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Renormalization-group transformation for quantum lattice systems at zero temperature

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We propose a new renormalization-group (RG) method applicable to quantum systems defined on a lattice at zero temperature. We apply our method to the one-dimensional Ising model in a transverse field and the two-dimensional triangular Ising model in a transverse field. In both cases, a nontrivial fixed point of our RG transformation is found. Critical exponents are calculated and compared with exact and previous RG results.

I. INTRODUCTION

In recent years there has been considerable interest in applying real-space renormalization-group (RG) methods to quantum systems defined on a lattice.[1–8] The pioneering work in this area was carried out by Wilson in his study of the Kondo problem.[2] Since this work, a great deal of effort has focused on quantum spin systems in d dimensions which map onto $(d+1)$-dimensional classical systems.[3,4,9] In particular, there have been a number of studies of the Ising model in a transverse field at zero temperature.[5–8] Our work is closest in spirit to the work of Drell *et al.*[5] Their idea was to divide the system into cells, determine the two lowest-lying energy eigenstates of the intracell Hamiltonian and use only the matrix elements of the Hamiltonian between these lowest states in mapping onto a new Hamiltonian. This idea is intuitively appealing, but it is not clear from their work how to systematize and improve their method. In this paper, we will show how to make contact with these ideas and how to systematically extend them.

This work has developed from the work carried out by one of us (GM) in the area of critical dynamics.[10] The relationship between the two problems centers about RG transformations mapping operators onto operators. A development similar to that in Sec. II of this paper has also been established in the critical dynamics case and will be discussed elsewhere.

In Sec. II and III of this paper we discuss a general formalism for developing RG transformations for quantum systems on a lattice. In Sec. IV we apply this method to one- and two-dimensional Ising models in a transverse field. Finally, in Sec. V, we end with a short discussion.

II. FORMAL DEVELOPMENT

Consider a quantum-mechanical Hamiltonian H_σ governing the behavior of a set of operators σ_m located on a lattice specified by the set of points $\{m\}$. We want to map that part of this Hamiltonian that corresponds to the lowest-energy states onto a new Hamiltonian H_μ describing the behavior of a new set of operators μ_p defined on a new lattice specified by the set of points $\{p\}$. This new lattice has a lattice spacing s (>1) times larger than the spacing of the original σ lattice. Physically we think of the index p as labeling cells which include s^d (d is the spatial dimension) sites on the original σ lattice. With a proper choice of mapping this transformation $H_\sigma \rightarrow H_\mu$ constitutes a renormalization-group transformation of the real-space variety.

We carry out this transformation by introducing a quantity $T[\mu|\sigma]$ which is a product of vectors in both the Hilbert spaces of the σ and μ operators and which satisfies the equation

$$H_\sigma T[\mu|\sigma] = H_\mu T[\mu|\sigma] \ , \tag{2.1}$$

with the normalization on the σ variables

$$\mathrm{Tr}_\sigma T[\mu|\sigma] T^\dagger[\mu|\sigma] = 1 \ , \tag{2.2}$$

where Tr_σ indicates the trace over the σ degrees of freedom. We can obtain an explicit expression for the new renormalized Hamiltonian H_μ by multiplying Eq. (2.1) on the right by $T^\dagger[\mu|\sigma]$, tracing over the σ degrees of freedom and using Eq. (2.2),

$$H_\mu = \mathrm{Tr}_\sigma H_\sigma T[\mu|\sigma] T^\dagger[\mu|\sigma] \ . \tag{2.3}$$

If $T[\mu|\sigma]$ is constructed to have components along the ground and first excited states of H_σ then, as we show in Secs. II and III, the ground-state and first excited-states energies of H_μ are the same as those for H_σ. Thus, the ground-state and first excited-state energies are preserved under our RG transformation.

Within our present prescription there is still a great deal of freedom in choosing the form of $T[\mu|\sigma]$. In practice one chooses $T[\mu|\sigma]$ such that H_μ "looks" as much as possible like H_σ. Thus far, we have succeeded in constructing useful transformations $T[\mu|\sigma]$ using perturbation-theory expansions where one breaks H_σ into an intracell piece H_σ^0 and an intercell coupling V_σ,

$$H_\sigma = H_\sigma^0 + \lambda V_\sigma \tag{2.4}$$

and then treating the λ term as a small perturbation. Clearly the initial problem is the choice of the appropriate $T_0[\mu|\sigma]$ corresponding to a given H_σ^0. We discuss the choice of $T_0[\mu|\sigma]$ in Sec. III. In this section we assume T_0 is known and satisfies the normalization condition

$$\mathrm{Tr}_\sigma T_0[\mu|\sigma]T_0{}^\dagger[\mu|\sigma] = 1 \ . \tag{2.5}$$

Once we have specified H_σ^0, V_σ, and T_0 we can proceed to carry out a systematic perturbation-theory analysis by writing

$$H_\mu = H_\mu^0 + \lambda H_\mu^{(1)} + \lambda^2 H_\mu^{(2)} + \cdots \ , \tag{2.6a}$$

$$\begin{aligned} T[\mu|\sigma] = T_0[\mu|\sigma] &+ \lambda T_1[\mu|\sigma] \\ &+ \lambda^2 T_2[\mu|\sigma] + \cdots \ , \end{aligned} \tag{2.6b}$$

and inserting these equations in Eq. (2.1) and equating powers of λ to obtain

$$H_\sigma^0 T_0 = H_\mu^0 T_0 \ , \tag{2.7a}$$

$$(H_\sigma^0 - H_\mu^0) T_1 = (H_\mu^{(1)} - V_\sigma) T_0 \ , \tag{2.7b}$$

$$(H_\sigma^0 - H_\mu^0) T_2 = (H_\mu^{(1)} - V_\sigma) T_1 + H_\mu^{(2)} T_0 \ , \tag{2.7c}$$

etc. This set of equations is very similar to the usual equations appearing in perturbation-theory treatments of the Schrödinger equation. An important difference is that the "eigenvalue" H_μ is now an operator. We satisfy the set of Eqs. (2.7) with the normalization (2.2) in two steps. First we introduce the quantity $\tilde{T}[\mu|\sigma]$ satisfying Eq. (2.7) with the normalization

$$\mathrm{Tr}_\sigma \tilde{T}[\mu|\sigma] T_0{}^\dagger[\mu|\sigma] = 1 \ . \tag{2.8}$$

This normalization is convenient since it allows one to avoid the difficulties associated with the zero eigenvalue of the operator $H_\sigma^0 - H_\mu^0$ when operating on any vector along $T_0[\mu|\sigma]$. We can then unambiguously solve the set of Eqs. (2.7) to obtain $\tilde{T}[\mu|\sigma]$ and the corresponding $\tilde{H}_\mu$ order by order in perturbation theory. We find, for example,

$$\tilde{H}_\mu^{(1)} = \mathrm{Tr}_\sigma V_\sigma T_0[\mu|\sigma] T_0{}^\dagger[\mu|\sigma] \ , \tag{2.9}$$

$$\tilde{H}_\mu^{(2)} = \mathrm{Tr}_\sigma V_\sigma T_1[\mu|\sigma] T_0{}^\dagger[\mu|\sigma] \ . \tag{2.10}$$

We note that $\tilde{T}$ will not, in general, satisfy Eq. (2.2). Instead there will exist some operator Δ defined by

$$\mathrm{Tr}_\sigma \tilde{T}[\mu|\sigma] \tilde{T}^\dagger[\mu|\sigma] = \Delta[\mu] \ . \tag{2.11}$$

This is not a problem however, since we can introduce an operator S, operating on the μ Hilbert space, that rotates $\tilde{T}$ into T,

$$T[\mu|\sigma] = S[\mu]\tilde{T}[\mu|\sigma] \tag{2.12}$$

clearly T satisfies Eq. (2.1), with H_μ now given by

$$H_\mu = S[\mu]\tilde{H}_\mu S^{-1}[\mu] \tag{2.13}$$

and we require that T satisfy Eq. (2.2) or, equivalently

$$S[\mu]\Delta[\mu]S^\dagger[\mu] = 1 \ . \tag{2.14}$$

It will be useful to note that $T[\mu|\sigma]$ also satisfies the normalization condition

$$\mathrm{Tr}_\sigma T[\mu|\sigma] T_0{}^\dagger[\mu|\sigma] = S[\mu] \ . \tag{2.15}$$

The renormalized Hamiltonian H_μ is not yet completely determined. The reason is that $S[\mu]$ in general is a sum of Hermitian and anti-Hermitian terms and we cannot determine both from Eq. (2.14). We need another condition.

Let us write T_0 in the general form

$$T_0[\mu|\sigma] = \sum_i |i\rangle \times |\mu_i\rangle \ , \tag{2.16}$$

where $\{|i\rangle\}$ is a subset of the eigenvalues of H_σ^0, with eigenvalues ϵ_i, and $\{|\mu_i\rangle\}$ is a set of states is the μ Hilbert space chosen to be complete so that Eq. (2.2) is satisfied to zeroth order. The zeroth-order Hamiltonians can be written

$$H_\sigma^0 = \sum_i |i\rangle \epsilon_i \langle i| + \sum_\alpha |\alpha\rangle \epsilon_\alpha \langle \alpha| \ , \tag{2.17a}$$

$$H_\mu^0 = \sum_i |\mu_i\rangle \epsilon_i \langle \mu_i| \ , \tag{2.17b}$$

where $\{|\alpha\rangle\}$ are the remaining eigenstates of H_σ^0 which we are not mapping onto the new space.

We now want to construct the full T to a given order of perturbation theory and the corresponding H_μ, so that Eq. (2.1) is satisfied. The most general form for T that one can construct to satisfy Eqs. (2.1) and (2.2) is of the form

$$T[\mu|\sigma] = \sum_i \Vert i\rangle\rangle \times \Vert \mu_i\rangle\rangle \ , \tag{2.18}$$

where $\Vert i\rangle\rangle$ is the eigenstate of H_σ that evolves from the unperturbed state $|i\rangle$ when the interaction is turned on. The set $\{||\mu_i\rangle\rangle\}$ is a complete set of orthonormal states in μ space, so that Eq. (2.2) is satisfied, but is otherwise arbitrary. The corresponding H_μ satisfies, due to Eq. (2.1),

$$H_\mu ||\mu_i\rangle\rangle = E_i ||\mu_i\rangle\rangle \quad , \tag{2.19}$$

where E_i is the eigenvalue of $||i\rangle\rangle$. Thus, the eigenvalues of H_μ are a subset of the eigenvalues of H_σ.

Let us insert Eqs. (2.16) and (2.18) into Eq. (2.15) and trace over the complete set of unperturbed σ states to obtain

$$\sum_{i,j} \langle j|i\rangle\rangle ||\mu_i\rangle\rangle \langle \mu_j| = S[\mu] \quad . \tag{2.20}$$

Taking the adjoint we find

$$\sum_{i,j} \langle\langle i|j\rangle |\mu_j\rangle \langle\langle \mu_i|| = S^\dagger[\mu] \quad . \tag{2.21}$$

Subtracting Eq. (2.21) from Eq. (2.20) and taking the matrix element between unperturbed μ states we obtain

$$\langle\mu_l|(S-S^\dagger|\mu_m\rangle = \sum_i (\langle m|i\rangle\rangle \langle\mu_l|\mu_i\rangle\rangle - \langle\langle i|l\rangle \langle\langle\mu_i|\mu_m\rangle) \quad . \tag{2.22}$$

We, of course, would like to choose

$$\langle m|i\rangle\rangle = \langle\langle\mu_i|\mu_m\rangle \tag{2.23}$$

since, from a renormalization-group point of view, we want the μ states to be "as much as possible" like the σ states. This choice is not possible because of normalization problems. The "sum rule" satisfied by this "ideal" mapping

$$\sum_i (\langle m|i\rangle\rangle \langle\mu_l|\mu_i\rangle\rangle - \langle\langle i|l\rangle \langle\langle\mu_i|\mu_m\rangle) = 0 \tag{2.24}$$

can, however, be preserved by choosing S to be Hermitian. With this choice our procedure for evaluating H_μ is completely determined. Let us consider a second-order calculation. We see then, due to the normalization conditions (2.5) and (2.8),

$$\mathrm{Tr}_\sigma \tilde{T}[\mu|\sigma]\tilde{T}^\dagger[\mu|\sigma] = \Delta(\mu) = 1 + \lambda^2 \mathrm{Tr}_\sigma \tilde{T}_1[\mu|\sigma] \times \tilde{T}_1^\dagger[\mu|\sigma] + O(\lambda^3) \quad . \tag{2.25}$$

It follows from Eqs. (2.14) and (2.24) that $S[\mu]$ is of the form

$$S[\mu] = 1 + \lambda^2 S^{(2)}[\mu] + O(\lambda^3) \quad . \tag{2.26}$$

Since S is Hermitian, this determines $S^{(2)}[\mu]$ as

$$S^{(2)}[\mu] = -\tfrac{1}{2}\mathrm{Tr}_\sigma \tilde{T}_1[\mu|\sigma]\tilde{T}_1^\dagger[\mu|\sigma] \quad . \tag{2.27}$$

The "renormalized" Hamiltonian is given then by Eqs. (2.13) and (2.26) as

$$H_\mu^0 = \tilde{H}_\mu^0 \quad , \tag{2.28a}$$

$$H_\mu^{(1)} = \tilde{H}_\mu^{(1)} \quad , \tag{2.28b}$$

and

$$H_\mu^{(2)} = \tilde{H}_\mu^{(2)} + [S^{(2)}[\mu], \tilde{H}_\mu^0] \quad . \tag{2.28c}$$

While $\tilde{H}_\mu^{(2)}$ is not in general Hermitian, $H_\mu^{(2)}$ is Hermitian. Note that we can write up to first order

$$H_\mu = \mathrm{Tr}_\sigma H_\sigma T_0[\mu|\sigma] T_0{}^\dagger[\mu|\sigma] \quad . \tag{2.29}$$

That is, to first order the new Hamiltonian is obtained by just taking the matrix elements of the old Hamiltonian with the subset of states $\{|i\rangle\}$. This corresponds to the Drell *et al.*[5] prescription.

III. CHOICE OF T_0

We have discussed in Sec. II a well-defined prescription for obtaining $T[\mu|\sigma]$ and H_μ to any order of perturbation theory starting from a T_0 satisfying Eq. (2.5). In this section, we address ourselves to the question of finding a useful T_0 to start our calculation. Let us assume that our original Hamiltonian is of the form

$$H_\sigma = \sum_m H_\sigma^0(m) + \sum_{m\neq n} V_{mn} A_\sigma^m A_\sigma^n \quad , \tag{3.1}$$

where $H_\sigma^0(m)$ and A_σ^m are operators constructed from σ at site m. We can then relabel every site m with a cell index p and a set of basis vectors within each cell $a\,[m \rightarrow (p,a)]$ and we can rewrite H_σ in the form

$$H_\sigma = \sum_p H_\sigma^0(p) + \sum_{p,p'}\sum_{a,a'} V_{pp'}(a,a') A_\sigma(p,a) A_\sigma(p',a') \quad , \tag{3.2}$$

where $H_\sigma^0(p)$ includes $H_\sigma^0(p,a)$ and any intracell component from the V term. $V_{pp'}(a,a')$ gives the coupling between cells. We then assume that T_0 is a product of factors for each cell, i.e.,

$$T_0[\mu|\sigma] = \prod_p T_0{}^p[\mu|\sigma] \quad . \tag{3.3}$$

Using Eq. (2.29), we find for the first-order renormalized Hamiltonian

$$H_\mu = \sum_p \mathrm{Tr}_\sigma H_\sigma^0(p) T_0{}^p[\mu|\sigma] T_0{}^p[\mu|\sigma]^\dagger + \sum_{p,p'}\sum_{a,a'} V_{pp'}(a,a') \times \bar{A}_\mu(p,a)\bar{A}_\mu(p',a') + O(\lambda^2) \quad , \tag{3.4}$$

where we have defined

$$\bar{A}_\mu(p,a) = \mathrm{Tr}_\sigma A_\sigma(p,a) T_0{}^p[\mu|\sigma] (T_0{}^p[\mu|\sigma])^\dagger \quad . \tag{3.5}$$

TABLE I. The eigenstates $|i\rangle$ and eigenvalues E_i for the cell Hamiltonian H_σ^0 given by Eq. (4.2b).

State	Energy
$\|0\rangle = \frac{1}{(1+a_0^2)^{1/2}}(\|\downarrow\downarrow\rangle + a_0\|\uparrow\uparrow\rangle)$	$E_0 = -(\epsilon_0^2+\Delta_0^2)^{1/2}$
$\|1\rangle = (2)^{-1/2}(\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle)$	$E_1 = -\Delta_0$
$\|2\rangle = (2)^{-1/2}(-\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle)$	$E_2 = +\Delta_0$
$\|3\rangle = \frac{1}{(1+a_0^2)^{1/2}}(-a_0\|\downarrow\downarrow\rangle + \|\uparrow\uparrow\rangle)$	$E_3 = +(\epsilon_0^2+\Delta_0^2)^{1/2}$
$a_0 = \frac{(\epsilon_0^2+\Delta_0^2)^{1/2}-\epsilon_0}{\Delta_0}$	

We want to choose $T_0{}^p[\mu|\sigma]$ such that

$$\bar{A}_\mu(p,a) = f_p(a) A_\mu^p \ , \tag{3.6}$$

where A_μ^p is formally the same operator as A_σ^m, except it operates on the μ Hilbert space and is defined on the p lattice, and $f_p(a)$ is a number. We have then that

$$H_\mu = \sum_p H_\mu^0(p) + \sum_{p,p'} \bar{V}_{pp'} A_\mu^p A_\mu^{p'} + O(\lambda^2) \ , \tag{3.7}$$

with

$$H_\mu^0(p) = \mathrm{Tr}_\sigma H_\sigma^0(p)\, T_0{}^p[\mu|\sigma] (T_0{}^p[\mu|\sigma])^\dagger \tag{3.8}$$

and

$$\bar{V}_{pp'} = \sum_{a,a'} f_p(a) f_{p'}(a') V_{pp'}(a,a') \ , \tag{3.9}$$

Since $H_\sigma^0(m)$ and $H_\mu^0(p)$ are single-site operators they can both be cast into the same form using the irreducible representations for the operators σ and μ. We then have that our RG transformation leaves the form of the Hamiltonian invariant to first order.

In choosing $T_0{}^p$ we diagonalize exactly the intracell Hamiltonian $H_\sigma^0(p)$, find its two lowest-lying eigenstates, $|0\rangle_p$ and $|1\rangle_p$, and define $T_0{}^p$ as

$$T_0{}^p[\mu|\sigma] = |0\rangle_p |\mu_0\rangle_p + |1\rangle_p |\mu_1\rangle_p \ , \tag{3.10}$$

where the states $|\mu_0\rangle_p$ and $|\mu_1\rangle_p$ form an orthonormal basis for the μ spin at cell p, and are chosen so as to satisfy Eq. (3.6). In the two cases we have studied, it is appropriate to choose the eigenstates of μ_z^p, i.e.,

$$|\mu_0\rangle_p = |\downarrow\rangle_p ; \ \ |\mu_1\rangle_p = |\uparrow\rangle_p \ . \tag{3.11}$$

The states $|i\rangle$ of Eq. (2.16) will then be all eigenstates of H_σ^0 that can be constructed by taking products of the cell states $|0\rangle_p$ and $|1\rangle_p$, in particular, the ground state and first excited states of the unperturbed system are among them. Assuming the levels do not cross when the interaction is turned on, the renormalized Hamiltonian H_μ will then have the same ground-state energy and first excited-state energy as H_σ.

IV. APPLICATIONS

As a first example, we consider the Ising model in a transverse field in one dimension governed by the Hamiltonian

$$H_\sigma = \tfrac{1}{2}\epsilon_0 \sum_m \sigma_z^m - \Delta_0 \sum_m \sigma_x^m \sigma_x^{m+1} \ , \tag{4.1}$$

where the σ_x^m and σ_z^m are the usual Pauli matrices and represent spin at site m. We divide the chain into cells of two spins each, so that we have for the

TABLE II. Comparison of our results for the one-dimensional Ising model in a transverse field with exact and previous RG results.

Approximation	y_c	Deviation of y_c from exact value	ν	β	$c(s)$
Exact	2	0	1	0.125	0.5
First order	2.553	28%	1.48	0.408	0.68
Second order	2.187	9.4%	0.882	0.170	0.52
First order[a]		22%	1.12	0.707	
Second order[a]		8.5%	0.949	−0.021	
First order[b]	1.636	18%	0.922	0.216	
Second order[b]	1.996	0.2%	0.888	1.21	
First order[c]	2.018	1%	1.13	$\lambda_h < 0$	
Second order[c]	1.780	11%	0.951	$\lambda_h < 0$	
Four-spins/cell[d]	2.211	11%	1.24	0.185	0.63
Seven-spins/cell[d]	2.106	5.3%	1.16	0.145	0.60

[a] Reference 12.

[b] Reference 7.

[c] Reference 8.

[d] Reference 6.

intracell and intercell Hamiltonian

$$H_\sigma = H_\sigma^0 + V_\sigma \ , \tag{4.2a}$$

$$H_\sigma^0 = \sum_p \{\tfrac{1}{2}\epsilon_0[\sigma_z(p,0) + \sigma_z(p,1)] - \Delta_0 \sigma_x(p,0)\,\sigma_x(p,1)\} \ , \tag{4.2b}$$

$$V_\sigma = -\Delta_0 \sum_p \sigma_x(p,1)\,\sigma_x(p+1,0) \ , \tag{4.2c}$$

where $\sigma(p,a)$ is the spin operator for site $a=0,1$ on cell p. The eigenstates and eigenvalues of the cell Hamiltonian are listed in Table I. Applying the formalism described previously to second order, we generate three-spin next-nearest-neighbor interaction terms of the form $K\mu_x^{p-1}\mu_z^p\mu_x^{p+1}$. Treating K as a second-order quantity, we get after n iterations the Hamiltonian

$$H_\mu^n = d_n \sum_p 1 + \tfrac{1}{2}\epsilon_n \sum_p \mu_z^p - \Delta_n \sum_p \mu_x^p \mu_x^{p+1} - K_n \sum_p \mu_x^{p-1}\mu_z^p\mu_x^{p+1} \ , \tag{4.3}$$

where the coefficients are determined by the recursion relations

$$\epsilon_{n+1} = E_1^n - E_0^n + b_n^2 c_n^2 \Delta_n^2 \left(\frac{1}{E_1^n} - \frac{1}{E_0^n} \right) \ , \tag{4.4a}$$

$$\Delta_{n+1} = b_n^2 \Delta_n - 2 b_n c_n K_n \ , \tag{4.4b}$$

$$K_{n+1} = -\tfrac{1}{2} b_n^2 c_n^2 \Delta_n^2 \left(\frac{1}{E_1^n} + \frac{1}{E_0^n} \right) \ , \tag{4.4c}$$

$$d_{n+1} = 2d_n + \tfrac{1}{2}(E_1^n + E_0^n) + \tfrac{1}{2}\Delta_n^2 \times \left[b_n^2 c_n^2 \left(\frac{1}{E_1^n} + \frac{1}{E_0^n} \right) + \frac{c_n^4}{E_0^n + E_1^n} \right] \ , \tag{4.4d}$$

with

$$b_n = \frac{1 + a_n}{[2(1 + a_n^2)]^{1/2}} \ , \tag{4.5a}$$

$$c_n = \frac{1 - a_n}{[2(1 + a_n^2)]^{1/2}} \ , \tag{4.5b}$$

and a_n and E_i^n as given in Table I, with ϵ_n, Δ_n replacing ϵ_0, Δ_0. We study the recursion relations for the ratios

$$y_n = \epsilon_n / \Delta_n \ , \tag{4.6a}$$

$$x_n = K_n / \Delta_n \ , \tag{4.6b}$$

and find a nontrivial fixed point at

$$y^* = 1.709\,854 \ , \tag{4.7a}$$

$$x^* = 0.269\,725 \ . \tag{4.7b}$$

The intersection of the critical surface with the Ising axis ($x=0$) gives

$$y_c = 2.187 \ . \tag{4.8}$$

This is to be compared with the exact value (Ref. 11) $y_c = 2$. The first-order result[5] is $y_c^{(1)} = 2.55$.

Linearizing the recursion relations around the fixed point and diagonalizing the resulting matrix we find the two eigenvalues

$$\lambda_T = 2.194 \ , \tag{4.9}$$

$$\lambda_2 = 0.259 \ , \tag{4.10}$$

which lead to the "thermal exponent"

$$\nu = \frac{\log 2}{\log \lambda_T} = 0.882 \ . \tag{4.11}$$

The exact value[11] is $\nu = 1$, while the first-order value is $\nu = 1.48$. For obtaining the magnetic eigenvalue we introduce an odd perturbation

$$V_\sigma' = h_0 \sum_m \sigma_x^m \ . \tag{4.12}$$

To lowest order, we obtain the recursion relation

$$h_{n+1} = 2 b_n h_n \ , \tag{4.13}$$

which yields, using the first-order fixed-point result for b_n,

$$\lambda_h = 2 b_n \big|_{y_c^{(1)}} = 1.65 \tag{4.14}$$

and from the relation

$$\frac{\beta}{\nu} = d - \frac{\log \lambda_h}{\log 2} \ , \tag{4.15}$$

we find

$$\beta = 0.408 \ . \tag{4.16}$$

This agrees with the value obtained by Drell *et al.*[5] via a direct calculation of the spontaneous magnetization. In the next order, we generate, after n iterations, a contribution to the Hamiltonian

$$H_\mu'^n = h_n \sum_p \mu_x^p + L_n \sum_p (\mu_x^p \mu_z^{p+1} + \mu_z^p \mu_x^{p+1}) \ , \tag{4.17}$$

with the coefficients satisfying the recursion relations

$$h_{n+1} = \left\{ 2b_n - \Delta_n b_n c_n^2 \left[\frac{2}{E_1^n + E_0^n} + \frac{1}{2}\left(\frac{1}{E_1^n} + \frac{1}{E_0^n} \right) \right] \right\} h_n - 2c_n(1 + b_n^2) L_n \ , \tag{4.18a}$$

$$L_{n+1} = -\tfrac{1}{2}\Delta_n b_n c_n^2 \left[\frac{2}{E_1^n + E_0^n} + \frac{1}{2}\left(\frac{1}{E_1^n} + \frac{1}{E_0^n} \right) \right] h_n \times b_n^2 c_n L_n \ . \tag{4.18b}$$

Diagonalizing the transformation using the first-order fixed-point results for the thermal parameters yields the two eigenvalues

$$\lambda_h = 1.75 \ , \tag{4.19a}$$

$$\lambda_2 = 0.61 \ , \tag{4.19b}$$

and

$$\beta = 0.170 \ . \tag{4.19c}$$

These values should be compared with the exact result $\beta = 0.125$. An important aspect of our calculation is that the values of all quantities computed are improved in going from first to second order.

The relation we use to connect the magnetic eigenvalue with β merits some discussion; our recursion relations at the fixed point leave the Hamiltonian invariant only up to an additive and a multiplicative constant

$$H_\mu^{n+1} = c(s) H_\mu^n + g(s) \ , \tag{4.20}$$

where s is the length scale change, $s = 2$ in our case. The multiplicative constant corresponds to the ratio of temperatures

$$c(s) = \lim_{T \to 0} T^n / T^{n+1} \tag{4.21}$$

Since the temperature corresponds to the length of a $(d+1)$-dimensional classical Ising system in the $(d+1)$ direction,[9] one expects at the fixed point[4] $c(s) = s^{-1}$. Given this assumption, one can calculate β from the relation

$$\frac{\beta}{\nu} = d + 1 - \frac{\log \lambda_h'}{\log s} \ , \tag{4.22}$$

where λ_h' is obtained from the recursion relations for the ratios of the coefficients of the Hamiltonian [as is done with Friedman's method,[3] because $c(s)$ cannot be obtained by his method]. We believe that it is consistent in a finite-order calculation to take the finite order $c(s)$ instead of assuming $c(s) = s^{-1}$, and indeed we obtain much better values in β by our method. For $c(s)$ we obtain from our calculation

$$c(s) = 0.68 \ \text{ to first order,} \tag{4.23}$$

$$c(s) = 0.52 \ \text{ to second order,} \tag{4.24}$$

which appears to be approaching the correct value $c(s) = \frac{1}{2}$.

In Table II, we summarize our numerical results and compare them with other RG calculations. Reference 12 is the Niemeijer–Van Leeuwen cumulant approximation for the two-dimensional Ising tri-

TABLE III. The eigenstates $\vert i\rangle$ and eigenvalues E_i for the cell Hamiltonian H_σ^0 for the two-dimensional Ising model in a transverse field.

State	Energy
$\vert 0\rangle = \frac{1}{(3+a_0^2)^{1/2}}(-a_0\vert\downarrow\downarrow\downarrow\rangle + \vert\downarrow\uparrow\uparrow\rangle + \vert\uparrow\downarrow\uparrow\rangle + \vert\uparrow\uparrow\downarrow\rangle)$	$E_0 = -\epsilon_0 - \Delta_0 - 2(\epsilon_0^2 + \Delta_0^2 - \epsilon_0\Delta_0)^{1/2}$
$\vert 1\rangle = \frac{1}{(3+b_0^2)^{1/2}}(-b_0\vert\uparrow\uparrow\uparrow\rangle + \vert\uparrow\downarrow\downarrow\rangle + \vert\downarrow\uparrow\downarrow\rangle + \vert\downarrow\downarrow\uparrow\rangle)$	$E_1 = \epsilon_0 - \Delta_0 - 2(\epsilon_0^2 + \Delta_0^2 + \epsilon_0\Delta_0)^{1/2}$
$\vert 2\rangle = (2)^{-1/2}(\vert\uparrow\downarrow\downarrow\rangle - \vert\downarrow\downarrow\uparrow\rangle)$	$E_2 = -\epsilon_0 + \Delta_0$
$\vert 3\rangle = (6)^{-1/2}(\vert\uparrow\downarrow\downarrow\rangle - 2\vert\downarrow\uparrow\downarrow\rangle + \vert\downarrow\downarrow\uparrow\rangle)$	$E_3 = -\epsilon_0 + \Delta_0$
$\vert 4\rangle = \frac{1}{(3+c_0^2)^{1/2}}(-c_0\vert\downarrow\downarrow\downarrow\rangle + \vert\downarrow\uparrow\uparrow\rangle + \vert\uparrow\downarrow\uparrow\rangle + \vert\uparrow\uparrow\downarrow\rangle)$	$E_4 = -\epsilon_0 - \Delta_0 + 2(\epsilon_0^2 + \Delta_0^2 - \epsilon_0\Delta_0)^{1/2}$
$\vert 5\rangle = (2)^{-1/2}(\vert\downarrow\uparrow\uparrow\rangle - \vert\uparrow\uparrow\downarrow\rangle)$	$E_5 = \epsilon_0 + \Delta_0$
$\vert 6\rangle = (6)^{-1/2}(\vert\downarrow\uparrow\uparrow\rangle - 2\vert\uparrow\downarrow\uparrow\rangle + \vert\uparrow\uparrow\downarrow\rangle)$	$E_6 = \epsilon_0 + \Delta_0$
$\vert 7\rangle = \frac{1}{(3+d_0^2)^{1/2}}(-d_0\vert\uparrow\uparrow\uparrow\rangle + \vert\uparrow\downarrow\downarrow\rangle + \vert\downarrow\uparrow\downarrow\rangle + \vert\downarrow\downarrow\uparrow\rangle)$	$E_7 = \epsilon_0 - \Delta_0 + 2(\epsilon_0^2 + \Delta_0^2 + \epsilon_0\Delta_0)^{1/2}$

$a_0 = 1 - 2\epsilon_0/\Delta_0 - 2[(\epsilon_0/\Delta_0)^2 - \epsilon_0/\Delta_0 + 1]^{1/2}$
$b_0 = 1 + 2\epsilon_0/\Delta_0 - 2[(\epsilon_0/\Delta_0)^2 + \epsilon_0/\Delta_0 + 1]^{1/2}$
$c_0 = 1 - 2\epsilon_0/\Delta_0 + 2[(\epsilon_0/\Delta_0)^2 - \epsilon_0/\Delta_0 + 1]^{1/2}$
$d_0 = 1 + 2\epsilon_0/\Delta_0 + 2[(\epsilon_0/\Delta_0)^2 + \epsilon_0/\Delta_0 + 1]^{1/2}$

TABLE IV. Comparison of our results for the two-dimensional Ising model in a transverse field with high-temperature expansions and previous RG results.

Approximation	ν	β
High-temperatures series	0.625	0.3125
Our calculation	0.724	0.394
First order[a]	0.967	1.271
Second order[a]	0.924	0.512
$2\times2\times2$ cell[b]	0.827	0.332
$3\times3\times3$ cell[b]	0.844	0.403
Reference 16	0.629	0.324

[a] Reference 4.

[b] Reference 14.

angular lattice to second order. References 7 and 8 are calculations by G. Um on the model discussed here using Friedman's technique to second order with two and three spins per cell, respectively. Our results are somewhat worse for y_c and ν but much better for β. Reference 6, by Jullien *et al.*, is a calculation using the method of Drell *et al.*, where the number of spins in a cell is varied from two to seven. We show two representative results, for four- and seven-spin cells (their results improve systematically with increasing cell size). Our results to second order are worse than the ones for seven-spin cells [except for $c(s)$] but better than the ones for four-spin cells. We believe that applying our method to second order with a bigger cell would significantly improve the accuracy of our results.

As a second example, we consider a two-dimensional Ising model on a triangular lattice in a transverse field, described by the Hamiltonian

$$H_\sigma=\epsilon_0\sum_m \sigma_z^m-\Delta_0\sum_{\langle m,n\rangle}\sigma_x^m\sigma_x^n \ . \tag{4.25}$$

We divide the lattice into three-spin cells as in Ref. 12 and diagonalize the intracell Hamiltonian. The resulting eigenstates and eigenvalues are listed in Table III. A calculation to first order yields

$$H_\mu^n=d_n\sum_p 1+\epsilon_n\sum_p \mu_z^p-\Delta_n\sum_{\langle p,p'\rangle}\mu_x^p\mu_x^{p'} \tag{4.26}$$

and the recursion relations

$$\epsilon_{n+1}=\tfrac{1}{2}(E_1^n-E_0^n) \ , \tag{4.27a}$$

$$\Delta_{n+1}=2\Delta_n\frac{(2-a_n-b_n)^2}{(3+a_n^2)(3+b_n^2)} \ , \tag{4.27b}$$

$$d_{n+1}=3d_n+\tfrac{1}{2}(E_1^n+E_0^n) \ . \tag{4.27c}$$

We find a nontrivial fixed point for $y_n=\epsilon_n/\Delta_n$ at $y^*=2.038$; the "thermal" eigenvalue gives

$$\lambda_T=2.1350 \tag{4.28}$$

and we get for the thermal exponent

$$\nu=0.724 \tag{4.29}$$

to be compared with the high-temperature expansion result for the three-dimensional Ising model[13]

$$\nu=0.625 \ . \tag{4.30}$$

For the magnetic eigenvalue, we introduce a magnetic perturbation and get to lowest order the recursion relation

$$h_{n+1}=\frac{3(2-a_n-b_n)}{[(3+a_n^2)(3+b_n^2)]^{1/2}}h_n \ , \tag{4.31}$$

which gives at the first-order fixed point the eigenvalue $\lambda_n=2.23$; the magnetic exponent is then

$$\beta=0.394 \tag{4.32}$$

to be compared with $\beta=0.3125$ from high-temperature series expansion.[13]

In Table IV we summarize our results and compare them with other RG calculations. Reference 4 (see also Ref. 3) is a calculation on the model discussed here using Friedman's technique. Our results for both exponents at lowest order are better than the second-order results of Ref. 4. Reference 14 is a calculation on the three-dimensional Ising model using the Niemeijer–Van Leeuwen cumulant expansion to second order with a variational parameter, as introduced by Kadanoff and Houghton.[15] Our results are comparable for the magnetic exponent but better for the thermal exponent. Reference 16 is a "lower-bound" variational calculation on the three-dimensional Ising model using the "one-hypercube approximation." This more sophisticated scheme gives better results than our lowest-order calculation.

V. SUMMARY AND DISCUSSION

We have proposed a new renormalization-group approach for quantum spin systems at zero temperature and have applied it to the one-dimensional Ising model in a transverse field to second order and to the two-dimensional triangular Ising model in a transverse field to first order. From the numerical results for the one-dimensional model, we see a substantial improvement in all numbers in going from first to second order. This systematic improvement does not seem to occur in any of the previous work on this model. The numbers for the two-dimensional case, even at lowest order, are more accurate than a previous calculation to second order for this model. They also compare well with calculations on the three-dimensional classical Ising model. Our method seems to be rather general and applicable to a variety of other problems. A point one must be careful about is that our method does not guarantee that the renormalized Hamiltonian has the same symmetry properties as the original one. The cell must be properly chosen in each problem so that at least the ground state has the correct symmetry. We have not encountered any such problem in the two models we studied, but it may appear in other models. Applications of our method to other problems are under way.

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