Exact renormalization group equations for the two-dimensional Ising model

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By explicit construction we show the existence of an exact renormalization group for triangular Ising models with only nearest-neighbor interactions. The recursion relations take the form of a set of three quasilinear first-order partial differential equations for the interactions. We determine a nontrivial fixed point and study the linearized flow around it. This yields the specific-heat exponent $\alpha = 0$, in agreement with the Onsager and Houtappel solutions, and demonstrates universality. The free energy is expressed as the trajectory integral of an explicitly given function of the interactions.

I. INTRODUCTION

The fundamental idea of the renormalization group (RG) approach to critical phenomena is to calculate the partition function of a system by successively thinning out its degrees of freedom. To this end an iteration procedure is set up in each step of which a certain fraction of the variables of the system is summed out. Upon such a partial trace operation a new system appears, similar to the original one, but with fewer degrees of freedom and with different, renormalized, interaction constants. The mapping of the original set of interaction constants, say, K, onto a renormalized set K' constitutes an RG transformation. Renormalization theory shows that the critical properties of a system are directly related to the fixed-point properties of the RG transformation, and that properties of the system away from criticality can be obtained by studying the RG transformation along its trajectory in K space.

If in one iteration step a finite fraction of all degrees of freedom is summed out, then the RG transformation will be a discrete transformation, and the trajectory in K space will be a discrete sequence of points. If, on the other hand, in each iteration only an infinitesimal fraction of the degrees of freedom is summed out, then the RG transformation will take a differential form and lead to a continuous flow in K space. Formal properties of the RG are most easily discussed if one has the equations in differential form. For such discussions one may consult the articles by Wilson[1] and Wegner and Houghton,[2] and the reviews by Wilson and Kogut,[3] Wilson,[4] Ma,[5] Fisher,[6] and Wegner.[7] Practical calculations with differential RG equations have, almost without exception,[8] been carried out in momentum space. In one iteration one integrates on an infinitesimal shell[7] of wave numbers q just below some cutoff wave number $q = \Lambda$, or, alternatively, uses Wilson's method[3] of incomplete integration.

In contrast to these momentum-space methods, the real-space approach[9] is directly inspired upon Kadanoff's original block-spin ideas[10]: it offers a technique of calculating the renormalized interactions between the block spins. Since a block always contains an integer number of spins, the block-spin method necessarily leads to discrete RG transformations. In practice the block-spin procedure cannot be carried through except in certain simple cases.[11] For the other cases of interest several approximation schemes have been developed[9] and extensively used.

The purpose of this paper is to show that real-space renormalization can be cast in differential form. That this is feasible was already reported in a short letter.[12] The intuitive idea, expressed in Sec. II, is to map a given lattice system into a system with infinitesimally larger lattice spacing, and thus out of registry with the first. In Sec. III we show how on the basis of this idea exact differential RG equations can be obtained for a triangular Ising model. Our equations for this model involve only three nearest-neighbor interactions, K_1, K_2, and K_3. When deriving the equations we are led to consider these interactions as position-dependent quantities. Thus our basic RG recursion relation, Eq. (3.16), is a set of three coupled partial differential equations for the functions K_i ($i = 1, 2, 3$). The independent variables are the two spatial coordinates $\vec{r}$ and a parameter t which measures the distance along the RG trajectory. In Sec. IV we discuss various properties of the equations. In Sec. V we find a nontrivial fixed-point solution. The fixed point corresponds to a nonhomogeneous lattice. In Sec. VI we expand the RG equations about the fixed point such as to obtain a linear flow problem. We isolate the eigenvalue equation in the temperature direction and find that it has a unique eigenvalue $y_T = 1$, in agreement with the exact results by Onsager[13] and Houtappel.[14] The corresponding space of eigenfunctions is infinitely degenerate. We interpret

this feature as an expression of universality within the class of triangular Ising lattices with anisotropic nearest-neighbor interactions. In Sec. VII we derive an expression for the free energy as the trajectory integral over an explicitly given function of the interactions. In Sec. VIII we present our conclusions and several further comments.

II. INTUITIVE BACKGROUND

We intend to construct a real-space RG transformation which reduces infinitesimally the number of degrees of freedom. Our idea is to accomplish this by relating the Hamiltonians of two similar lattices which differ infinitesimally in lattice constant. To be specific we consider a triangular lattice $\mathfrak{L}$ of Ising spins $\frac{1}{2}$, which has lattice constant a and the shape of an equilateral triangle with side of length L. Hence there are $L/a+1$ lattice sites along each side. This system will be mapped onto a similar triangular lattice $\mathfrak{L}'$, also of length L, but with one less lattice site along each side. Thus the lattice constant of $\mathfrak{L}'$ is larger than that of $\mathfrak{L}$ by the fractional amount $\delta a/a = a/(L-a)$. The number of spins in $\mathfrak{L}$ and $\mathfrak{L}'$ also differ by a fraction of order a/L, which becomes infinitesimal in the limit $L \to \infty$. Figure 1 shows the systems $\mathfrak{L}$ and $\mathfrak{L}'$, denoted by circles and crosses, respectively.

Let s and s' stand for the spin configurations on $\mathfrak{L}$ and $\mathfrak{L}'$, respectively. Suppose that a Hamiltonian $\mathcal{H}(s)$ is given on $\mathfrak{L}$. The mapping of $\mathcal{H}(s)$ onto the renormalized Hamiltonian $\mathcal{H}'(s')$ on $\mathfrak{L}'$ then takes the usual form[15]

$$e^{\mathcal{H}'(s')} = \sum_s P(s', s) e^{\mathcal{H}(s)} . \tag{2.1}$$

Here $P(s', s)$ is a projection operator (weight factor) that satisfies the normalization condition

$$\sum_{s'} P(s', s) = 1 \tag{2.2}$$

in order that the transformation preserve the partition function. We remark that the present formulation allows for $\mathcal{H}$ and $\mathcal{H}'$ to contain spin-independent terms. Now let us introduce the Hamiltonian $\mathcal{H}_c(s', s)$ of the combined primed and unprimed systems as

$$e^{\mathcal{H}_c(s', s)} = P(s', s) e^{\mathcal{H}(s)} . \tag{2.3}$$

The transformation (2.1) together with the normalization condition (2.2) can then be written in the symmetric form

$$e^{\mathcal{H}'(s')} = \sum_s e^{\mathcal{H}_c(s', s)} , \tag{2.4a}$$

$$e^{\mathcal{H}(s)} = \sum_{s'} e^{\mathcal{H}_c(s', s)} . \tag{2.4b}$$

Equations (2.4) show that the coupling Hamiltonian $\mathcal{H}_c(s', s)$ induces the RG transformation. The form of $\mathcal{H}_c(s', s)$ is at our disposal, and we shall choose a local Hamiltonian, coupling only nearby spins. Intuitively we expect that pairs of primed and unprimed spins which are quite close to one another, such as occur near the vertices in Fig. 1, should be coupled more strongly by $\mathcal{H}_c$ than pairs which are not so close, as in the center. Thus we are prepared to allow for spatially dependent coupling constants in $\mathcal{H}_c$. This is, in fact, the only way to express that the spatial relationship between the two lattices $\mathfrak{L}$ and $\mathfrak{L}'$ varies from place to place.

From Eqs. (2.4) it is clear that if $\mathcal{H}_c$ were symmetric in s and s', then $\mathcal{H}'$ would equal $\mathcal{H}$ and the RG transformation would reduce to the identity. However, if $\mathcal{H}_c$ fails infinitesimally from being symmetric in s and s', then it will generate an infinitesimal difference between $\mathcal{H}'$ and $\mathcal{H}$, and therefore a flow in Hamiltonian space. It turns out that the infinitesimal deviation from symmetry which is needed is provided precisely by the spatial gradients of the coupling constants. When the couplings vary only appreciably over distances of the order L, then $\mathcal{H}' - \mathcal{H}$ becomes small of the order a/L, and therefore infinitesimal in the limit $L \to \infty$. The spatial dependence of $\mathcal{H}_c$ will lead us to consider Hamiltonians $\mathcal{H}$ and $\mathcal{H}'$ that are also spatially dependent. It is in the space of such Hamiltonians that we shall be able to construct differential RG equations and study fixed points and other flow properties.

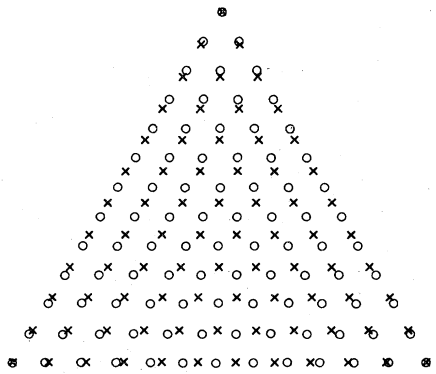

FIG. 1. Triangular lattices $\mathfrak{L}$ (the unrenormalized system: circles) and $\mathfrak{L}'$ (the renormalized system: crosses) have sides of equal length L. $\mathfrak{L}$ has one more lattice site along each side than $\mathfrak{L}'$.

III. CONSTRUCTION OF THE RENORMALIZATION EQUATIONS

We shall now make concrete the ideas discussed in Sec. II and actually construct a set of differential RG equations. It is instructive to do this in three steps. We first couple the two systems shown in Fig. 2, which have the same lattice constant a. They are governed by the spatially dependent Hamiltonians $\mathcal{H}(s)$ and $\tilde{\mathcal{H}}(s')$. The transformation from $\mathcal{H}(s)$ to $\tilde{\mathcal{H}}(s')$ results from two applications of the standard star-triangle transformation,[16,17] which here, due to the space dependence of the interactions, yields new and interesting results. The second step is to uniformly dilate the lattice of crosses by a factor $L/(L-a)$, so that we obtain the situation of Fig. 1. This operation transforms $\tilde{\mathcal{H}}(s')$ to the renormalized Hamiltonian $\mathcal{H}'(s')$, so that at this point the desired equations governing the Hamiltonian flow are obtained. Lastly we derive the boundary conditions appropriate to the flow equations, thereby completing the construction of our RG transformation.

A. Star-triangle flow equations

We begin by considering a given Hamiltonian $\mathcal{H}(s)$ on the lattice $\mathfrak{L}$ (circles) of Fig. 2. We shall map this Hamiltonian onto a new one $\tilde{\mathcal{H}}(s')$ defined on $\tilde{\mathfrak{L}}$ (crosses). We consider the case in which $\mathcal{H}(s)$ contains only nearest-neighbor interactions and, in the light of Sec. II, shall allow these to be spatially dependent. Thus we take $\mathcal{H}(s)$ to be of the form

$$\mathcal{H}(s)=\sum_{\vec{R}}{}^{\times}[K_1(\vec{R})s(\vec{R}-a\vec{e}_2)s(\vec{R}-a\vec{e}_3)+K_2(\vec{R})s(\vec{R}-a\vec{e}_3)s(\vec{R}-a\vec{e}_1)+K_3(\vec{R})s(\vec{R}-a\vec{e}_1)s(\vec{R}-a\vec{e}_2)]. \quad (3.1)$$

Here $\sum^{\times}$ indicates that the vector $\vec{R}\equiv(X,Y)$ runs

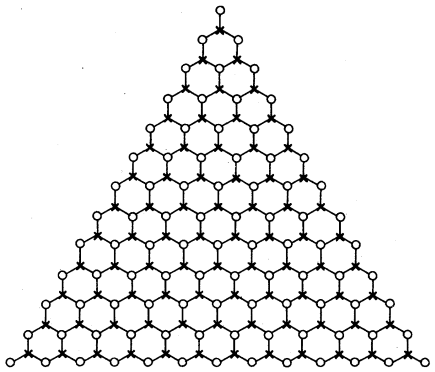

FIG. 2. Triangular lattices $\mathfrak{L}$ (circles) and $\tilde{\mathfrak{L}}$ (crosses) have sides of lengths L and $L-a$, respectively. A uniform dilation by a factor $L/(L-a)$ carries $\tilde{\mathfrak{L}}$ over into the lattice $\mathfrak{L}'$ of Fig. 1. The bonds shown (solid lines) are the couplings p_i of the coupling Hamiltonian $\mathcal{H}_c$.

through the centers of all up-triangles of $\mathfrak{L}$, and we have introduced the vectors $\vec{e}_i$ defined by

$$\vec{e}_1=(0,-\tfrac{1}{3}\sqrt{3}),\quad \vec{e}_2=(\tfrac{1}{2},\tfrac{1}{6}\sqrt{3}),\quad \vec{e}_3=(-\tfrac{1}{2},\tfrac{1}{6}\sqrt{3}) \quad (3.2)$$

[see Fig. 3(a)]. The relation between the index i of the interactions K_i (and of the couplings p_i to be introduced shortly) and their orientation in space is shown in Fig. 3(b). Note that each K_i has been labeled by the coordinate $\vec{R}$ of the center of the up-triangle to which it belongs.

We must now specify a Hamiltonian $\mathcal{H}_c(s',s)$ which couples the spins s and s' and can be made to satisfy the condition (2.4b). Once this is done the transformed Hamiltonian $\tilde{\mathcal{H}}$ will follow from Eq. (2.4a), where it takes the place of $\mathcal{H}'$. We take $\mathcal{H}_c(s',s)$ of the form

$$\mathcal{H}_c(s',s)=\sum_{\vec{R}}{}^{\times}[p_1(\vec{R})s(\vec{R}-a\vec{e}_1)+p_2(\vec{R})s(\vec{R}-a\vec{e}_2)+p_3(\vec{R})s(\vec{R}-a\vec{e}_3)]s'(\vec{R})-\sum_{\vec{R}}{}^{\times}g(p_1(\vec{R}),p_2(\vec{R}),p_3(\vec{R})). \quad (3.3)$$

Note that, apart from a spin-independent term, $\mathcal{H}_c(s',s)$ is just the Hamiltonian of a honeycomb lattice with spatially dependent nearest-neighbor couplings p_i. Each p_i has been labeled by the coordinate of the center of the upward pointing star to which it belongs. Equation (2.4b) imposes a relation between the p's and the K's. Upon summing out the primed spins we find the connection

$$K_i(\vec{R})=F(p_i(\vec{R}),p_j(\vec{R}),p_k(\vec{R})),\quad i,j,k \text{ cyclic}, \quad (3.4a)$$

$$F(p_1,p_2,p_3)=\tfrac{1}{4}\ln\frac{\cosh(p_1+p_2+p_3)\cosh(-p_1+p_2+p_3)}{\cosh(p_1-p_2+p_3)\cosh(p_1+p_2-p_3)}, \quad (3.4b)$$

and

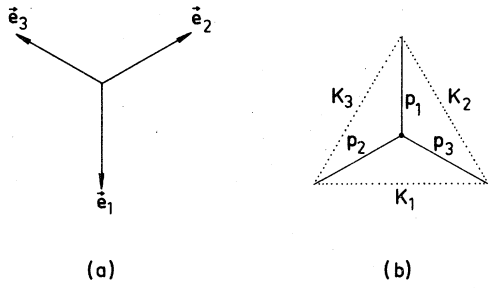

FIG. 3. (a) Vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. (b) The spatial orientation of the interactions K_i and the couplings p_i.

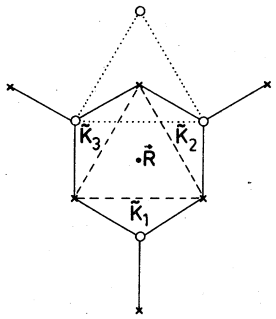

FIG. 4. Bonds p_i (solid lines) involved in the calculation of the interactions $\tilde{K}_i$ (dashed lines) in a triangle of $\tilde{\mathcal{L}}$ centered at $\vec{R}$. Also shown is a triangle of $\mathcal{L}$, centered at $\vec{R}-a\vec{e}_1$ (dotted lines).

$$g(p_1,p_2,p_3) = \ln 2 + \tfrac{1}{4}\ln[\cosh(p_1+p_2+p_3)\cosh(-p_1+p_2+p_3) \times \cosh(p_1-p_2+p_3)\cosh(p_1+p_2-p_3)]. \tag{3.5}$$

Equation (3.4) is the well-known star-triangle (ST) transformation.[16,17] Here it relates, for every up-triangle of $\mathcal{L}$, a triplet of couplings p to a triplet of interactions K. In this paper we shall limit ourselves to non-negative values of the K_i and the p_j.

With the coupling Hamiltonian $\mathcal{H}_c(s',s)$ now completely determined we substitute it into Eq. (2.4a) (with $\mathcal{H}'$ replaced by $\mathcal{H}$) and carry out the sum on the spins s. The result is a triangular lattice Hamiltonian $\tilde{\mathcal{H}}(s')$ on $\tilde{\mathcal{L}}$ with nearest-neighbor interactions $\tilde{K}_i(\vec{R})$. Our convention for assigning coordinates to the $\tilde{K}_i$ is analogous to the one adopted for the K_i, i.e., each bond K_i will have the coordinate $\vec{R}$ corresponding to the center of the up-triangle of $\tilde{\mathcal{L}}$ of which it is a part. Every $\tilde{K}_i(\vec{R})$ is generated by a ST transformation in which three p bonds enter with three different coordinate arguments. Upon tracing these back (see, e.g., Fig. 4) we find

$$\tilde{\mathcal{H}}(s') = \sum_{\vec{R}} [\tilde{K}_1(\vec{R})s'(\vec{R}-a\vec{e}_2)s'(\vec{R}-a\vec{e}_3)+\tilde{K}_2(\vec{R})s'(\vec{R}-a\vec{e}_3)s'(\vec{R}-a\vec{e}_1)+\tilde{K}_3(\vec{R})s'(\vec{R}-a\vec{e}_1)s'(\vec{R}-a\vec{e}_2)] + \sum_{\vec{R}}{}^{0} g(p_1(\vec{R}+a\vec{e}_1),p_2(\vec{R}+a\vec{e}_2),p_3(\vec{R}+a\vec{e}_3)) - \sum_{\vec{R}}{}^{\times} g(p_1(\vec{R}),p_2(\vec{R}),p_3(\vec{R})), \tag{3.6}$$

where

$$\tilde{K}_i(\vec{R}) = F(p_i(\vec{R}+2a\vec{e}_i),p_j(\vec{R}+a\vec{e}_i+a\vec{e}_j), p_k(\vec{R}+a\vec{e}_i+a\vec{e}_k)), \quad i,j,k \text{ cyclic}. \tag{3.7}$$

In Eq. (3.6) the first summation (no superscript) runs through the centers of the up-triangles of $\tilde{\mathcal{L}}$, and the second summation (superscript 0) runs through all lattice sites of $\mathcal{L}$.

The Hamiltonians $\mathcal{H}$ of Eq. (3.1) and $\tilde{\mathcal{H}}$ of Eq. (3.6) both describe triangular lattices with inhomogeneous nearest-neighbor couplings. A discussion of the spin-independent term that arises in Eq. (3.6) will be postponed until Sec. VII. We shall now focus on the relationship between the interactions K_i and $\tilde{K}_j$. The transformation from the K_i to the $\tilde{K}_j$ is given implicitly by Eqs. (3.4) and (3.7). Let us now assume that the three functions $p_i(\vec{R})$ vary on a scale set by a large distance $L \gg a$. Then $\vec{R}$ can be considered a continuous variable and in Eq. (3.7) we can Taylor expand the p_l about $\vec{R}$ and F about $(p_i(\vec{R}),p_j(\vec{R}),p_k(\vec{R}))$. Comparison with Eq. (3.4a) then yields

$$\tilde{K}_i(\vec{R}) = K_i(\vec{R}) + a\sum_l (\vec{e}_i+\vec{e}_l)Q_{il}\cdot\nabla p_l + O(a^2/L^2), \tag{3.8}$$

where

$$Q_{il} \equiv \frac{\partial K_i}{\partial p_l} = \frac{\partial F(p_i,p_j,p_k)}{\partial p_l}, \quad i,j,k \text{ cyclic}. \tag{3.9}$$

The matrix Q and its inverse occur repeatedly in subsequent calculations. Their elements are given explicitly in Appendix A. The gradient of p_l in Eq. (3.8) is easily converted into the gradients of the

K_j by means of

$$\vec{\nabla} p_l = \sum_j Q_{lj}^{-1} \vec{\nabla} K_j . \tag{3.10}$$

Since the $\vec{\nabla} p_l$, and therefore also the $\vec{\nabla} K_j$, are of order $1/L$, we see from (3.8) and (3.10) that $\tilde{K}_i(\vec{R})$ differs from $K_i(\vec{R})$ only by terms of order a/L. This sets the scale for the rate of change along the trajectory obtained by iterating Eqs. (3.8) and (3.10). We therefore define the successive iterates of three initial functions $K_i(\vec{R})$ by

$$K_i(\vec{R}, 0) = K_i(\vec{R}) , \tag{3.11a}$$

$$K_i(\vec{R}, t) = K_i(\vec{R}, t - a/L) + a \sum_l (\vec{e}_i + \vec{e}_l) Q_{il} Q_{lj}^{-1} \cdot \vec{\nabla} K_j + O(a^2/L^2) , \quad i = 1, 2, 3 . \tag{3.11b}$$

We shall now consider the K_i as functions of a rescaled coordinate $\vec{r} \equiv \vec{R}/L$. Upon setting $\delta t \equiv a/L$, Taylor expanding Eq. (3.11b) in a/L, and taking the limit of vanishing a/L, we obtain

$$\frac{\partial K_i(\vec{r}, t)}{\partial t} = \sum_j \vec{D}_{ij} \cdot \vec{\nabla} K_j , \quad i = 1, 2, 3, \tag{3.12}$$

with the set of two matrices $\vec{D}$ given by

$$\vec{D}_{ij}(K) = \sum_l (\vec{e}_i + \vec{e}_l) Q_{il} Q_{lj}^{-1} \tag{3.13}$$

In Eq. (3.13) and in what follows, K is shorthand for the triplet (K_1, K_2, K_3). The operator $\vec{\nabla}$ in Eq. (3.12) now differentiates with respect to $\vec{r}$. We shall refer to the three equations (3.12) as the *star-triangle flow equations*. They were derived without reference to the particular shape of the lattice of Fig. 2, and hence apply, just as well, to any infinite triangular lattice system in which a characteristic length L exists. For given initial conditions at $t = 0$ they determine the trajectory $(K_1(\vec{r}, t), K_2(\vec{r}, t), K_3(\vec{r}, t))$ of an inhomogeneous triangular Ising spin-$\frac{1}{2}$ system in the space of function triplets $(K_1(\vec{r}), K_2(\vec{r}), K_3(\vec{r}))$. For various reasons (e.g., in connection with inhomogeneities, temperature gradients) it may be of interest to study Eq. (3.12). However, this equation is not an RG equation. To construct RG recursion relations we must carry out one further step, which will subtly alter the form of Eq. (3.12).

B. Isotropic dilation

The above transformation from $\mathcal{H}(s)$ to $\tilde{\mathcal{H}}(s')$ relates two Hamiltonians on large but finite triangularly shaped lattices. The lattice $\tilde{\mathcal{L}}$ has the same lattice constant as $\mathcal{L}$, so that the transformation $\mathcal{H} \to \tilde{\mathcal{H}}$ preserves the density of degrees of freedom. In this way it differs crucially from an RG transformation, which dilutes the degrees of freedom. We notice that the linear dimension of the lattice $\tilde{\mathcal{L}}$ is smaller than that of $\mathcal{L}$ by a factor $(L - a)/L$. Therefore we expect to obtain a renormalization transformation by scaling $\tilde{\mathcal{L}}$ up to the size of $\mathcal{L}$. This means performing an isotropic dilation of the crossed lattice in Fig. 2 by a factor $L/(L - a)$ with respect to the origin (chosen in the center of the lattice), so that we obtain the situation of Fig. 1. This dilation carries the couplings $\tilde{K}_i$ at $\vec{R}$ to $[L/(L - a)]\vec{R}$. We therefore define the *renormalized* couplings K_i' by

$$K_i'([L/(L - a)]\vec{R}) = \tilde{K}_i(\vec{R}) . \tag{3.14}$$

We shall combine this equation with Eqs. (3.8) and (3.10). Taylor expanding in a/L and using that $\vec{\nabla} K_i'$ and $\vec{\nabla} K_i$ are equal to leading order in a/L we find for $K_i'(\vec{R})$ the expression

$$K_i'(\vec{R}) = K_i(\vec{R}) + a \sum_j \sum_l (\vec{e}_i + \vec{e}_l) Q_{il} Q_{lj}^{-1} \cdot \vec{\nabla} K_j - \frac{a}{L} \vec{R} \cdot \vec{\nabla} K_i + O\left(\frac{a^2}{L^2}\right) . \tag{3.15}$$

If we define iterates of the transformation (3.15) in analogy to Eqs. (3.11), and as before take the limit $a/L \to 0$ with $a/L = \delta t$ and $\vec{R}/L = \vec{r}$, we arrive at the differential recursion relations

$$\frac{\partial K_i(\vec{r}, t)}{\partial t} = \sum_j \vec{D}_{ij} \cdot \vec{\nabla} K_j - \vec{r} \cdot \vec{\nabla} K_i , \tag{3.16}$$

where $\vec{D}$ is given by Eq. (3.13). Equations (3.16) are our RG flow equations. They differ from the ST flow equations (3.12) by the dilation term $-\vec{r} \cdot \vec{\nabla} K_i$. The parameter t, which measures the distance along the RG trajectory, is easily shown to be related to the areal density of degrees of freedom ρ by

$$\rho(t) = \rho(0) e^{-2t} . \tag{3.17}$$

Equations (3.16) have to be solved for given initial conditions at $t = 0$, within an equilateral triangle with side of unit length centered at the origin. From their construction it is clear that Eqs. (3.16) make no sense beyond the border of this domain. Therefore, it only remains to specify the appropriate boundary conditions.

C. Boundary conditions

Let us denote the lower, right, and left boundaries of the triangular domain by the indices 1, 2, and 3, respectively. The presence of the boundaries does not affect the inverse ST transformation that carries $\mathcal{H}(s)$ to $\mathcal{H}_c(s', s)$. However, when we transform from $\mathcal{H}_c(s', s)$ to $\tilde{\mathcal{H}}(s')$, each bond $\tilde{K}_i$ on the ith edge arises from the summation on an un-

primed spin that is connected by only two bonds, a p_j and a p_k bond (i,j,k cyclic), to the rest of the lattice. If the bonds $\vec{K}_i$ on the ith edge are to be given by the same equation (3.7) which applies in the interior, we must impose the boundary conditions

$$p_i = 0 \text{ on the } i\text{th edge.} \tag{3.18}$$

From Eqs. (3.18) and (3.4) we obtain in terms of the K variables the equivalent boundary conditions

$$K_j = K_k = 0 \text{ on the } i\text{th edge, } i,j,k \text{ cyclic.} \tag{3.19}$$

These are all the boundary conditions that we can derive mathematically. In Sec. IV we shall see that for solutions of physical interest these boundary conditions necessarily lead to singular behavior near the edges and at the vertices.

IV. SOME PROPERTIES OF THE RG EQUATIONS

A. Transformation properties

The RG equations (3.16) have been formulated for the three functions K_i. On several occasions it will be useful to write them in other variables. Let in general X_j, where $j = 1, 2, 3$, be a new set of variables defined in terms of the K_i by some nonlinear transformation. Then one easily verifies that the functions X_j satisfy the RG equations

$$\frac{\partial X_i}{\partial t} = \sum_j \vec{B}_{ij} \cdot \vec{\nabla} X_j - \vec{r} \cdot \vec{\nabla} X_i , \tag{4.1}$$

with $\vec{B}_{ij}$ given by

$$\vec{B}_{ij} = \sum_l \sum_m \frac{\partial X_i}{\partial K_l} \vec{D}_{lm} \frac{\partial K_m}{\partial X_j} . \tag{4.2}$$

A special case of such a transformation is the (inverse) star-triangle transformation itself, by which we can convert the K_i into the p_j. In view of Eqs. (4.1), (4.2), and (3.10), we find for the p variables the equations

$$\frac{\partial p_i}{\partial t} = \sum_j (Q^{-1}\vec{D}Q)_{ij} \cdot \vec{\nabla} p_j - \vec{r} \cdot \vec{\nabla} p_i . \tag{4.3}$$

By inserting the explicit expression (3.13) for $\vec{D}$ we obtain the property

$$(Q^{-1}\vec{D}Q)_{ij} = \vec{D}_{ji} . \tag{4.4}$$

B. Invariant subspace $\mathcal{C}$

1. Duality

The analysis of the RG equations is facilitated if we utilize the well-known duality relation between the triangular and the hexagonal lattice. We therefore begin this subsection by a brief summary of the relevant formulas. For a detailed discussion we refer to Syozi.[17] One defines the variables $\bar{p}_j$ and $\bar{K}_j$ by

$$\sinh 2\bar{p}_j = 1/\sinh 2K_j , \tag{4.5a}$$

$$\sinh 2p_j = 1/\sinh 2\bar{K}_j , \quad j = 1, 2, 3 . \tag{4.5b}$$

The variables K_j and $\bar{p}_j$ are pairwise dual in the usual sense of duality for Ising models, and so are the variables $\bar{K}_j$ and p_j. One can show[17] that the $\bar{K}_i$ are again the ST transforms of the $\bar{p}_j$. We express the cycle of transformations between K, p, $\bar{K}$, and $\bar{p}$ in shorthand notation as

$$K = R_{ST}(p), \quad p = R_D(\bar{K}), \quad \bar{K} = R_{ST}(\bar{p}), \quad \bar{p} = R_D(K) . \tag{4.6}$$

Inspection shows that the product transformation $R_{ST}R_D$ maps high values K onto low values $\bar{K}$, and conversely. Similarly $R_D R_{ST}$ is a weak-coupling–strong-coupling transformation for the hexagonal lattice.

The star-triangle equations (3.4) take a particularly simple and useful form in terms of the new variables u_i and v_i defined by

$$u_i = \sinh 2K_j \sinh 2K_k , \tag{4.7a}$$

$$v_i = (\sinh 2p_j \sinh 2p_k)^{-1} , \quad i,j,k \text{ cyclic.} \tag{4.7b}$$

With the aid of some algebra Eqs. (3.4) can be rewritten equivalently as

$$v_i = k^2(u) u_i \tag{4.8a}$$

or

$$u_i = k^2(v) v_i , \tag{4.8b}$$

where

$$k^{-2}(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2x_3 + 2[(x_1 + x_2x_3)(x_2 + x_3x_1)(x_3 + x_1x_2)]^{1/2} . \tag{4.9}$$

We note that $k^2(u) = k^{-2}(v)$. Points in $K_1K_2K_3$ space that are invariant under $R_{ST}R_D$ will be of special interest. From Eqs. (4.5) and (4.7)–(4.9) one derives that such points satisfy the following relations, which are all equivalent,

$$K_i = \bar{K}_i , \quad i = 1, 2, 3 , \tag{4.10a}$$

$$p_i = \bar{p}_i , \quad i = 1, 2, 3 , \tag{4.10b}$$

$$u_i = v_i , \quad i = 1, 2, 3 , \tag{4.10c}$$

$$k^2 = 1 , \tag{4.10d}$$

$$u_1 + u_2 + u_3 = 1 , \tag{4.10e}$$

$$v_1 + v_2 + v_3 = 1 . \tag{4.10f}$$

2. RG equations and duality

The RG equations for the $\bar{K}_i$ can be constructed according to the general formulas given above.

For the derivatives $\partial\bar{K}_i/\partial K_j$ one finds by chain rule differentiation either via the p_k or via the $\bar{p}_k$ the expressions

$$\frac{\partial\bar{K}_i}{\partial K_j} = -\bar{Q}_{ij}\frac{\tanh 2\bar{p}_j}{\tanh 2K_j}, \tag{4.11}$$

$$= -\frac{\tanh 2\bar{K}_i}{\tanh 2p_i} Q^{-1}_{ij}, \tag{4.12}$$

respectively, where $\bar{Q}_{ij} \equiv \partial\bar{K}_i/\partial\bar{p}_j$. By symmetry one also has the analogs of Eqs. (4.11) and (4.12) that arise when all barred variables are replaced with unbarred ones, and vice versa. Upon using all these equations in Eqs. (4.1) and (4.2) with $X = \bar{K}$, and employing the expression (3.13) for $\vec{\mathrm{D}}$, one finds that the $\bar{K}_i$ satisfy the RG equations

$$\frac{\partial\bar{K}_i}{\partial t} = \sum_j \vec{\mathrm{D}}_{ij}(\bar{K})\cdot\vec{\nabla}\bar{K}_j - \vec{r}\cdot\vec{\nabla}\bar{K}_i . \tag{4.13}$$

Here the matrices $\vec{\mathrm{D}}$ have the same functional form as before, but now depend on the arguments $\bar{K}$ instead of K.[18] Hence *the* RG *flow operator commutes with the transformation* $R_{\mathrm{ST}}R_D$. In particular, if at $t=0$ we have that $K_i(\vec{r},0) = \bar{K}_i(\vec{r},0)$ for all $\vec{r}$ and for $i=1,2,3$, then this property remains true for all $t>0$. Thus by the equivalence of Eqs. (4.10a) and (4.10e), we have found a class $\mathcal{C}$ of triplets of functions, viz.,

$$\mathcal{C} = \left\{(K_1(\vec{r}), K_2(\vec{r}), K_3(\vec{r}))\,\middle|\, \sum_{i=1}^{3} u_i(\vec{r}) = 1\right\}, \tag{4.14}$$

which is invariant under the flow. We remark that Eq. (4.10e) is precisely the condition found by Houtappel and others[14] for the critical surface in K_1-K_2-K_3 space of an infinite homogeneous triangular Ising model. Therefore, the systems in class $\mathcal{C}$ are characterized by the fact that for every $\vec{r}$ they are locally critical (in Houtappel's sense), but with a critical triplet of interactions that varies with $\vec{r}$. We shall refer to relation (4.10e) as Houtappel's surface. Similarly, Eq. (4.10f), which is an alternative characterization of $\mathcal{C}$, is Houtappel's equation for the critical surface in p_1-p_2-p_3 space of an infinite homogeneous hexagonal Ising model.

The commutation of the RG flow and $R_{\mathrm{ST}}R_D$ implies that if $(K_1^*(\vec{r}), K_2^*(\vec{r}), K_3^*(\vec{r}))$ is a fixed-point solution (i.e., satisfies $\partial K_i^*(\vec{r})/\partial t = 0$), then $(\bar{K}_1^*(\vec{r}), \bar{K}_2^*(\vec{r}), \bar{K}_3^*(\mathrm{r}))$ is also a fixed point solution. Therefore fixed points come in pairs. An exception are those in class $\mathcal{C}$, which are invariant under $R_{\mathrm{ST}}R_D$. If one assumes that only one nontrivial fixed point exists, then it should be located in $\mathcal{C}$. However, duality arguments cannot help us to actually determine such a fixed point. In Sec. V we shall show by explicit verification that $\mathcal{C}$ does indeed contain a fixed-point solution.

C. Boundary conditions and duality

We can now extend our discussion of the boundary conditions. If we write Eq. (4.8b) in the form $\sinh 2K_i = k(v)/\sinh 2p_i$, we see that we have the equivalence

$$K_i = \infty \leftrightarrow p_i = 0 \tag{4.15}$$

as long as

$$0 < k^2 < \infty . \tag{4.16}$$

This restriction on k^2 is natural: the cases $k^2=0$ and $k^2=\infty$ correspond to degenerate situations where some of the interactions are infinite or zero, so that the lattice is not truly two dimensional. We shall henceforth assume that $0<k^2<\infty$ for all $\vec{r}$. Equation (4.15) then implies that the boundary conditions (3.18) are equivalent to the boundary conditions

$$K_i = \infty \text{ on the } i\text{th edge}. \tag{4.17}$$

From Eqs. (4.5) and (4.15) we see that the pair of equations (3.19) and (4.17) is invariant under the transformation $R_{\mathrm{ST}}R_D$ so that the two solutions $K(\vec{r},t)$ and $\bar{K}(\vec{r},t)$ satisfy the same boundary conditions.

A special consequence of the combined boundary conditions (3.19) and (4.17) is that, e.g., $\lim K_1(\vec{r})$ does not exist as $\vec{r}$ approaches one of the bottom vertices of the domain. This limit is infinite along the bottom edge, but is zero along the other edges. Therefore, solutions to the flow equations which in addition satisfy $0<k^2<\infty$ are necessarily highly singular in the vertices. Equation (4.16) guarantees, however, that at the vertices and near the edges such solutions still represent two-dimensional systems, albeit in the limit of extreme anisotropy. To conclude this subsection we mention that on the edges we can express condition (4.16) and the boundary conditions together in terms of the u_i as

$$u_i = 0,$$
$$0 < (u_j + u_k)^2 < \infty \text{ on the } i\text{th edge}. \tag{4.18}$$

D. Matrices $\vec{\mathrm{D}}$

In this subsection we collect some properties of the matrices $\vec{\mathrm{D}}$ that will be helpful in later calculations. We define the normal vector $\xi = (\xi_1, \xi_2, \xi_3)$ to Houtappel's surface by

$$\xi_i = A\frac{\partial}{\partial K_i}\sum_l u_l$$
$$= 2A\cosh 2K_i(\sinh 2K_j + \sinh 2K_k),\ i,j,k \text{ cyclic}, \tag{4.19}$$

where A is a normalization factor that need not be

specified. From the definition of the class $\mathcal{C}$ we have that

$$\sum_i \xi_i \vec{\nabla} K_i = 0 \quad \text{for all } K \equiv (K_1, K_2, K_3) \in \mathcal{C} . \quad (4.20)$$

The invariance of $\mathcal{C}$ under the flow can be expressed as $\sum_i \xi_i \partial K_i / \partial t = 0$ for all $K \in \mathcal{C}$. With Eqs. (3.16) and (4.20) this gives

$$\sum_i \sum_j \xi_i \vec{D}_{ij} \cdot \vec{\nabla} K_j = 0 , \quad K \in \mathcal{C} . \quad (4.21)$$

This is true if and only if ξ is a left eigenvector of both components of $\vec{D}$, i.e., if there exists a vector $\vec{\mu}$ such that

$$\sum_i \xi_i \vec{D}_{ij} = \vec{\mu} \xi_j , \quad j = 1, 2, 3; \quad K \in \mathcal{C} . \quad (4.22)$$

With the aid of the explicit expressions for ξ and $\vec{D}$ one can verify Eq. (4.22) and calculate $\vec{\mu}$. In fact, one has a stronger result which may be stated as follows. Define

$$D^l_{ij} = -\sqrt{3} \vec{D}_{ij} \cdot \vec{e}_l - \tfrac{1}{3}\sqrt{3}\delta_{ij} , \quad l = 1, 2, 3. \quad (4.23)$$

The three matrices D^l are not independent, but satisfy the relation

$$\sum_l D^l_{ij} = -\sqrt{3}\delta_{ij} . \quad (4.24)$$

Corresponding to Eq. (4.23) we have the inverse relation

$$\vec{D}_{ij} = -\tfrac{2}{3}\sqrt{3} \sum_l D^l_{ij} \vec{e}_l . \quad (4.25)$$

The stronger result is that ξ is a left eigenvector of each of the three D^l, i.e.,

$$\sum_i \xi_i D^l_{ij} = \mu_l \xi_j , \quad j = 1, 2, 3; \quad l = 1, 2, 3; \quad K \in \mathcal{C} . \quad (4.26)$$

For the eigenvalues μ_l we find in Appendix B

$$\mu_l = \tfrac{1}{2}\sqrt{3}(u_l - 1) \quad (4.27)$$

so that

$$\vec{\mu} = -\sum_l u_l \vec{e}_l . \quad (4.28)$$

In subsequent calculations it is sometimes useful to employ a vector η defined by

$$\eta_j = \sum_k Q_{jk} \xi_k . \quad (4.29)$$

In view of the special relation (4.4) between $\vec{D}$ and its transpose we have from Eqs. (4.26) and (4.29) that

$$\sum_j D^l_{ij} \eta_j = \mu_l \eta_i , \quad i = 1, 2, 3; \quad l = 1, 2, 3; \quad K \in \mathcal{C} . \quad (4.30)$$

which in Cartesian coordinates takes a form analogous to Eq. (4.22). In conclusion we remark that Eqs. (4.22)–(4.30) depend on $\vec{r}$ only through the K_i.

V. FIXED-POINT SOLUTION

A fixed-point solution of the RG flow equations (3.16) is by definition a solution $(K_1^*(\vec{r}), K_2^*(\vec{r}), K_3^*(\vec{r}))$ of

$$\sum_j \vec{D}_{ij} \cdot \vec{\nabla} K_j - \vec{r} \cdot \vec{\nabla} K_i = 0 . \quad i = 1, 2, 3, \quad (5.1)$$

which satisfies the boundary conditions (3.19) and (4.17). We note that any set of space-independent constants (K_1^0, K_2^0, K_3^0) satisfies Eqs. (5.1) but not the boundary conditions.

A nontrivial fixed point is most easily described in terms of the u_i of Eq. (4.7a) and the vector $\vec{e}_i$ of Eq. (3.2). Inspection of the boundary conditions (4.18) and the condition $u_1 + u_2 + u_3 = 1$ which defines class $\mathcal{C}$ and consideration of points of special symmetry (center, vertices, midpoints of edges) suggest a fixed point in $\mathcal{C}$ given by

$$u_i^*(\vec{r}) = \tfrac{1}{3} - 2\vec{r} \cdot \vec{e}_i , \quad i = 1, 2, 3. \quad (5.2)$$

The corresponding expression for $K_i^*(\vec{r})$ is

$$K_i^*(\vec{r}) = \tfrac{1}{2} \operatorname{arcsinh} \left(\frac{(\frac{1}{3} - 2\vec{r} \cdot \vec{e}_j)(\frac{1}{3} - 2\vec{r} \cdot \vec{e}_k)}{(\frac{1}{3} - 2\vec{r} \cdot \vec{e}_i)} \right)^{1/2} , \quad i,j,k \text{ cyclic.} \quad (5.3)$$

One easily verifies that expressions (5.2) and (5.3) satisfy their respective boundary conditions. With the aid of straightforward but tedious algebra we can show (see Appendix C) that Eq. (5.3) is indeed a solution of Eq. (5.1), so that we have found a nontrivial fixed point in $\mathcal{C}$. In Fig. 5 we show a qualitative plot of the contour lines of $K_1^*(\vec{r})$. Those of $K_2^*(\vec{r})$ and $K_3^*(\vec{r})$ are obtained by rotating the domain over $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$, due to the triangular symmetry of Eqs. (5.2) and (5.3). We see that the fixed-point solution describes a lattice which is isotropic triangular in the center of the domain, but deforms continuously away from the center,

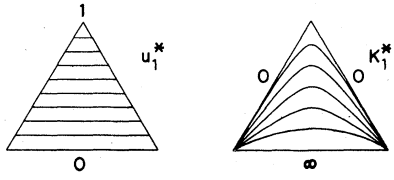

FIG. 5. Qualitative contour maps of the fixed-point functions $u_1^*(\vec{r})$ and $K_1^*(\vec{r})$.

and becomes extremely anisotropic near the edges. In particular, along a straight line from the center to a vertex, the lattice deforms from triangular to square.

We note that if we combine Eqs. (5.2) and (4.28) and use the identity

$$\vec{r} = 2 \sum_i (\vec{r} \cdot \vec{e}_i)\vec{e}_i , \tag{5.4}$$

we obtain

$$\vec{\mu}^* = \vec{r} , \tag{5.5}$$

where the asterisk denotes evaluation at the fixed point.

VI. LINEARIZED FLOW AROUND THE FIXED POINT

In RG theory the physically observable critical properties of a system are related to the behavior of the linearized flow around the fixed point. There is, in particular, a direct connection between the critical exponents y of a system and the eigenvalues λ of the linearized transformation. For the case of a discrete RG transformation with rescaling factor b the connection

$$\lambda = b^y \tag{6.1}$$

is well known.[19] To see how the y arise in the present case we consider a small deviation $\psi_i(\vec{r})$ from the fixed-point solution $K_i^*(\vec{r})$. It is a straightforward matter to linearize the RG flow equations (3.16) about K^*. After rearranging indices we find that ψ evolves according to the equation

$$\frac{\partial \psi_i}{\partial t} = \sum_j T_{ij}^*(\vec{r}, \vec{\nabla})\psi_j , \tag{6.2}$$

with the operator T^* given by

$$T_{ij}^* = (\vec{D}_{ij}^* - \vec{r}\delta_{ij}) \cdot \vec{\nabla} + \sum_l \left(\frac{\partial \vec{D}_{il}}{\partial K_j}\right)^* \cdot \vec{\nabla} K_l^* . \tag{6.3}$$

The asterisk denotes again evaluation at the fixed point. One notices that T^* is the sum of a differential operator and an ordinary matrix. If T^* acts during ∇t, the result is that an initial deviation $\psi_i(\vec{r}, 0)$ is mapped onto a deviation $\psi_i(\vec{r}, \Delta t)$ given by

$$\psi_i(\vec{r}, \Delta t) = \sum_j (\delta_{ij} + \Delta t\, T_{ij}^*)\psi_j(\vec{r}, 0) + O(\Delta t^2), \tag{6.4}$$

which corresponds to a transformation with rescaling factor $b = 1 + \Delta t$. By using this value of b in Eq. (6.1) we see that the eigenvalues λ of the linear operator in Eq. (6.4) have the form $\lambda = 1 + \Delta t y$. It follows that the critical exponents y are the eigenvalues of T^*, a relation of a type well known for differential RG equations.[20]

The analysis of the eigenvalues and eigenfunctions of T^* is simplified by our knowledge of the invariant subspace $\mathcal{C}$ discussed in Sec. IV. In terms of the variables ψ we have that the linear subspace tangent to $\mathcal{C}$ in K^* and defined by

$$\sum_i \xi_i^*(\vec{r})\psi_i(\vec{r}) = 0 \quad \text{for all } \vec{r}, \tag{6.5}$$

is invariant under T^*. Let us introduce the scalar product

$$(\varphi, \psi) = \int_\Delta d\vec{r} \sum_i \varphi_i(\vec{r})\psi_i(\vec{r}) , \tag{6.6}$$

where the integral is on the triangular domain. The subspace of functions φ such that

$$\varphi_i(\vec{r}) = f(\vec{r})\xi_i^*(\vec{r}) , \tag{6.7}$$

where f is arbitrary, is then orthogonal to the subspace characterized by Eq. (6.5), and hence is invariant under the adjoint operator $\tilde{T}^*$. On the basis of general RG arguments we expect the fixed-point solution to be stable within the space $\mathcal{C}$, but unstable in directions away from it. We are first of all interested in the behavior of the flow in the unstable (temperaturelike) directions, which for the adjoint flow problem correspond to vectors φ of the type (6.7). Therefore we consider the eigenvalue problem

$$\sum_i \tilde{T}_{ji}^* f \xi_i^* = y f \xi_j^* , \quad j = 1, 2, 3, \tag{6.8}$$

which is a set of three necessarily equivalent equations for the scalar function $f(\vec{r})$. The expression for $\tilde{T}^*$ is obtained via an integration by parts (see Appendix D, where also the conditions for the boundary terms to vanish are investigated). Substitution of the result in Eq. (6.8) gives us the eigenvalue equation

$$y f \xi_j^* = -\sum_i \vec{\nabla} \cdot [f \xi_i^* (\vec{D}_{ij}^* - \vec{r}\delta_{ij})] + \sum_i f \xi_i^* \sum_l \left(\frac{\partial \vec{D}_{il}}{\partial K_j}\right)^* \cdot \vec{\nabla} K_l^* . \tag{6.9}$$

If we now employ Eqs. (4.22) and (5.5), we find that the first member on the right-hand side of Eq. (6.9) vanishes. The remaining eigenvalue equation is trivially solved. Labeling the eigenfunctions by a parameter $\vec{\rho}$ we can write

$$f_{\vec{\rho}}(\vec{r}) = \delta(\vec{r} - \vec{\rho}) . \tag{6.10}$$

The corresponding eigenvalues are given by

$$y_{\vec{\rho}} = \sum_i \frac{\xi_i^*}{\xi_j^*} \sum_l \left(\frac{\partial \vec{D}_{il}}{\partial K_j}\right)^* \cdot \vec{\nabla} K_l^* \Big|_{\vec{r}=\vec{\rho}} , \tag{6.11}$$

which should be independent of j. Upon inserting

the explicit expression for $\vec{D}$ one can show (see Appendix E) that

$$y_{\vec{p}} = 1 \tag{6.12}$$

for all $\vec{p}$. We conclude that in the temperature direction there is an infinitely degenerate eigenvalue $y_{\vec{p}} \equiv y_T = 1$. The value 1 agrees with the exact results first obtained by Onsager[13] and Houtappel,[14] viz, $\alpha = 2 - d/y_T = 0$ (where $d = 2$ is the dimension of the system). In fact, Eq. (6.12) is a statement of universality, since for different values of $\vec{p}$ the fixed-point solution (5.3) describes different critical systems, which by (6.12) all have the exponent $y_T = 1$.

VII. FREE ENERGY

Quite generally a differential RG equation leads to an expression for the free energy as an integral along an RG trajectory.[21] In this section we shall derive the formulas appropriate to our case.

The equivalence of the Hamiltonians of Eqs. (3.1) and (3.6) means that

$$\sum_s e^{\mathcal{H}(s)} = \sum_{s'} e^{\tilde{\mathcal{H}}(s')} = \sum_{s'} e^{\mathcal{H}'(s')}. \tag{7.1}$$

The second equality holds trivially because $\tilde{\mathcal{H}}$ and $\mathcal{H}'$ describe the same system, the difference consisting only in the labeling of the interaction constants. We define $F_L\{K(\vec{R})\}$ as the reduced free energy (i.e., the logarithm of the partition function) of a Hamiltonian with couplings $K_i(\vec{R})$ and with zero spin-independent term, defined on a triangular lattice with side of length L. By taking the logarithm of Eq. (7.1) and using the explicit expressions for $\mathcal{H}$ and $\tilde{\mathcal{H}}$, we obtain

$$F_L\{K(\vec{R})\} - F_{L-a}\{\tilde{K}(\vec{R})\} = (a/L) G_L\{K(\vec{R})\}, \tag{7.2}$$

where F_L refers to the lattice of circles ($\mathfrak{L}$) and F_{L-a} to the lattice of crosses ($\tilde{\mathfrak{L}}$), and where

$$\begin{aligned}\frac{a}{L} G_L\{K(\vec{R})\} &\equiv \sum_{\vec{R}}{}^{0}\, g(p_1(\vec{R}+a\vec{e}_1), p_2(\vec{R}+a\vec{e}_2), p_3(\vec{R}+a\vec{e}_3)) \\ &\quad - \sum_{\vec{R}}{}^{\times}\, g(p_1(\vec{R}), p_2(\vec{R}), p_3(\vec{R})).\end{aligned} \tag{7.3}$$

Here $\sum^0$ runs through all sites of $\mathfrak{L}$ and $\sum^\times$ through all sites of $\tilde{\mathfrak{L}}$. Equation (7.2) is an exact difference equation for the unknown free-energy functional F_L. We shall derive a differential equation from it by expansion in powers of aL, as in Sec. III. In the limit $a/L \to 0$ the interactions $K_i(\vec{R})$ converge to continuous functions which, after scaling the coordinates down to the domain of the unit triangle, we denoted as $K_i(\vec{r})$. We shall assume here that F_L has an asymptotic expansion of the type

$$F_L\{K(\vec{R})\} = \frac{1}{2}\left(\frac{L}{a}\right)^2 f\{K(\vec{r})\} + \frac{L}{a} f_1\{K(\vec{r})\} + O(1). \tag{7.4a}$$

Here f is the free energy per spin averaged over the triangular domain. The analogous expansion for F_{L-a} is

$$\begin{aligned}&F_{L-a}\{\tilde{K}(\vec{R})\} \\ &\quad = \frac{1}{2}\left(\frac{L-a}{a}\right)^2 f\{K'(\vec{r})\} + \frac{L-a}{a} f_1\{K'(\vec{r})\} + O(1),\end{aligned} \tag{7.4b}$$

where we used that in rescaled coordinates the $\tilde{K}_i$ and K_i' are the same functions. Let in a way analogous to Eqs. (7.4) the functional $c\{K(\vec{r})\}$ be defined by

$$G_L\{K(\vec{R})\} = \frac{1}{2}\left(\frac{L}{a}\right)^2 c\{K(\vec{r})\} + O(L/a). \tag{7.5}$$

We substitute Eqs. (7.4) and (7.5) into Eq. (7.2), divide by $\frac{1}{2}(L/a)$, and take the limit $a/L \to 0$. Taking into account that $K'(\vec{r}, t) = K(\vec{r}, t + a/L)$, we obtain

$$\frac{\partial f}{\partial t} = 2f - c, \tag{7.6}$$

which is the desired differential equation for the average free energy per spin. The solution

$$\begin{aligned}&f\{K(\vec{r}, t)\} \\ &\quad = \int_t^\infty e^{2(t-\tau)}[c\{K(\vec{r}, \tau)\} - 2f\{K(\vec{r}), \infty)\}]\, d\tau \\ &\qquad + f\{K(\vec{r}, \infty)\}\end{aligned} \tag{7.7}$$

reduces for $t = 0$ the free energy of the initially given system.

From Eq. (7.3) and the definition (7.5) we can find an explicit expression for the inhomogeneous term c in Eq. (7.6). We rescale the spatial coordinates in the first sum of Eq. (7.3) by a factor $1/L$ and those in the second sum by $1/(L-a)$. Both sums can then be converted into integrals over the unit triangle. Expanding in a/L, as in Sec. III, we find that the leading order terms cancel. We assume without rigorous justification that a similar cancellation occurs for the errors that we make in passing from sums to integrals. The result, then, is that $c\{K(\vec{r})\}$ takes the form

$$c\{K(\vec{r})\} = \tfrac{4}{3}\sqrt{3} \int_\Delta d\vec{r} \left(2g + \vec{r}\cdot\vec{\nabla} g + \sum_l Q_{mn}\vec{e}_l \cdot \vec{\nabla} p_l\right), \tag{7.8}$$

where we used that $\partial g/\partial p_l = Q_{mn}$ with l, m, n cyclic.

Substitution of Eq. (7.8) into Eq. (7.7) yields our final result, namely, an expression for the free energy as the trajectory integral of an explicitly given function. The integral can in principle be calculated once the trajectory $K(\vec{r},t)$ has been solved from the RG equations. As a special case the fixed-point free energy follows trivially from Eq. (7.7) as

$$f\{K^*(\vec{r})\} = \tfrac{1}{2} c\{K^*(\vec{r})\} . \tag{7.9}$$

Finally, the additivity of the free energy guarantees that for arbitrary initial state $K(\vec{r})$ the expression for f obtained from (7.7) and (7.8) should be equal to the average over the domain of Houtappel's free energy $f_H(K_1,K_2,K_3)$ for a homogeneous system,[14] i.e.,

$$f\{K(\vec{r})\} = \tfrac{4}{3}\sqrt{3} \int_{\Delta} d\vec{r}\, f_H(K_1(\vec{r}),K_2(\vec{r}),K_3(\vec{r})). \tag{7.10}$$

Thus one obtains in principle the Onsager and Houtappel expressions for the free energy by evaluating Eq. (7.7) for a homogeneous initial state.

VIII. CONCLUSION

We have constructed exact RG equations, Eqs. (3.16), for the triangular spin-$\frac{1}{2}$ Ising model with three nearest-neighbor interactions K_i. The very existence of such equations is a remarkable fact for which there was no *a priori* evidence. Perhaps the most unexpected feature that arises in the derivation of the equations is that the interactions have to be considered as functions of a spatial coordinate $\vec{r}$, confined to a triangular domain. The flow operator $\vec{D}$ in the RG equations results from a local transformation of the lattice, based on the standard star-triangle transformation. As a consequence, the matrices $\vec{D}_{ij}$, defined by Eq. (3.13) together with Eqs. (3.9) and (3.4), are nonlinear but analytic functions of the interactions. The RG equations resemble classical equations of motion, as was hypothesized by Wilson[1] in an early discussion of the renormalization group.

An analytic solution of the RG equations seems out of reach, except possibly for certain special initial states. However, we have determined a nontrivial fixed-point solution, Eq. (5.3), which describes a particular space-dependent state. The fixed point lies in an invariant critical subspace $\mathcal{C}$, defined by Eq. (4.14), which is most easily located by a duality argument. We showed the fixed point to be unstable in the temperature-like directions (i.e., the directions away from $\mathcal{C}$) with eigenvalue $y_T = 1$, in agreement with the classical results of Onsager and Houtappel. The infinite degeneracy of this eigenvalue is an expression of universality within the class of anisotropic nearest-neighbor Hamiltonians on the triangular lattice.

Our approach has the characteristics of all RG methods: it focuses on the critical properties of the system, which are the first to emerge from the calculation. Only in the Sec. VII did we consider the free energy over the whole temperature range, and expressed it as the trajectory integral of an explicitly given function. This integral expression is just a starting point. Further investigation might show that in certain special cases it can be evaluated analytically.

Even our treatment of the fixed-point properties of the equations has not been complete. In particular, one would like to know the stability properties of the fixed point (5.3) with respect to perturbations in the critical subspace.[22] Also, one would like to study the flow around any other fixed points that can be found; of special interest are the two trivial ones at $K^* = 0$ and $K^* = \infty$, which one expects to be the attractors of the flow outside the critical subspace.

Many questions of a mathematical nature remain. First, the validity of the differential equations rests upon certain smoothness conditions on the functions K_i and p_j. It would be good to have a better idea of these conditions, especially in view of the singular behavior enforced by the boundary conditions. Second, once the equations are established, it remains to rigorously determine their properties. The mathematical literature offers few results that are ready to apply. It would surely be interesting to have some general statements on the evolution of given initial states. Especially, an initial state that is subcritical for some $\vec{r}$ and supercritical for other $\vec{r}$ (in the sense of Sec. IV B2) will supposedly be the object of competition between the two trivial fixed points. An unanswered question is whether such competition will lead to the appearance of discontinuities in the solution, a phenomenon known to occur quite generally for hyperbolic equations.

Finally we would like to remark on a few extensions of our work that we have considered, and on some further connections. It is natural to attempt to construct RG equations for a magnetic field term added to the Hamiltonian. However, we have not succeeded in finding a finite set of closed equations for the odd interactions. On the other hand, several other models can be solved by our methods. The seemingly trivial one-dimensional Ising model is instructive, as not suprisingly the RG transformation can be carried out in full detail, even in the presence of a magnetic field, and the free energy and correlation functions can be calculated.[23] In one dimension the RG procedure

leaves one with an extra degree of freedom per lattice site, which one can exploit by demanding that the Hamiltonians $\mathcal{H}$ and $\mathcal{H}'$ (but not $\mathcal{H}_c$) be translationally invariant. Another instructive case is the two-dimensional Gaussian model, which will be discussed in a separate publication.[24]

At first sight application of our method to the square Ising lattice does not seem possible. We should like, however, to draw attention to a remarkable paper by Baxter and Enting,[25] who calculate the partition function of the homogeneous Ising model using only the star-triangle transformation. In their argument these authors employ a transformation that carries a hexagonal lattice to a square lattice. It appears that similar lattice restructuring transformations can be used to convert a not necessarily homogeneous square lattice, and probably any planar lattice, into a triangular lattice. Via this indirect way the methods of this paper also apply to other two-dimensional lattices.

The points of view of Baxter and Enting's paper on the one hand, and of our RG approach on the other, are strongly divergent. The important common feature is that both demonstrate that the star-triangle transformation contains all that is needed to determine the free energy. One would guess, therefore, that RG flow equations will exist for any system to which a star-triangle transformation applies. Whether one can also find such equations for other systems is an open question.

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APPENDIX A: THE MATRIX Q

It is easy to verify that Q is a symmetric matrix given by

$$Q_{ii}=\tfrac{1}{4}[\tanh(p_1+p_2+p_3)-\tanh(-p_1+p_2+p_3)-\tanh(p_1-p_2+p_3)-\tanh(p_1+p_2-p_3)],\quad i=1,2,3, \tag{A1}$$

$$Q_{ij}=\tfrac{1}{4}[\tanh(p_i+p_j+p_k)+\tanh(-p_i+p_j+p_k)+\tanh(p_i-p_j+p_k)-\tanh(p_i+p_j-p_k)],\quad i,j,k \text{ cyclic}. \tag{A2}$$

Although Eqs. (A1) and (A2) express the matrix Q in terms of p_1, p_2, p_3, we may equally well think of it as a function of K_1, K_2, K_3.

It is convenient to express the matrix elements of Q in terms of the quantities $\sinh 2p_i$ and $\cosh 2p_i$. This is most easily done if one multiplies Eqs. (A1) and (A2) by a quantity C defined as

$$C=4\cosh(p_1+p_2+p_3)\cosh(-p_1+p_2+p_3)\times\cosh(p_1-p_2+p_3)\cosh(p_1+p_2-p_3). \tag{A3}$$

After applying the appropriate formulas for sums and products of hyperbolic functions, one then finds

$$Q_0\equiv Q_{ii}=-\sinh 2p_1\sinh 2p_2\sinh 2p_3/C, \tag{A4a}$$

$$Q_{jk}=\sinh 2p_i(\cosh 2p_i+\cosh 2p_j\cosh 2p_k)/C,\quad i,j,k \text{ cyclic}, \tag{A4b}$$

where C can be rewritten

$$C=\cosh^2 2p_1+\cosh^2 2p_2+\cosh^2 2p_3+2\cosh 2p_1\cosh 2p_2\cosh 2p_3-1. \tag{A5}$$

The matrix Q^{-1} is obtained from Q by an elementary matrix inversion. One finds

$$Q^{-1}_{ii}=(Q_0^2-Q_{jk}^2)/\det Q,\quad i=1,2,3, \tag{A6a}$$

$$Q^{-1}_{ij}=Q^{-1}_{ji}=(Q_{ik}Q_{jk}-Q_0Q_{ij})/\det Q,\quad i,j,k \text{ cyclic}, \tag{A6b}$$

where we used the notation defined in Eq. (A4a).

In the critical surface, defined by each one of the relations (4.10), the above expressions are simplified. Useful relationships that follow directly from Eqs. (4.7b) and (4.10f) are

$$\sinh 2p_i=(\sinh 2p_j+\sinh 2p_k)/(\sinh 2p_j\sinh 2p_k-1), \tag{A7a}$$

$$\cosh 2p_i=\cosh 2p_j\cosh 2p_k/(\sinh 2p_j\sinh 2p_k-1),\quad i,j,k \text{ cyclic}. \tag{A7b}$$

Using these relations we obtain at criticality the expressions

$$C=1/Q_0^2, \tag{A8}$$

$$Q_{jk}=-Q_0\cosh 2p_i,\quad i,j,k \text{ cyclic}, \tag{A9}$$

$$\det Q=-Q_0, \tag{A10}$$

with

$$Q_0 = Q_{ii} = -1/(\sinh 2p_1 \sinh 2p_2 \sinh 2p_3)$$
$$= -1/(\sinh 2p_1 + \sinh 2p_2 + \sinh 2p_3)\,. \quad \text{(A11)}$$

For the matrix elements of Q^{-1} we find

$$Q^{-1}_{ii} = Q_0 \sinh^2 2p_i\,, \quad \text{(A12a)}$$

$$Q^{-1}_{jk} = -Q_0 \cosh 2p_i \sinh 2p_j \sinh 2p_k\,,$$
$$i,j,k \text{ cyclic}. \quad \text{(A12b)}$$

APPENDIX B: THE MATRICES D^l

We consider the matrices D defined by Eq. (4.23) for $l = 1, 2, 3$. With the aid of Eq. (3.13) and the scalar product relation

$$\vec{e}_i \cdot \vec{e}_l = \tfrac{1}{6}(3\delta_{il} - 1)\,, \quad \text{(B1)}$$

we find that

$$D^l_{ij} = -\tfrac{1}{2}\sqrt{3}(Q_{il}Q^{-1}_{lj} + \delta_{il}\delta_{lj})\,. \quad \text{(B2)}$$

By straightforward diagonalization one finds that D^l has the three eigenvalues

$$\mu_{l\pm} = -\tfrac{1}{2}\sqrt{3}\,[1 \mp (Q_{ll}Q^{-1}_{ll})^{1/2}]\,, \quad \text{(B3)}$$

$$\mu_{l0} = 0. \quad \text{(B4)}$$

At criticality we obtain from Eqs. (B3), (A6a), (A10), (A9), (A4a), and (A11) the result $\mu_{l+} \equiv \mu_l = -\tfrac{1}{2}\sqrt{3}[1 - (\sinh 2p_m \sinh 2p_n)^{-1}]$, with l, m, n cyclic. From this one gets Eq. (4.27) by employing the critical surface relations between K and p,

$$\sinh 2K_i = 1/\sinh 2p_i, \quad K \in \mathcal{C}, \quad \text{(B5a)}$$

$$\cosh 2K_i = 1/\tanh 2p_i, \quad K \in \mathcal{C}, \quad \text{(B5b)}$$

which follow directly from Eqs. (4.7) and (4.10c). One can now show explicitly that in the critical surface Eq. (4.26) holds, i.e., that the eigenvectors of the D^l belonging to μ_l are independent of l and equal to the vector ξ defined in Eq. (4.19). This is most easily verified in terms of the variables $\sinh 2p_i$ and $\cosh 2p_i$. Useful relations are again Eqs. (A7) and (B5). We remark that the eigenvectors belonging to μ_{l-} and μ_{l0} do depend on l, so that the matrices D^l do not commute.

APPENDIX C: THE FIXED-POINT SOLUTION

In this Appendix we show that the quantity I_i defined by

$$I_i = \sum_j \vec{D}_{ij}(K^*) \cdot \vec{\nabla} K^*_j - \vec{r} \cdot \vec{\nabla} K^*_i \quad \text{(C1)}$$

vanishes, which means that the functions $K^*_i(\vec{r})$ given by Eq. (5.3) satisfy the fixed-point equation (5.1). We shall use the abbreviations

$$S_i = \sinh 2K_i, \quad C_i = \cosh 2K_i, \quad \text{(C2a)}$$

$$S'_i = \sinh 2p_i, \quad C'_i = \cosh 2p_i. \quad \text{(C2b)}$$

From Eq. (5.2) we find the relation $\vec{\nabla} u^*_l = -2\vec{e}_l$, whence we have via chain rule differentiation

$$\vec{\nabla} K^*_j = -\frac{1}{C_j} \sum_l \frac{\partial S_l}{\partial u_l} \vec{e}_l\,. \quad \text{(C3)}$$

In this equation as well as in the remaining ones of this Appendix, all quantities are understood to be evaluated at the fixed point. We substitute (C3) in (C1) and eliminate $\vec{r} \cdot \vec{e}_l$ in favor of u^*_l by means of Eq. (5.2). We furthermore eliminate $\vec{D} \cdot \vec{e}_l$ with the aid of the definition (4.23) and substitution of the explicit expression (B2) for D^l. The result is that we find

$$2I_i = \sum_l \sum_j C_i Q_{il} Q^{-1}_{lj} \frac{1}{C_j} \frac{\partial S_l}{\partial u_l} + \frac{\partial S_i}{\partial u_i}$$
$$- \sum_l \frac{\partial S_l}{\partial u_l} + \sum_l u_l \frac{\partial S_l}{\partial u_l}\,. \quad \text{(C4)}$$

In order to evaluate this expression we use

$$\frac{\partial S_i}{\partial u_l} = \tfrac{1}{2} S_i (1 - 2\delta_{il})/u_l\,, \quad \text{(C5)}$$

which follows from the definitions (C2a) and (4.7a). For the last terms in Eq. (C4) we thus find

$$\frac{\partial S_i}{\partial u_i} - \sum_l \frac{\partial S_l}{\partial u_l} + \sum_l u_l \frac{\partial S_l}{\partial u_l} = \frac{1}{2}\left(S_i - \frac{1}{S_j} - \frac{1}{S_k}\right),$$
$$i,j,k \text{ cyclic}\,. \quad \text{(C6)}$$

In the first term of Eq. (C4) we use the explicit expressions (A9), (A11), and (A12) for the matrix elements of Q and Q^{-1}. The critical surface relation (4.10f) and the derived expressions (A7) are again used several times. Relations (B5) serve to convert the variables C_i, S_i and C'_i, S'_i into each other. Via straightforward algebra one obtains the intermediate result

$$\sum_j Q^{-1}_{lj} \frac{1}{C_j} \frac{\partial S_l}{\partial u_l} = \frac{1}{2}\left(\frac{S'_l}{C'_l} + C'_n \frac{S'_m}{C'_m} + C'_m \frac{S'_n}{C'_n}\right),$$
$$l, m, n \text{ cyclic} \quad \text{(C7)}$$

and finally shows that I_i vanishes.

APPENDIX D: THE ADJOINT OPERATOR $\tilde{T}^*$

The adjoint operator $\tilde{T}^*$ is defined by $(\tilde{T}^*\varphi, \psi) = (\varphi, T^*\psi)$ for all φ and ψ. Upon employing expression (6.3) for T^* and the definition (6.6) of the

scalar product, and performing an integration by parts, we obtain

$$(\varphi, T^*\psi) = \int_\Delta d\vec{r} \sum_i \sum_j \Big[-\vec{\nabla} \cdot [\varphi_i (\vec{D}^*_{ij} - \vec{r}\delta_{ij})]\psi_j + \varphi_i \sum_l \left(\frac{\partial \vec{D}_{ij}}{\partial K_l}\right)^* \cdot \vec{\nabla} K^*_l \psi_j \Big] + \int_{\partial\Delta} dl \sum_i \sum_j \varphi_i (\vec{D}^*_{ij} - \vec{r}\delta_{ij}) \cdot \hat{n} \psi_j \cdot \quad \text{(D1)}$$

The second integral in Eq. (D1), which we shall denote by I_B, is a line integral along the boundary $\partial\Delta$ of the domain, $\hat{n}$ being the outward pointing normal vector. From the first integral we obtain the expression for $\tilde{T}^*$ used in Eq. (6.9), provided that I_B vanishes. We shall investigate the conditions for this to happen. To simplify I_B we use the relation $\hat{n} = \sqrt{3}\vec{e}_l$ on the l-th border $\partial_l\Delta$, and employ Eqs. (4.23), (B2), and (5.2) for $\vec{D} \cdot \vec{e}_l$, D^l, and $\vec{r} \cdot \vec{e}_l$, respectively. The result that we find is

$$I_B = \sum_l \int_{\partial_l\Delta} dl \sum_i \sum_j \varphi_i [Q^*_{il} Q^{-1}_{lj}{}^* + \delta_{il}\delta_{lj} - (1 - u^*_l)\delta_{ij}]\psi_j . \quad \text{(D2)}$$

It is convenient to define the triangular coordinate

$$x_l = \tfrac{1}{6}\sqrt{3} - \sqrt{3}\,\vec{r} \cdot \vec{e}_l , \quad \text{(D3)}$$

which gives the distance of $\vec{r} \in \Delta$ to the l-th edge. One has from Eqs. (D3) and (5.2)

$$u^*_l = \tfrac{2}{3}\sqrt{3} x_l . \quad \text{(D4)}$$

As $\vec{r}$ approaches the l-th edge ($x_l \to 0$), u^*_l tends to zero, whereas u^*_m and u^*_n assume finite nonzero values (l, m, n cyclic). From the equations of Appendix A and Eqs. (4.7) and (4.10c) we can obtain, via the u^*_i, the limiting behavior of the matrices Q^* and $Q^{-1}{}^*$ as $x_l \to 0$. Using these results in Eq. (D2), we finally find that the integrand along the lth border vanishes provided φ and ψ belong to the space of functions χ such that

$$x_l^{1/2}\chi_l \to 0, \quad \chi_m \to 0, \quad \chi_n \to 0 \ \text{ as } x_l \to 0. \quad \text{(D5)}$$

In terms of the deviations $\delta u_i \equiv u_i - u^*_i$ we have the equivalent conditions

$$x_l^{-1/2}\delta u_l \to 0, \quad x_l^{1/2}\delta u_m \to 0, \quad x_l^{1/2}\delta u_n \to 0 \ \text{ as } x_l \to 0, \quad \text{(D6)}$$

which follow from Eq. (D5) and relations (C2a) and (C5) between K and u. For deviations ψ of the special type (6.7) we can replace Eq. (D5) by a boundary condition on the function f. Using Eq. (4.19) we find

$$A\{K^*(\vec{r})\} \sinh 2K^*_l(\vec{r}) f(\vec{r}) \to 0 \ \text{ as } x_l \to 0, \quad \text{(D7)}$$

a relation which is indeed satisfied by the solutions (6.10).

APPENDIX E: THE CRITICAL EXPONENT y_T

We evaluate the right-hand side of Eq. (6.11) by expressing all quantities involved in terms of the functions S'_i and C'_i defined by Eq. (C2b). The expression for ξ^*_i/ξ^*_j follows from Eqs. (4.19) and (B5); the expression for $\vec{\nabla}K^*_l$ follows from Eqs. (C3), (C5), (4.7a), and (B5); the expression for $\vec{D}_{il}$ follows from the definition (3.13) and the formulas of Appendix A. We convert $\partial\vec{D}_{il}/\partial K_j$ into derivatives of $\vec{D}_{il}$ with respect to the p_m by chain rule differentiation. Employing Eq. (B1) and performing the summation that occurs in the definition of $\vec{D}_{il}$, we find

$$y_{\vec{p}} = -\frac{1}{4}\sum_i \frac{C'_i(S'_{i'} + S'_{i''})}{C'_j(S'_{j'} + S'_{j''})} \sum_l \frac{1}{C'_l} \sum_n S'_{n'}S'_{n''}(1 - 2\delta_{ln}) \times \sum_m \frac{\partial}{\partial p_m}(Q_{in}Q^{-1}_{nl})Q^{-1}_{jm}, \quad \text{(E1)}$$

where (i, i', i''), etc., are cyclic triplets, and where it is understood that the right-hand side has to be evaluated at the fixed point $K^*(\vec{p})$. We remark that the right-hand side contains no explicit $\vec{p}$ dependence. For a further evaluation we need the p derivatives of the matrices Q and Q^{-1}, taken on the critical surface. By straightforward calculation we find, on the critical surface,

$$\frac{\partial Q}{\partial p_1} = \frac{2}{C}\begin{bmatrix} C'_1(2 - S'_2S'_3) & C'_2(2 - S'_3S'_1) & C'_3(2 - S'_1S'_2) \\ C'_2(2 - S'_3S'_1) & C'_1(2 - S'_2S'_3) & S'_1S'_2 + S'_2S'_3 + S'_3S'_1 - 2 \\ C'_3(2 - S'_1S'_2) & S'_1S'_2 + S'_2S'_3 + S'_3S'_1 - 2 & C'_1(2 - S'_2S'_3) \end{bmatrix} \quad \text{(E2)}$$

and

$$\frac{\partial Q^{-1}}{\partial p_1} = -\frac{2}{C}\begin{bmatrix} C'_1S'^2_1S'_2S'_3 & 0 & 0 \\ 0 & -C'_1S'^3_2S'_3 & S'^2_2S'^2_3 \\ 0 & S'^2_2S'^2_3 & -C'_1S'_2S'^3_3 \end{bmatrix}, \quad \text{(E3)}$$

where C is given by Eqs. (A8) and (A11). The derivatives with respect to p_2 and p_3 follow from Eqs. (E2) and (E8) by cyclic symmetry. The evaluation of $y_{\vec{p}}$ from Eq. (E1) with the aid of Eqs. (E2) and (E3) is a matter of tedious but straightforward algebra. In the calculations one frequently uses Eqs. (A7b) to eliminate products and ratios of the C_i', and the critical surface relation (4.10f) in the form $S_1'+S_2'+S_3'=S_1'S_2'S_3'$. In the end the result (6.12) is obtained.

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