sign for the other. Collecting finally gives

$$\langle\psi^\dagger(1)\psi^\dagger(2)\psi(3)\psi(4)\rangle=\langle\psi^\dagger(1)\psi(4)\rangle\langle\psi^\dagger(2)\psi(3)\rangle-\langle\psi^\dagger(1)\psi(3)\rangle\langle\psi^\dagger(2)\psi(4)\rangle\ , \quad (2.27)$$

the desired result. Generalization to n-point Fermi function is tedious but straightforward. It is found that only pairwise contractions on angles survive the $N\to\infty$ limiting procedure, and the operators O_Ω correctly give all signs, since they automatically insure the operator anticommutation property. Any correlation function with an unequal number of creation or destruction operators vanishes, due to an extra infrared singularity in the exponent.

The general form for these n-point functions with alternating creation and destruction operators, can be written

$$\langle\psi^\dagger(\Omega_1,1)\psi(\Omega_2,2)\cdots\psi^\dagger(\Omega_{2n-1},2n-1)\psi(\Omega_{2n},2n)\rangle = (8\pi^3 k_F\alpha^{-1})^n \exp\left(-ng(0)-\sum_{i<j}^{2n}(-)^{i-j}\delta_{\Omega_i,\Omega_j}g(\Omega_i,\vec{R}_{ij},\tau_{ij})\right). \quad (2.28)$$

A finite result in the $N\to\infty$ limit leads to the constraint on the angular labels Ω_i that they must be pairwise equal. The signs in the exponent are such that no odd (even) label be equal to another odd (even) label. Otherwise the exponent is equal to minus intuity and the corresponding contraction vanishes. In the original Fermi representation, this statement is just the vanishing of $\langle\psi^\dagger\psi^\dagger\rangle$-type contractions. The signs of the contraction come from the O_Ω, which are commuted through to make use of $O_\Omega O_\Omega^\dagger=1$, and Eq. (2.24) insures that the signs are appropriate for fermions. This completes the proof that the n-point functions for the Tomonaga fermions are correctly given by this boson representations.

This construction can be used in applications where interactions are present, for in the interaction representation, the relevant Green's functions are those computed with these operators. The boson representation provides the extra flexibility in solving for correlation functions exactly if the boson structure of the theory is known. In Sec. III, the extension to the Dirac equation is discussed, and applications of this result are to be given later.

III. DIRAC EQUATION

It is interesting to apply the methods of Sec. II to find a boson representation for massless Dirac particles. At first thought, it might appear meaningless to construct Dirac operators from a system whose ground state, namely, the Fermi sea, is obviously not Lorentz invariant. Nonetheless, it is possible, and this implies that there is a set of states, within the Fermi sea, which can be used to construct Lorentz invariant operators. It is shown here how to extract this set of states and how the boson representation can be constructed without any reference to a Fermi sea, surely a more appealing situation.

The key technological development of Sec. II involved the use of the radial phase field, given by Eqs. (2.12), defined for the Fermi sea. It is possible to define an equivalent field directly from an ordinary massless boson problem, quantized in a box,

$$\mathcal{H}_B=\sum_{\vec{k}}|\vec{k}|(\alpha^\dagger_{\vec{k}}\alpha_{\vec{k}}+\beta^\dagger_{\vec{k}}\beta_{\vec{k}}), \quad (3.1)$$

where

$$[\alpha_{\vec{k}},\alpha^\dagger_{\vec{k}'}]=[\beta_{\vec{k}},\beta^\dagger_{\vec{k}'}]=\delta_{\vec{k},\vec{k}'},\quad [\alpha_{\vec{k}},\beta^\dagger_{\vec{k}}]=0,$$

and the $\vec{k}$ vectors are defined by periodic boundary conditions. The radial phase field in this representation is defined by

$$\phi_1(\Omega,\hat{k}\cdot\vec{x})=\left(\frac{V}{8\pi^3}\right)^{1/2}\int_0^\infty e^{-\alpha k/2}k^{1/2}\,dk \times(\alpha_{\vec{k}}e^{ik(\hat{k}\cdot\vec{x})}+\text{H.c.}),$$

$$\phi_2(\Omega,\hat{k}\cdot\vec{x})=-\left(\frac{V}{8\pi^3}\right)^{1/2}\int_0^\infty e^{-\alpha k/2}k^{1/2}\,dk \times(\beta^\dagger_{\vec{k}}e^{ik(\hat{k}\cdot\vec{x})}+\text{H.c.}), \quad (3.2)$$

where V is the volume of the box and the angular integration over Ω is omitted. It is necessary to insert the cutoff α^{-1} in these integrals, which is taken to infinity in the final result. This definition differs from the canonical boson field by a factor of k^{-1}, and these objects do not describe local observables. It will turn out that appropriate functions of these fields do.

The calculation of an expectation value of these fields gives the same result as contained in Eq. (2.18), since both have the same property,

$$\langle\phi_1^2-\phi_1(\Omega,\hat{k}\cdot\vec{x},t)\phi_1(\Omega',\hat{k}'\cdot\vec{x}',t')\rangle = \int_0^\infty k^{-1}e^{-\alpha k}\,dk(\delta_{\Omega,\Omega'}e^{ik(\hat{k}\cdot\vec{R}-\tau)}-1), \quad (3.3)$$

with an identical result for the "two" fields, except $\vec{R}\to-\vec{R}$. Consequently, these fields can be used to construct the Tomonaga fermions of Sec. II, and the radial excitations discussed in that section are seen to correspond to these bosons.

It is possible to construct a similar representation for massless Dirac fermions. In a convenient representation the massless Hamiltonian is

$$\mathcal{H}_F=\sum_{\vec{k},\sigma,\sigma'} a^\dagger_{\vec{k},\sigma}(\vec{k}\cdot\vec{\sigma})a_{\vec{k},\sigma'}-b^\dagger_{\vec{k},\sigma}(\vec{k}\cdot\vec{\sigma})b_{\vec{k},\sigma'}\,, \tag{3.4}$$

where the $\vec{\sigma}$ are Pauli matrices, the $a_{k\sigma}$ ($b_{k\sigma}$) operators are two-component anticommutating spinor fields, representing the upper (lower) two entries in the four-component Dirac spinor. The propagator $G(\vec{k}, E)$ for the a fields is given by

$$G(\vec{k},E)=(E+\vec{k}\cdot\vec{\sigma})/(E^2-k^2) \tag{3.5}$$

and the correlation functions $G(\vec{R},t)$, which are defined by

$$G(\vec{R},\tau)=\int\frac{d^3k}{8\pi^3}\int\frac{dE}{2\pi i}e^{i\vec{k}\cdot\vec{R}-iE\tau}G(\vec{k},E)\,,$$

$$=-i\int\frac{d\Omega}{8\pi^3}\frac{1+\hat{k}\cdot\vec{\sigma}}{(\hat{k}\cdot\vec{R}-\tau)^3}\,, \tag{3.6}$$

where the remaining integration is over the angles of $\vec{k}$. The two-component operators a and b do not couple in the absence of a mass, and the two fields can be treated separately. Consider the a part separately. The Hamiltonian can be diagonalized by a transformation

$$U=e^{iS}$$

$$S=i\sum_{\vec{k},\sigma,\sigma'}\frac{\theta_{\vec{k}}}{2}a^\dagger_{\vec{k},\sigma}\hat{V}_{\vec{k}}\cdot\vec{\sigma}a_{\vec{k},\sigma'}\,, \tag{3.7}$$

with the vector $\vec{k}=k(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)$, and the unit vector $\hat{V}_{\vec{k}}=(\theta,-\sin\varphi,\cos\varphi)$. The angles θ and φ specify the direction of the vector $\vec{k}$. The diagonal Hamiltonian is

$$\mathcal{H}'=\sum_{\vec{k}}ka'^\dagger_{\vec{k}}\sigma_3 a'_{\vec{k}}\,, \tag{3.8}$$

and the diagonal operators $a'_k=Ua_kU^\dagger$ are related to the original a_k through

$$a_{\vec{k}}=[\cos(\tfrac{1}{2}\theta)1-i\hat{V}\cdot\vec{\sigma}\sin(\tfrac{1}{2}\theta)]a'_{\vec{k}}\,. \tag{3.9}$$

The boson representation is similar to this equation, and is given by

$$\psi(\Omega,\hat{k}\cdot\vec{x})=e^{i\theta\hat{V}\cdot\vec{\sigma}}\psi'(\Omega,\hat{k}\cdot\vec{x})\,, \tag{3.10}$$

where $\psi'(\Omega,\hat{k}\cdot\vec{x})$ is defined to be the two-component operator

$$\psi'(\Omega,\hat{k}\cdot\vec{x})=(2\pi\alpha)^{-3/2}\begin{pmatrix}\exp[\sqrt{3}\phi_1(\Omega,\hat{k}\cdot\vec{x})]\\ \exp[\sqrt{3}\phi_2(\Omega,\hat{k}\cdot\vec{x})]\end{pmatrix} \tag{3.11}$$

and Ω, representing the two angles θ and φ, is restricted to be in the upper hemisphere. The expectation value $\langle\psi^\dagger_\sigma(x,t)\psi_{\sigma'}(\vec{x}',t')\rangle$ is found using the Baker-Hausdorf formula, with Eq. (3.3), along with $\psi(\vec{x},t)=\int_u d\Omega\,\psi(\Omega,\hat{k}\cdot\vec{x})$, integrating over the upper hemisphere, to give

$$\langle\psi^\dagger(\vec{x},t)\psi(\vec{x}',t')\rangle=-i\int\frac{d\Omega}{8\pi^3}\frac{1+\hat{k}\cdot\vec{\sigma}}{(\hat{k}\cdot\vec{R}-\tau)^3}\,. \tag{3.12}$$

In deriving Eq. (3.12), the contribution from ϕ_2 fields, integrated over the upper hemisphere, extends the angular integration to the lower half sphere through the reflection $\hat{k}\to-\hat{k}$.

It is seen that Eq. (3.12) is the same as Eq. (3.6), and it can be verified that the other ordering, $\psi\psi^\dagger$, is correctly reproduced by Eq. (3.11), proving the pair functions equal. For the general functions, it is helpful to study the equations of motion.

The ordinary Fermi operator for the upper two components satisfies the equation of motion

$$\frac{id}{dt}\psi(\vec{x},t)=-i\vec{\nabla}\cdot\vec{\sigma}\psi(\vec{x},t)\,, \tag{3.13}$$

or, in Fourier-transform representation,

$$\frac{id}{dt}a_{\vec{k}}=\vec{k}\cdot\vec{\sigma}a_{\vec{k}}\,, \tag{3.14}$$

and the lower two components obey the same equations with a minus sign on the right-hand side. These follow from the nondiagonalized form of the Hamiltonian; the equations in the diagonal representation are simply

$$\frac{id}{dt}a_{\vec{k}}=k\sigma_3 a_{\vec{k}}\,, \tag{3.15}$$

and it is this equation that is most directly related to the boson representation. There is no fundamental difference between these representations, since the solution to Eq. (3.14),

$$a_{\vec{k}}(t)=e^{i\vec{k}\cdot\vec{\sigma}t}a_{\vec{k}}(0)\,, \tag{3.16}$$

can be obtained from the solution to Eq. (3.15) by applying the diagonalizing transformation, Eq. (3.7). The solution to Eq. (3.15),

$$a'_{\vec{k}}(t)=e^{ik\sigma_3 t}a_{\vec{k}}\,, \tag{3.17}$$

leads to

$$a_{\vec{k}}(t)=e^{iS}e^{ik\sigma_3 t}e^{-iS}a_{\vec{k}}(0)\,, \tag{3.18}$$

and it can be verified that $e^{iS}e^{ik\sigma_3 t}e^{-iS}=e^{i\vec{k}\cdot\vec{\sigma}t}$ as required.

However, the boson representation is most understandable for this diagonal representation, and the ψ' operators of (3.10) are solutions in this representation. The transformation e^{iS} involves only matrices and angles; there is no dependence on the $|k|$. For that reason, the Lorentz transformations to the diagonalizing frame are easily constructed for the boson representation.

The boson operator $\psi'(\Omega,\hat{k}\cdot\vec{x},t)$ in Eq. (3.10) satisfies the equation

$$\frac{id}{dt}\psi'(\Omega,\hat{k}\cdot\vec{x},t)=i\sigma_3(\hat{k}\cdot\vec{\nabla})\psi'(\Omega,\hat{k}\cdot\vec{x},t)\,, \qquad (3.19)$$

and the transformed operator $\psi=\exp[i(\theta/2)\hat{V}\cdot\vec{\sigma}]\psi'$ satisfies the equation of motion

$$\frac{id}{dt}\psi(\Omega,\hat{k}\cdot\vec{x},t)=i(\hat{k}\cdot\vec{\sigma})(\hat{k}\cdot\vec{\nabla})\psi(\Omega,\hat{k}\cdot\vec{x},t)\,, \qquad (3.20)$$

where use is made of the property

$$\exp[i(\tfrac{1}{2}\theta)\hat{V}\cdot\vec{\sigma}]\sigma_3\exp[-i(\tfrac{1}{2}\theta)\hat{V}\cdot\vec{\sigma}]=\hat{k}\cdot\vec{\sigma}.$$

This equation of motion is the same as Eq. (3.14), since $\hat{k}\cdot\vec{\nabla}=k$ under a Fourier transform.

Although the boson representation satisfies the equations of motion, these operators do not anticommute. It is necessary to construct an additional operator to correct this deficiency. The example from Sec. II provides the key, and the correct operator is given by multiplying $\psi(\Omega,\hat{k}\cdot\vec{x})$ from Eq. (3.11) by the ordering operator O_Ω,

$$\psi(\Omega,\hat{k}\cdot\vec{x})=e^{i\theta\hat{V}\cdot\vec{\sigma}}\psi'(\Omega,\hat{k}\cdot\vec{x})O_\Omega\,, \qquad (3.21)$$

where O_Ω is the ordering operator encountered in the Tomonaga case, Eq. (2.22), with the following modifications. To construct the appropriate operator for the Dirac equation, it is necessary to use the relation between the ρ boson operators of Sec. II, and the $\alpha_{\vec{k}}$ boson operators. Comparing the commutation relations

$$[\rho_1(-k,\Omega),\rho_1(k',\Omega')]=k\delta(k-k')\delta_{\Omega,\Omega'}\,,$$
$$[\alpha_{\vec{k}},\alpha^\dagger_{\vec{k}'}]=\delta_{\vec{k},\vec{k}'}=8\pi^3V^{-1}k^{-2}\delta(k-k')\delta_{\Omega,\Omega'}\,, \qquad (3.22)$$

where $\vec{k}=(k,\Omega)$, leads to the identification

$$\alpha_{\vec{k}}=(8\pi^3/k^3V)^{1/2}\rho_1(-k,\Omega)\,, \qquad (3.23)$$

with $\alpha^\dagger_{\vec{k}}$ given by the opposite sign of k ($|k|$ in the parenthesis). The $\rho_1(\Omega)$ operator needed in the ordering operator O_Ω then follows immediately. The result is

$$O_\Omega=\exp(i\sqrt{3}J_\Omega)\,,$$
$$J_\Omega=\frac{\pi}{2}\sum_{\Omega'=1}^{\Omega-1}[\rho_1(\Omega')+\rho_2(\Omega')]\,, \qquad (3.24)$$

with $\rho_1(\Omega)$ the limit $k\to 0$ of $(k^3V/8\pi^3)^{1/2}\alpha_{\vec{k}}$, for fixed Ω. Since these definitions preserve the commutation relations in the ρ language, it follows that

$$O_\Omega O^\dagger_\Omega=1$$
$$[O_\Omega,O_{\Omega'}]=[O_\Omega,\mathcal{H}]=0$$
$$O_{\Omega'}\exp[\sqrt{3}\phi(\Omega,\hat{k}\cdot\vec{x})]O^\dagger_{\Omega'}=\begin{cases}-\exp[\sqrt{3}\phi(\Omega,\hat{k}\cdot\vec{x})]\,, & \Omega'>\Omega\\ +\exp[\sqrt{3}\phi(\Omega,\hat{k}\cdot\vec{x})]\,, & \Omega'\leq\Omega\,;\end{cases} \qquad (3.25)$$

the sign change above for $\Omega'>\Omega$ results from the commutator of J_Ω with ϕ_1 and is a factor of $3i\pi$ in the exponent. The steps to prove the anticommutation property are identical to those leading to Eq. (2.25), and the conclusion is

$$(\exp[\sqrt{3}\phi(\Omega,\hat{k}\cdot\vec{x})]O_\Omega\,,\ \exp[\sqrt{3}\phi(\Omega',\hat{k}\cdot\vec{x}')]O_{\Omega'})_+=0\,, \qquad (3.26)$$

which implies $(\psi(\vec{x}),\psi(\vec{x}'))_+=0$.

This property can be used to prove that the n-point fermion expectation values are correctly given by the boson operator in Eq. (3.21). The proof is exactly the same as in Sec. II, reconstructed with the new operators. The steps are (i) anticommutation relations are satisfied, (ii) only pairwise contractions due to the infrared singularity, and (iii) the pair functions, as in Eq. (3.12), are correct.

To conclude this section, it is necessary to remark that the operators J_Ω, used in Eq. (3.24), in fact, include the extra degrees of freedom needed to obtain anticommutation with the b operators—the other two components of the Dirac spinor—of Eq. (3.1). This can be done because in Eq. (3.12), only the upper hemisphere of the operators was used. The lower hemisphere is available, and the two lower components of the spinor operator in boson form is given by

$$\psi_L(\vec{x})=(2\pi\alpha)^{-3/2}\int_L d\Omega\,\psi(\Omega,\hat{k}\cdot\vec{x})\,, \qquad (3.27)$$

where the integral in the sense of Eq. (2.14) is over the lower hemisphere, $\psi(\Omega,\hat{k}\cdot\vec{x})$ is given by Eq. (3.11), and O_Ω is from Eq. (3.24). This, together with the upper two components given by Eq. (3.11), completes the boson representations for the massless Dirac spinor. Since the mass term or any currents are defined in terms of these field operators and matrices, we conclude that this representation works for the interacting massive Dirac equation, as discussed in Sec. IX.

IV. POSSIBLE APPLICATIONS

As emphasized in the introduction, a boson representation is useful if it provides insight, solutions, or perhaps some formal rigor for problems of real physical interest. None of these has been yet demonstrated, and the reader is perhaps justified in retaining a residual skepticism about the ultimate utility of this exercise. However, there are several interesting possibilities in answer to this, and it is the task of this section to enumerate some of these. It is the belief of the author that some, if not all, will provide interesting application of the methods introduced in Secs. I–III and that the duality ideas suggested by Bloch and Tomonaga will have a major impact on our understanding of fields and particles.

The constructions here have been restricted to the massless, or gapless, fermion problem. It is not remiss to repeat that the mass term of the Dirac equation, or a BCS type of term giving a gap at the Fermi level, can be written in the boson representation. The proof that this is correct is contained in the Sec. III, for if the n-point functions are given correctly, it follows trivially that the equations of motion including the mass term are correct. Hence the solution of these equations is the same as that of the conventional Fermi representation. The mass term in this representation is given by

$$m_0\psi^\dagger_{(\vec{x})}\beta\psi(\vec{x})$$

$$= m_0 \int_U d\Omega \int_L d\Omega' [\psi^\dagger(\Omega, \hat{k}\cdot\vec{x})\psi(\Omega', \hat{k}'\cdot\vec{x}) + \text{H.c.}] , \tag{4.1}$$

where m_0 is the mass, and

$$\beta = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and this representation is the generalization of the sine-Gordon duality with the Dirac equation, familiar in two dimensions. It is obvious that the mass term is not local in terms of the ϕ fields, although the solution of this equation certainly is local. An implication of this example is that locality of the ultimate physical theory does not necessarily require locality at every step along the way.

The question of locality arises naturally in extensions of the simple Dirac model to more interesting cases with interactions present. Just as the mass term can be constructed in this boson representation, so can the current operators. The algebra from Sec. III leads to the result

$$\vec{J}(\vec{x}) = \psi^\dagger(\vec{x})\vec{\alpha}\psi(\vec{x}) ,$$

$$= \int_U d\Omega \int_U d\Omega' \psi^\dagger(\Omega, \hat{k}\cdot\vec{x})\vec{\sigma}\psi(\Omega', \hat{k}'\cdot\vec{x})$$

$$- \int_L d\Omega \int_L d\Omega' \psi^\dagger(\Omega, \hat{k}\cdot\vec{x})\vec{\sigma}\psi(\Omega', \hat{k}'\cdot\vec{x}) , \tag{4.2}$$

where

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}.$$

This representation for currents, of course, contains no obvious advantage over the conventional definition, for any interaction constructed from currents is just as nonrenormalizable. However, it could be possible to consider a larger class of interactions, which are not constructed from canonical free fields, but from fields with different dimensions. Such fields are obviously suggested by this boson representation and involve changing the $\sqrt{3}$ to some more complicated function. It is not known if it is possible to generate a new class of theories that possess the desired properties of Lorentz invariance, locality, etc., from this starting point, but certainly the intuition of two dimensions would suggest that it would be interesting to try.

An examination of the current operator, given above, or the single fermion field, given in Sec. III, reveals an interesting possibility for constructing composite operators with an internal symmetry. For a given fermion field, many components of a boson field can be used, as long as the combinations are canonical. For example, a "flavor" index on the boson field can be introduced, and the various canonical combinations can be constructed. With N flavors, there are N orghogonal combinations, and each combination can be used in the boson representation to construct a Fermi field (along with a suitable generalization of the ordering operator, of course). There are many more combinations of the boson operators that give independent Fermi fields, which suggests that an underlying algebra is naturally realized with this construction.

This algebraic question is important, for the conventional quark picture requires the introduction of an extra fermion label, the color index, in order to construct hadrons that have the correct Fermi statistics. Since this requires that all states be color singlets, the prime function of the extra degrees of freedom is to enable the construction of composite particles from some underlying unobserved fermions.

The suggestion here is that there exist operators satisfying the appropriate group algebra, but not decomposable into more fundamental fields—that is, they cannot be written as a direct product of three Fermi fields. The existence of such operators would obviate the necessity of introducing separate quark fields, and the question of quark confinement would be moot, for no single quark states would exist in the Hilbert space. Nonetheless, the composite feature of these operators, within the boson representation, means that many of the desirable features of the quark picture could remain.

A further interesting application arises from the fact that the boson Hamiltonian of Sec. III can represent a lattice phonon Hamiltonian. The obvious suggestion that from the acoustic phonons in a lattice it is possible to construct a fermion state, has implications for defect theory and the analysis of nonlinear lattice theories. An example of an interesting type of nonlinearity is given by the mass term Eq. (4.1), when expressed in boson operator language.

Here it is necessary to remark on the evidently nonlocal nature of Eq. (4.1). In Fermi description, it is obviously local, and since the boson language is identical to the Fermi, one must conclude that appearances are deceiving. To the extent that some interactions have evident desirable properties, one must conclude that desirability is in the eye of the beholder, that is, dependent on representation. Despite appearances, there is some physics underlying these observations. It is presumably true that many realistic nonlinear lattice problems will have some nonlinearities that are relevant, and others irrelevant—in the sense of Wilson. Good candidates for those that are relevant are those that give use to new nonlinear degrees of freedom, or solitons. Solitons are just another word for those Fermi degrees of freedom constructed from the boson fields.

A further application is to the interacting electron gas. The RPA for this problem was one of the historical first attempts to formalize the concept of bosonization.[12] It seems, at first sight, to be related to the representation found here. However, this first sight is misleading. For the free-particle case, the density operator spectrum differs from the boson spectrum of Sec. II. It is, of course, trivial to calculate the particle-hole spectrum exactly. Since the boson construction of fermion operators is exact, it follows that the particle-hole spectrum, calculated in boson language, is also exact. At no stage does the treatment of particle-hole pairs as bosons appear, and it is therefore difficult to relate this bosonization to the RPA bosonization.

To be more precise, consider the density-density correlation function $\langle\rho(x,t)\rho\rangle$ for the Fermi sea of Sec. II. The RPA result for this is

$$\langle\rho(\vec{x},t)\rho(\vec{x}',t')\rangle=\frac{C}{R^2}\left(\frac{1}{(R-\tau)^2}+\frac{1}{(R+\tau)^2}\right), \qquad (4.3)$$

where $C=(k_F/2\pi)^2$, whereas, to leading order in $(k_FR)^{-1}$ and $k_F\tau$, the exact result is

$$\langle\rho(\vec{x},t)\rho(\vec{x}',t)\rangle =\frac{C}{R^2}\left(\frac{1}{(R-\tau)^2}+\frac{1}{(R+\tau)^2}+\frac{2i\cos 2k_FR}{R^2-\tau^2}\right). \qquad (4.4)$$

As is well known, the RPA does not include the correct large-momentum excitations responsible for the phase factors. The boson representation of Sec. II, however, reproduces Eq. (4.4), not the RPA result.

In the purely one-dimensional Luttinger model, a similar difference appears. The RPA reproduces the first term in the density-density correlation function, and it is necessary to find the $2k_F$ processes separately. The RPA operators are of the type $:\psi_1^\dagger\psi_1:=\rho_1$ (in the notation of the Luttinger model), while the $2k_F$ processes come from $\psi_1^\dagger\psi_2$-type operators. Defining $\rho=\psi^\dagger\psi$ where $\sqrt{2}\psi=\psi_1+\psi_2$, the Luttinger model results for RPA and exact are, respectively,

$$\langle\rho(x,t)\rho\rangle=\frac{1}{(x-t)^2}+\frac{1}{(x+t)^2},$$

$$=\frac{1}{(x-t)^2}+\frac{1}{(x+t)^2}+\frac{2i\cos 2k_Fx}{x^2-t^2}, \qquad (4.5)$$

analogous to the case in three dimensions. In general dimension, $C=(k_F/2\pi)^{d-1}$, and the above expression is multiplied by $R^{-(d-1)}$.

In three dimensions the separation into appropriate generalizations of ρ and $\psi_1^\dagger\psi_2$ is not yet known; only the total combination summed over angles has a simple boson representation (in one dimension, the sum over "angles" has only two terms, $\psi_1+\psi_2$). It would be interesting, indeed, even helpful in generalizing beyond the RPA, if a simple boson representation which separates small momentum from large momentum could be found. In such a representation, it would be possible to separately parametrize, and perhaps solve, the large-momentum-transfer part of the model. The further development of these ideas will be reported in future publications.

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