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Dynamics of vortex pairs in superfluid films

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The linear response of a thermal ensemble of vortex pairs to an oscillating applied force is calculated by solving a Fokker-Planck equation. The result differs quantitatively but not qualitatively from an estimate given in the work of Ambegaokar, Halperin, Nelson, and Siggia.

I. INTRODUCTION

In a recent note[1] Halperin, Nelson, Siggia, and one of the present authors (V.A.) have given a dynamical generalization of the very beautiful Kosterlitz-Thouless picture[2,3,4] of superfluidity in thin ^{4}He films. In that picture, the superfluid-to-normal fluid transition is due to an unbinding of pairs of vortices of opposite sign in a sea of other such vortex pairs. In Ref. 1, the dynamics of the motion of a quantized vortex was made explicit, by balancing Magnus and drag forces against fluctuating thermal forces. The resulting Langevin equation for the self-consistent motion of a vortex pair contained, in its long-time or equilibrium behavior, the Kosterlitz-Thouless transition. By thus uncovering the dynamics which for thermodynamical purposes can be hidden under a partition function, one was able to include the effect of time-dependent perturbations, such as that due to an oscillating substrate. Estimates of the inertial and dissipative response of free vortices and of bound vortex pairs to small oscillating substrate velocities were thereby deduced. Rather satisfying fits to experimental curves taken by Bishop and Reppy[5] have resulted.[5,6]

The present brief communication is addressed very particularly to the estimate made in Ref.1 for the frequency-dependent response of bound pairs. We write down and solve the Fokker-Planck equation describing the motion of a bound pair, screened by other bound pairs, under the influence of an applied oscillating force. The resulting solution differs quantitatively from the estimate previously given, but it does not change the qualitative form of the result. The effect on numerical fits of theory to experiment is to change the values of certain fitted parameters in a direction which seems more physically reasonable.

In the opinion of the authors, the main contribution being made in this note is pedagogic and aesthetic: the calculation is particularly transparent, and the solution involves simple functions of the sort beloved by nineteenth-century mathematicians.

II. CALCULATION

The Langevin equation on which the present calculation is built is given in Eq. 11 of Ref. 1. We write it as follows:

$$\frac{d\vec{r}}{dt} = -\frac{2D}{k_B T}\frac{\partial U}{\partial \vec{r}} + \vec{\eta}(t) \quad , \tag{1}$$

where $\vec{r}$ is the vector separating a vortex pair, D is a diffusion constant, k_B is Boltzmann's constant, T is the temperature, U is the potential describing the screened interaction between the pair as well as the interaction with the driving force, and $\vec{\eta}$ is a Gaussian noise source obeying

$$\langle \eta^\alpha(t)\eta^\beta(t')\rangle = 4D\,\delta_{\alpha\beta}\delta(t-t') \quad .$$

We will write the potential $U(\vec{r})$ in the language of the equivalent problem of the diffusive motion of charged rods,

$$U(\vec{r}) = 2q^2\int_a^r \frac{dr}{r\tilde{\epsilon}(r)} - q\,\delta\vec{E}\cdot\vec{r} - 2\mu_0 \quad . \tag{2}$$

The transcription between the charge q, the (small) macroscopic electric field $\delta\vec{E}$, and the parameters of the vortex problem can, for example, be read off by comparing Eqs. (1) and (2) above with Eq. (11) of Ref. 1. The other symbols not yet defined in Eq. (2) are μ_0 the chemical potential, a the core radius, and $\tilde{\epsilon}(r)$ the static length and temperature dependent dielectric constant of Kosterlitz.[3] The "electric field" will be taken to vary sinusoidally in time with circular frequency ω. We note that the *static* dielectric constant occurs in Eq. (2) because it accounts for the screening effect of smaller pairs which adiabatically follow the electric field due to the larger pair being considered in Eq. (1).

As a first step in deriving the macroscopic frequency-dependent dielectric constant of the system, we must calculate the linear effect of the electric field on the distribution function of pairs. From Eq. (1) one obtains in the usual way a Fokker-Planck

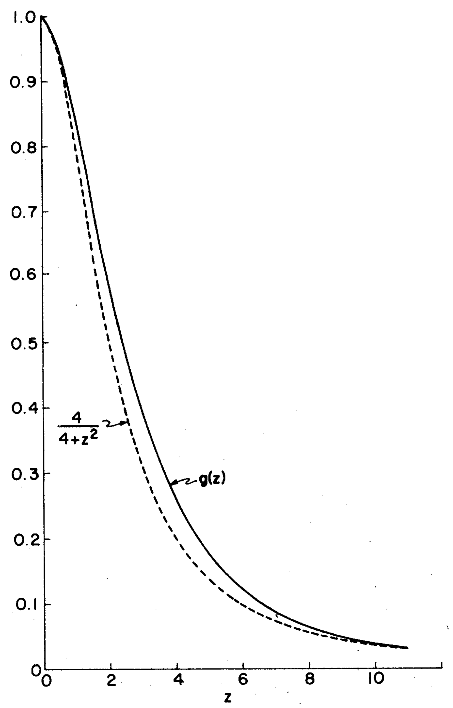

FIG. 1. Solid line shows the exact solution $g(z)$, given in Eq. (12) to Eq. (9). The dashed line is the approximate solution obtained by ignoring the derivative terms in Eq. (9).

equation for $\Gamma(\vec{r},t)$, the number of pairs per unit film area per unit area of separation in the neighborhood of $\vec{r}$;

$$\frac{d\Gamma}{dt}=\frac{2D}{k_BT}\frac{\partial}{\partial\vec{r}}\cdot\left(\frac{\partial U}{\partial\vec{r}}\Gamma\right)+2D\frac{\partial^2\Gamma}{\partial r^2} \quad . \tag{3}$$

In principle Eq. (3) could be used to calculate the nonlinear frequency-dependent response of pairs, in which case the destabilizing effect of a finite field on pairs larger than a critical size would have to be treated by appropriate boundary conditions.[7] Here, however, we are limiting ourselves to calculating the linear response which is described by the dielectric constant

$$\epsilon(\omega)=1+4\pi q\int d^2r\,\tfrac{1}{2}\vec{r}\cdot\frac{\delta\Gamma}{\delta\vec{E}} \quad . \tag{4}$$

We now write $U(\vec{r})=U_0-q\,\delta\vec{E}\cdot\vec{r}$ and $\Gamma=\Gamma_0+\delta\Gamma$, where Γ_0 is the equilibrium distribution function $a^4\Gamma_0=\exp(-U_0/k_BT)$. Substituting these expansions into Eq. (3) and keeping linear terms we obtain

$$-i\omega\delta\Gamma=-\frac{2D}{k_BT}\frac{\partial}{\partial\vec{r}}\cdot[q\,\delta\vec{E}\Gamma_0]+\frac{2D}{k_BT}\times 2q^2\frac{\partial}{\partial\vec{r}}\cdot\left(\frac{\vec{r}}{r^2\tilde{\epsilon}(r)}\delta\Gamma\right)+2D\frac{\delta^2}{\delta r^2}\delta\Gamma \quad . \tag{5}$$

The angular dependence of this equation can be removed by the decomposition

$$\delta\Gamma(r,\theta,\omega)=\sum_{l=-\infty}^{\infty}\delta\Gamma_l(r,\omega)\exp(il\theta) \quad .$$

We note that only $\delta\Gamma_1$ is coupled to the external force, and that only this quantity occurs in Eq. (4) for $\epsilon(\omega)$. It is now convenient to write

$$\delta\Gamma_1=\Gamma_0\frac{q\,\delta Er}{2k_BT}g(r,\omega) \quad . \tag{6}$$

The factor split off from $\delta\Gamma_1$ is the static response function, i.e., $g=1$ would correspond to a local equilibrium approximation.

By straightforward substitution, g is found to obey the differential equation

$$r^2g''+r\left(3-\frac{2q^2}{k_BT\tilde{\epsilon}(r)}\right)g'-\left(-\frac{i\omega r^2}{2D}+\frac{2q^2}{k_BT\tilde{\epsilon}(r)}\right)g+\frac{2q^2}{k_BT\tilde{\epsilon}(r)}=0 \quad . \tag{7}$$

Furthermore, by substituting Eq. (6) into Eq. (4) and using the self-consistent relationship[2,4] that determines $\tilde{\epsilon}(r)$, we find

$$\epsilon(\omega)=1+\int_a^\infty dr\,\frac{d\tilde{\epsilon}}{dr}\,g(r,\omega) \quad . \tag{8}$$

Equation (8) is exactly the form intuited for $\epsilon(\omega)$ in Ref. 1 where g was approximated by $2Dr^{-2}/(2Dr^{-2}-i\omega)$.

For the purpose at hand it is not necessary to solve the complicated differential Eq. (7). We observe that $\tilde{\epsilon}(r)$ is a weakly varying function of r when T is near T_c. As a first approximation we therefore replace $2q^2/k_BT\tilde{\epsilon}(r)$ by $2q^2/k_BT_c\epsilon_c=4$, where $\epsilon_c\equiv\tilde{\epsilon}(\infty,T_c)$ and we have used the famous universal result[8] of the Kosterlitz scaling equations. This replacement allows one to make a simple calculation for g, and, finally, to verify that the errors introduced by the approximation are indeed small.

With the above approximation, the differential equation we must solve takes the simple form

$$z^2g''(z)-zg'(z)-(4+z^2)g(z)+4=0 \quad , \tag{9}$$

where we have introduced the variable $z^2\equiv-i\omega r^2/2D$. The required solution has to approach 1 for $z\to0$ (statics), and go to zero for $z\to\infty$. If one were to neglect the terms involving derivatives in Eq. (9), one would obtain a solution of the form given in Eq. (8) of Ref. 1 with, however, $2D/r^2$ replaced by $8D/r^2$. We shall see below that the main effect of the derivative terms is to cause a further increase in the characteristic length at which $g(z)$ drops to half its $z=0$ value. (see Fig. 1).

A particular integral of Eq. (9) is easily generated as a power series in z. The result is

$$g_P(z) = 1 + \sum_{m=1}^{\infty} \left(\frac{z}{2}\right)^{2m} \{[m(m-1)-1] \times [(m-1)(m-2)-1] \ldots [-1]\}^{-1}$$

$$= 1 - \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^4 - \left(\frac{z}{2}\right)^6 \cdot \frac{1}{5} - \left(\frac{z}{2}\right)^8 \cdot \frac{1}{5 \cdot 11} - \ldots \qquad (10)$$

The homogeneous part of Eq. (9) can also be solved by the power-series method. The solutions can be written in terms of modified Bessel functions of the first kind, usually called I_ν, of orders $\nu = \pm (5)^{1/2}$. These functions have the series expansion[9]

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\Gamma(\nu+k+1)} , \qquad (11)$$

where Γ is the Gamma function. The solution of the homogeneous part of Eq. (9) regular at the origin turns out to be $zI_{(5)^{1/2}}(z)$. Thus the general solution of Eq. (9) regular at the origin is

$$g(z) = g_P(z) + \kappa z I_{(5)^{1/2}}(z) \qquad (12)$$

where κ is a constant. Since

$$\lim_{z\to 0}[zI_{(5)^{1/2}}(z)] = 0 ,$$

the $z=0$ boundary condition has been satisfied. It remains to adjust κ to obtain the required behavior at infinity, i.e., $g(z) \to 0$. The asymptotic behavior of the series (10), which converges for all z, can be shown to be proportional to that of $zI_{(5)^{1/2}}(z)$. It therefore suffices to match the most singular behavior of the two functions on the right of Eq. (12). Now the I_ν's of all order have the asymptotic behavior[10]

$$I_\nu(z) \to \frac{e^z}{(2\pi z)^{1/2}} \left(1 + 0\left(\frac{1}{z}\right) + \ldots\right) , \qquad (13)$$

and we easily see by comparing the series (10) and (11) for $\nu = 1$ that the most singular part of g_P behaves like

$$g_P(z) \to -\left(\prod_{m=3}^{\infty} \frac{m(m-1)}{m(m-1)-1}\right) zI_1(z) . \qquad (14)$$

Thus the required constant κ is given by the large parenthesis in Eq. (14). Numerical evaluation gives $\kappa = 3.3707\ldots$. With this value of κ the function (12) is found by numerical calculation to behave like $4/z^2$ for large z as one would expect from the differential equation (9).

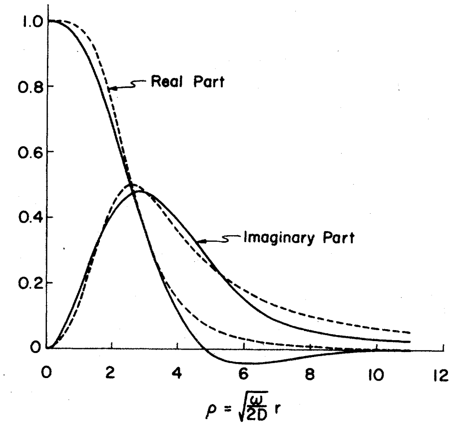

FIG. 2. Solid lines represent the real and imaginary parts of the exact solution $g(z^2 = -i\omega r^2/2D)$. The dashed lines show the real and imaginary parts of the simple Ansatz of Ref. 1, $g = \gamma/(-i\omega + \gamma)$. The cut-off length of this approximate solution has been matched to the exact solution, requiring $\gamma \approx 14D/r^2$.

Figure 1 shows the result of the calculation for $g(z)$, and, for purposes of comparison, the function $4/(4+z^2)$ corresponding to neglecting the derivatives of g in Eq. (9).

In Fig. 2 the full lines are plots of the real and imaginary parts of $g(z^2 = -i\omega r^2/2D)$. We note that the shapes of these curves are very similar to those that follow from the simple ansatz of Ref. 1 quoted below Eq. (8). The dashed lines show such curves with the relaxation time adjusted to match the cut-off length of the correct solution. The matching requires the replacement of $2D/r^2$ by roughly $14D/r^2$.

III. DISCUSSION AND CONCLUSIONS

We now investigate the goodness of the approximation $\tilde{\epsilon}(r) \to \epsilon_c$ in Eq. (7). The space and temperature dependence of $\tilde{\epsilon}$ is discussed in Refs. 1, 3, and 6. We first note that for $(\omega r^2/2D) \ll 1$, $g \approx 1$ is a solution of the differential equation (7) regardless of the spatial dependence of $\tilde{\epsilon}$. In the experimental situation discussed in Ref. 5 and analyzed in Ref. 6, $\omega \approx 10^4$ sec^{-1}, and D and a (the core radius) were found to be consistent with the physically reasonable values of 10^{-4} cm^2/sec and 10^{-8} cm, respectively. For these values, the inequality written above which defines the region in which the r dependence of $\tilde{\epsilon}(r)$ is irrelevant becomes $\ln(r/a) < l_D$, where we define

$$l_D \equiv \ln[(2D/\omega a^2)^{1/2}] \approx 10 .$$

Now, quite generally one has[1,3,6]

$$\tilde{\epsilon}(r) = \epsilon_c[1 - \tfrac{1}{2}x(l)] \quad , \tag{15}$$

where $x(l)$ is the scaling parameter introduced by Kosterlitz, and $l \equiv \ln(r/a)$. For $T \leq T_c$, the scaling equations[3] yield

$$x(l) = \tfrac{1}{2}x(T) \coth \tfrac{1}{2}x(T)l \quad , \tag{16}$$

where $x(T) \propto [|1-(T/T_c)|]^{1/2}$ determines the correlation length[1] $\xi_-(T) \approx a \exp[1/x(T)]$. The region of most rapid variation of $x(l)$, and therefore of $\tilde{\epsilon}(r)$, occurs for $l < l_- \equiv \ln(\xi_-/a)$. Except extremely close to T_c, we have $l_- << l_D$, and thus, because of the considerations of the preceding paragraph, the region of most rapid variation of $\tilde{\epsilon}(r)$ plays no role in determining $g(z)$. Even at T_c, where $x(l) = l^{-1}$, the exclusion of the region $l < l_D$ in Eq. (15) has the effect that the correction to $\tilde{\epsilon}(r)$ is less than 10%. For $l > \ln(\xi_-/a)$ the correction to ϵ_c in Eq. (15) is determined by $\frac{1}{4}x(T)$. For the temperature region of interest in the application of the dynamic theory,[5,6] this number is less than 0.07, again a small correction.

For $T > T_c$, Eq. (15) remains valid but for small $x(l)$ Eq. (16) is changed to[3,6]

$$x(l) \approx \tfrac{1}{2}x(T) \cot \tfrac{1}{2}x(T)l \quad .$$

There are now two regions of rapid variation of $x(l)$, $l < l_-$ and $l \sim l_+ \equiv \ln(\xi_+/a)$, where $\xi_+ \approx a \exp[2\pi/x(T)]$. The first region is of no importance, the reasons being the same as for the $T \leq T_c$ regime discussed above. However, since $\xi_+ >> \xi_-$ for corresponding $|T-T_c|$, the condition $l_+ < l_D$, which would allow one to ignore the r dependence of $\tilde{\epsilon}(r)$, is not very restrictive, and for the fit described in Ref. 6 leaves a region of $(T-T_c) < 0.01\,^\circ$K unaccounted for. Well within this temperature region, where $l_+ >> l_D$, the zero-order solution for $g(z)$ shows that regions of $r \sim \xi_+$ correspond to very small values of $\delta\Gamma$. The region of temperatures where $l_+ \approx l_D$ is more difficult to deal with precisely. However, as was shown in Ref. 6, in this region of temperatures the dynamics is already dominated by free vortices. The errors arising from replacing Eq. (7) by Eq. (9) are therefore small in all regimes.

We conclude therefore that the formulas (9a) and (9b) of Ref. 1 for the real and imaginary parts of the dynamical dielectric constant due to bound pairs are unchanged by the considerations of this paper, except for the replacement of the length $(2D/\omega)^{1/2}$ by something closer to $(14D/\omega)^{1/2}$.

We may ask, finally, how this change affects the fit reported in Ref. 6. The only changes are that $\ln[(14D/a^2\omega)^{1/2}]$—instead of $\ln[2D/a^2\omega)^{1/2}]$—is determined to be ≈ 12; and that F the coefficient of the free vortex contribution to the dielectric constant, is changed from ~ 0.18 to ~ 1.21. The first change is insignificant, and, since we had expected F to be $0(1)$, the second change is in a physically reasonable direction.

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