Cylindrical Josephson tunneling

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(Received 14 March 1978)

Josephson-tunneling experiments between small cylindrical superconductors are proposed. The essential idea is that the wave functions in the superconductors must be of the form $\psi_1 = |\psi_\infty| f(r) e^{in_1\theta}$ and $\psi_2 = |\psi_\infty| f(r) e^{i(n_2\theta+\delta_0)}$ to guarantee that they are single valued. This, together with the Josephson relation $J = J_0 \sin\delta$, where J_0 is the maximum Josephson current and δ is the phase difference between the superconductors, leads to the conclusion that the Josephson current will be zero for $n_1 \neq n_2$. For $n_1 = n_2$ the current will be proportional to $f_1^2 f_2^2$ in the thin-film approximation. If the inner cylinder is solid and of higher transition temperature then $J_0 \propto f_2^2$ providing a means for determining the order parameter as a function of magnetic field. The ratio of cylinder radius to coherence length determines, in large measure, the variation of J_0 with field. Specific examples are given.

In the usual Josephson-tunneling experiment the superconducting pairs tunnel through a thin oxide which separates two plane-parallel superconductors which are typically thin films. The current density is given by the Josephson relation $J = J_0 \sin\delta$, where δ is the phase difference between the wave functions on the two sides of the junction and J_0 is the maximum Josephson current density. When a magnetic field is applied parallel to the plane of the junction the phase difference becomes $\delta(x) = \delta(0) + 2\pi\Phi(x)/\Phi_0$, where $\Phi(x)$ is the magnetic flux which penetrates the superconductor between $x = 0$ and x, and Φ_0 is the flux quantum. This leads to a Josephson current

$$I_J = AJ_0 \sin\delta_0 (\Phi_0/\pi\Phi) \sin(\pi\Phi/\Phi_0) \quad ,$$

where Φ is the flux enclosed within the penetration region over the whole width of the junction and A is the junction area.[1]

In this geometry the phase difference $\delta(x)$ may assume *any* value and this is the essential difference between the flat singly connected junction and the junction between two multiply connected superconductors.

We wish to consider two cylinders, one within the other, separated by a thin insulating barrier in an axial magnetic field. The requirement that the wave functions be single valued in the cylinders demands that they be of the form

$$\psi_1 = |\psi_{1\infty}| f_1(r) e^{in_1\theta} \quad ,$$

$$\psi_2 = |\psi_{2\infty}| f_2(r) e^{i(n_2\theta+\delta_0)} \quad ,$$

where n_1 and n_2 are integers (0,1,2,...) which express the fluxoid quantum states of the cylinders. The Josephson current density between the cylinders will now be

$$J_J = J_0(H) \sin[(n_2 - n_1)\theta + \delta_0] \quad .$$

Integrating around the junction to obtain the total current we have

$$I_J = lrJ_0(H) \int_0^{2\pi} \sin[(n_2 - n_1)\theta + \delta_0]\, d\theta$$
$$= \begin{cases} 0, & n_1 \neq n_2 \ , \\ 2\pi r l J_0(H) \sin\delta_0, & n_1 = n_2 \ . \end{cases} \tag{1}$$

In the case of a solid inner cylinder and a thick outer cylinder as considered by Tilley[2] the *flux* between the two will be quantized $\Phi = (n_2 - n_1)\Phi_0$ and the tunneling current can have only two values, its maximum value ($n_1 = n_2$) and zero ($n_1 \neq n_2$). We wish to extend this idea to tunneling between small thin cylinders. In this case, as will become clear later, the order parameters and the maximum Josephson current may depend strongly upon the axial magnetic field. It is this property that gives the junctions a certain utility in fundamental investigations of superconductivity.

This is the basic idea. It is now necessary to apply the Ginsburg-Landau theory to this situation to answer two questions. (i) Will the Gibbs function sometimes be minimized by having $n_2 = n_1$ and sometimes by having $n_2 \neq n_1$? (ii) Can the system achieve the minimum Gibbs function state or will it get stuck in some metastable states?

We will see that the answer to question (i) is "yes" and that the answer to question (ii) is sometimes "yes" and sometimes equivocal. The uncertainty in the last instance arises because all solutions to the Ginsburg-Landau equations minimize the Gibbs function with respect to the current density in the cylinders. Therefore, the cylinders may not be able to make the transition from a state which is a local minimum with respect to the current to another state of lower Gibbs function. This would lead to no uncertainty if the cylinders were infinitely long as assumed in the theory but the presence of ends may allow them to make a transition from a state which is a local minimum of the Gibbs function. There are, in any case, experimental means for inducing the sample to achieve the lowest Gibbs function state even when it is inclined to get trapped in a higher state.

Except for its application to dual cylinders the treatment which follows is like the Douglass[3] treatment of a single cylinder which when applied to the Little-Parks[4] experiment is equivalent to Tinkham's[5] treatment of that experiment.

If the Ginsburg-Landau (GL) wave function is written in cylindrical symmetry as $\psi = |\psi_\infty| f(r) e^{in\theta}$ then the GL differential equation becomes, in cylindrical coordinates,[1]

$$f - f^3 - \xi^2 \left[\left(\frac{n}{r} - \frac{2\pi A_\theta}{\Phi_0} \right)^2 f - \frac{1}{r}\frac{d}{dr}\left(r \frac{df}{dr} \right) \right] = 0 \ . \quad (2)$$

The current density may be expressed as

$$\begin{aligned} J_\theta &= -\frac{dH_z(r)}{dr} = -\frac{1}{\mu_0}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA_\theta) \right] \\ &= \frac{eh|\psi_\infty|^2 f^2}{m}\left[\frac{n}{r} - \frac{2\pi A_\theta}{\Phi_0} \right] \\ &= \frac{\Phi_0}{2\pi}\frac{f^2}{\mu_0 \lambda^2}\left[\frac{n}{r} - \frac{2\pi A_\theta}{\Phi_0} \right] \ , \end{aligned} \quad (3)$$

where

$$|\psi_\infty|^2 = -\alpha/\beta, \quad \lambda^2 = m/2\mu_0 e^2 |\psi_\infty|^2 \ ,$$

$$\xi = \Phi_0 / 2(2)^{1/2} \pi H_c(T) \lambda, \quad \Phi_0 = h/2e \ ,$$

and α and β are the usual GL parameters.

We will consider a system of two concentric cylinders of mid radius r_1 and r_2 in the thin-film approximation

$$\lambda_1 = \lambda_2 >> d_1 = d_2 = d, \quad \xi_1 = \xi_2 >> d \ .$$

In this case, the current density and the order parameter may be considered constant through the film thickness and the last term in the GL differential equation may be neglected, yielding an algebraic equation in f.

In an applied axial field $H_z = H$ the total vector potential at the two films may be written (the currents and vector potentials have only θ components)

$$\begin{aligned} A(r_1) &= \tfrac{1}{2}\mu_0 r_1 (H + J_1 d + J_2 d) \ , \\ A(r_2) &= \tfrac{1}{2}\mu_0 r_2 (H + (r_1^2/r_2) J_1 d + J_2 d) \ , \end{aligned} \quad (4)$$

so that

$$\begin{aligned} J_1 &= \frac{\Phi_0 f_1^2}{2\pi\mu_0\lambda^2}\left[\frac{n_1}{r_1} - \frac{2\pi A(r_1)}{\Phi_0} \right] \ , \\ J_2 &= \frac{\Phi_0 f_2^2}{2\pi\mu_0\lambda^2}\left[\frac{n_2}{r_2} - \frac{2\pi A(r_2)}{\Phi_0} \right] \ , \end{aligned} \quad (5)$$

and

$$\begin{aligned} f_1^2 &= 1 - \xi^2 [n_1/r_1 - 2\pi A(r_1)/\Phi_0]^2 \ , \\ f_2^2 &= 1 - \xi^2 [n_2/r_2 - 2\pi A(r_2)/\Phi_0]^2 \ . \end{aligned} \quad (6)$$

Equations (4) and (5) may be solved for J_1 and J_2. We write out the expression for J_1 explicitly

$$J_1 = \frac{\Phi_0 (2\lambda^2 + f_2^2 r_2 d) f_1^2}{\pi\mu_0 [2\lambda^2(2\lambda^2 + f_2^2 r_2 d) + (2\lambda^2 + f_2^2 r_2 d) f_1^2 r_1 d - f_1^2 f_2^2 r_1 r_2 d^2 (r_1/r_2)^2]}$$

$$\times \left[\left(\frac{n_1}{r_1} - \frac{\mu_0 \pi r_1 H}{\Phi_0} \right) - \frac{f_2^2 r_1 d}{2\lambda^2 + f_2^2 r_2 d}\left(\frac{n_2}{r_2} - \frac{\mu_0 \pi r_2 H}{\Phi_0} \right) \right] \ .$$

This and a similar expression for J_2 when substituted into Eqs. (6) allow a self-consistent determination of f_1, f_2, J_1, and J_2. However, the equations would require numerical methods for their solution. Because of this complexity and because we shall be primarily interested in very small cylinders we shall from this point limit our consideration to cylinders for which $\lambda^2 >> r_2 d > r_1 d$. This allows us to write

$$\begin{aligned} J_1 &\cong \frac{\Phi_0 f_1^2}{2\pi\mu_0\lambda^2}\left[\frac{n_1}{r_1} - \frac{\pi\mu_0 r_1 H}{\Phi_0} \right] \\ &= \frac{\Phi_0}{2\pi\mu_0\lambda^2}\frac{f_1^2}{r_1}\left[n_1 - \frac{\Phi(r_1)}{\Phi_0} \right] \ , \end{aligned}$$

where $\Phi(r_1)$ is the flux of $B = \mu_0 H$ within the radius r_1. In this approximation we have

$$J_1 \cong \frac{\Phi_0 f_1^2}{2\pi\mu_0\lambda^2}\frac{1}{r_1}\left[n_1 - \frac{\Phi(r_1)}{\Phi_0}\right] ,$$
$$J_2 \cong \frac{\Phi_0 f_2^2}{2\pi\mu_0\lambda^2}\frac{1}{r_2}\left[n_2 - \frac{\Phi(r_2)}{\Phi_0}\right] , \tag{7}$$

$$f_1^2 \cong 1-(\xi/r_1)^2[n_1-\Phi(r_1)/\Phi_0]^2 ,$$
$$f_2^2 \cong 1-(\xi/r_2)^2[n_2-\Phi(r_2)/\Phi_0]^2 . \tag{8}$$

These are not now self-consistent results since the self-consistency has been lost in the approximations.

It can be shown that if f satisfies the GL differential equation then the Gibbs function for the superconductor is given by[6,3]

$$G = G_n - \tfrac{1}{2}\mu_0 H_c^2 \int f^4\, dv + \tfrac{1}{2}\mu_0 \int M^2\, dv , \tag{9}$$

where M is the magnetization and G_n is the Gibbs function of the normal metal. For our case the magnetization is given by

$$M = (J_1+J_2)d, \quad r \le r_1 ,$$
$$M = J_2 d, \quad r_1 \le r \le r_2 , \tag{10}$$
$$M = 0, \quad r > r_2 .$$

From Eqs. (5) and (6) we have

$$J_{1,2}^2 = (\Phi_0^2/4\pi^2\mu_0^2\xi^2\lambda^4) f_{1,2}^4 (1-f_{1,2}^2) . \tag{11}$$

Using Eqs. (9)–(11) and $\alpha|\psi_\infty|^2 = -\mu_0 H_c$ we obtain

$$G - G_n = \pi d l \mu_0 H_c^2 \left[-r_1 f_1^4 - r_2 f_2^4 + (r_2^2 d/\lambda^2) f_2^4(1-f_2^2) + (r_1^2 d/\lambda^2)\{f_1^4(1-f_1^2) \pm 2 f_1^2 f_2^2 [(1-f_1^2)(1-f_2^2)]^{1/2}\}\right] . \tag{12}$$

The last two terms are the magnetic energy. The plus sign preceding the square root applies when J_1 and J_2 are in the same direction and the minus sign when they are in opposite directions. In the approximation in which we are working ($\lambda^2 >> r\, d$) this magnetic contribution is negligible, so we have

$$(G-G_n)/\pi d l \mu_0 H_c^2 = -r_1 f_1^4 - r_2 f_2^4 = g_1 + g_2 .$$

We see that to minimize the Gibbs function we must maximize f^2. From Eqs. (7) this would require that n_1 and n_2 change to the next higher value when $\Phi(r_{1,2})/\Phi_0 = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc. Since $r_2 > r_1$ this leads to the expectation that the outer cylinder will shift to the next higher quantum number at a smaller field than the inner cylinder. We can be certain that this will occur spontaneously only if the order parameter and g_2 are driven to zero. To keep the sample in the minimum Gibbs-function state each cylinder must change its quantum number when $n_{1,2} - \Phi(r_{1,2})/\Phi_0$ is $\frac{1}{2}$. For $g_{1,2}$ to be zero at this point Eqs. (8) require that $\xi/r_{1,2} \ge 2$.

We now wish to obtain an expression for the Josephson current. To do this we will use the results of Ambegaokar and Baratoff[7] for J_0. These must be modified to take account of the magnetic field dependence of the energy gaps which we obtain from Eqs. (8) and the fact that $\Delta(T,H) \propto |\psi_\infty|^2 f^2$. When the energy gaps in the two films are unequal the expression given by Ambegaokar and Baratoff requires numerical evaluation. Near T_c where $\Delta(T,H) << \pi k T$ their expression for J_0 can be written

$$J_0(T,H) \cong \frac{\pi\Delta(T,0)}{2eR_n}\tanh\left[\frac{\Delta(T,0)}{kT}\right] f_1^2 f_2^2 , \tag{13}$$

where R_n is the normal-state junction resistance per square meter. This expression applies when $\Delta_1(T,0) = \Delta_2(T,0)$ but allows for unequal energy gaps in the axial magnetic field. Combining Eqs. (1), (8), and (13) we have

$$I_J = \begin{cases} 2\pi r l\left[\dfrac{\pi\Delta(T,0)}{2eR_n}\right]\tanh\left[\dfrac{\Delta(T,0)}{kT}\right]\left[1-\left(\dfrac{\xi}{r_1}\right)^2\left(n_1-\dfrac{\Phi(r_1)}{\Phi_0}\right)^2\right]\left[1-\left(\dfrac{\xi}{r_2}\right)^2\left(n_2-\dfrac{\Phi(r_2)}{\Phi_0}\right)^2\right] , & n_1 = n_2 , \\ 0, \quad n_1 \ne n_2 . \end{cases} \tag{14}$$

As an example of this result we will calculate I_J for two tin films for which $d = 10^{-7}$ m, $r_2 = 5\times10^{-7}$ m, and $r_1 = 4\times10^{-7}$ m at a relative temperature $T/T_c = t = 0.988$ ($T_c = 3.72$ K, $T = 3.677$ K). At this temperature $\Delta(T) = 1.74\,[1.76\,\Delta(0)](1-t)^{1/2} = 0.331\, kT_c$ and the maximum Josephson current is from Eq. (13) $J_0(T,0) = 0.04\, J_0(0,0)$. Using the values $\lambda_L(0) = 3.5\times10^{-8}$ m and $\xi_0 = 2.3\times10^{-7}$ m we calculate $\xi(t)$ and $\lambda(t)$ in the dirty limit

$$\xi(t) = 0.855[\xi_0 l/(1-t)]^{1/2} ,$$
$$\lambda(t) = \lambda_L(0)[\xi_0/2.66 l(1-t)]^{1/2} .$$

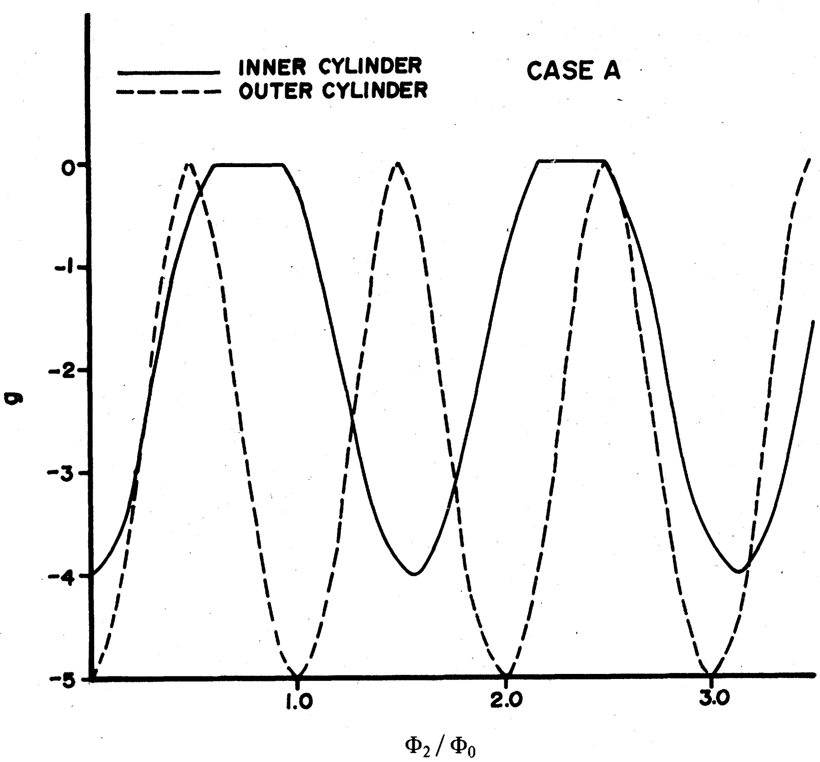

FIG. 1. Reduced Gibbs function for two cylinders of thickness 10^{-7} m and radius $r_1=4\times10^{-7}$ m, $r_2=5\times10^{-7}$ m for case a. $l=6.9\times10^{-8}$ m, $\xi=10^{-6}$ m.

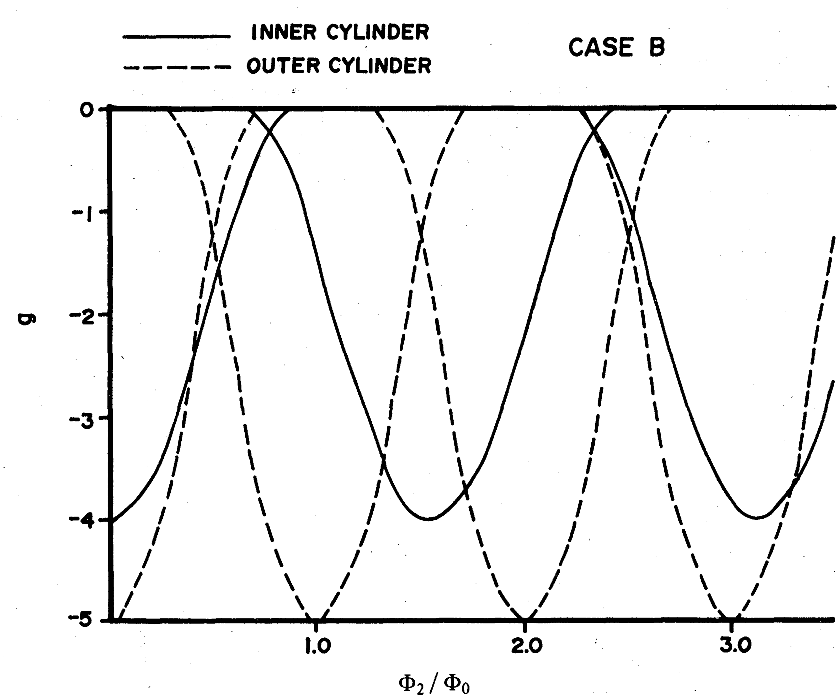

FIG. 2. Reduced Gibbs function for the two cylinders. Case b. $l=3.45\times10^{-8}$ m, $\xi=7.07\times10^{-7}$ m.

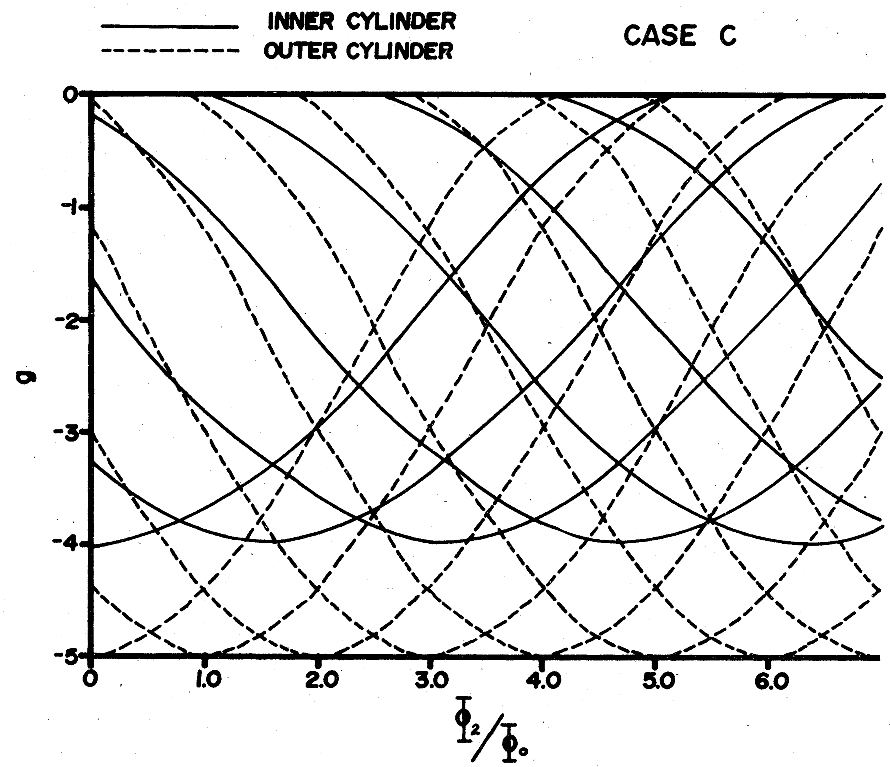

FIG. 3. Reduced Gibbs function for the cylinders. Case c. $l=10^{-9}$ m, $\xi=1.2\times10^{-7}$ m.

We consider mean free paths (a) 6.90×10^{-8} m, (b) 3.45×10^{-8} m, and (c) 10^{-9} m to show three different types of behavior. The reduced Gibbs functions are shown in Figs. 1–3. Case (a) makes $(\xi/r_2)^2=4$ and g_2 is driven to zero when $\Phi(r_2)/\Phi_0=\frac{1}{2}$ so that the outer film will certainly switch from $n_2=0$ to $n_2=1$ at this point. Similarly g_1 is driven to zero when $\Phi_2/\Phi_0=0.675$ at which point $\Phi_1/\Phi_0=0.4$. So for this

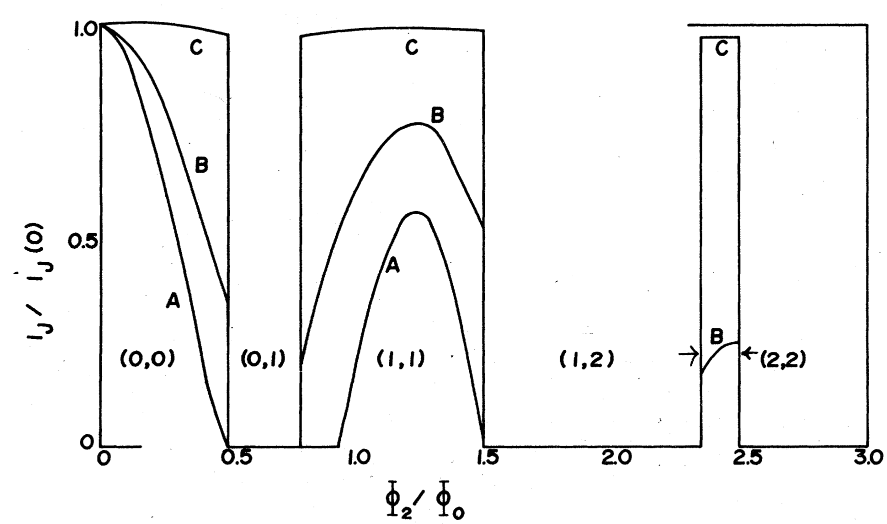

FIG. 4. Josephson-tunneling current for three cases if the cylinders are always in the lowest available state.

case the sample is always in the minimum Gibbs function state. The Josephson current for this case is shown as curve A in Fig. 4. Curve A does not reappear beyond $\Phi_2/\Phi_0 = 1.5$ because for larger fields the cylinders are never in the same quantum state with both order parameters nonzero. The second case shown in curve B is more complicated since $(\xi/r_2)^2 = 2$ and $(\xi/r_1)^2 = 3.25$. Curve B shows the behavior when there is some mechanism which causes the cylinders always to assume their minimum Gibbs function state. This mechanism might be end effects or some briefly applied current or field which momentarily drives both f_1^2 and f_2^2 to zero. For curve C the mean free path is 10^{-9} m so that $(\xi/r_1)^2$ is 0.0906. Again we make the presumption that the sample is always in the minimum Gibbs function state.

It is instructive to reconsider case b when *there is no* auxiliary means of forcing the sample into the lowest available state. Then cylinders can switch quantum states only when g_1 and g_2 become zero (Fig. 2). This is shown in Fig. 5. This leads to discontinuities in the Josephson current since f_1^2 and f_2^2 change discontinuously from zero to finite values when the cylinders switch from meta-stable states to the states of lowest Gibbs function. The results are shown for both increasing (heavy solid curve) and decreasing (lighter curve) magnetic field.

The parameters used in this example place rather stringent but achievable requirements upon the experiment. These requirements can be considerably relaxed. By driving the superconductors briefly into the normal state one can force them to assume their lowest state. This should allow the effects to be demonstrated on much larger cylinders. Further the cylinders may be deposited upon an insulated superconducting wire of high transition temperature, say niobium. Since the magnetic field is excluded from the interior of the wire the factor $(\xi/r_1)^2$ should be replaced by $\xi^2/2r_1\lambda$, where λ is the penetration depth in the wire. If this penetration depth is 5×10^{-8} m the wire may be 4 μm in diameter and the results for the cylinders of tin will be essentially identical to those in the examples. This technique will, therefore, allow the experimenter to increase the cylinder diameter by a factor of 5 without changing the results shown in the figures.

Further one may tunnel from such a wire into a thin cylinder. In this case the wire is always in state $n_1 = 0$ so that the tunneling occurs only when the cylinder is in state $n_2 = 0$. Since the order parameter in the wire would be unaffected by the small magnetic fields the Josephson current becomes $J(T,H) = J(T,0)f_2^2$. This would be an ideal arrangement for measuring the change in the film's order parameter as a function of the magnetic field, since the Josephson current is proportional to a single order parameter.

With a cylindrical junction the energy gap can be measured directly as a function of axial magnetic field by observing the quasiparticle tunneling characteristic. Since the Josephson current can also be determined one can obtain the empirical equivalent of Eq. (13) or its modification above. It should be noted that Eq. (13) and the fundamental ideas presented here are more general than the Ginsburg-Landau theory which was used to calculate the order parameters and Gibbs functions. The junction provides a

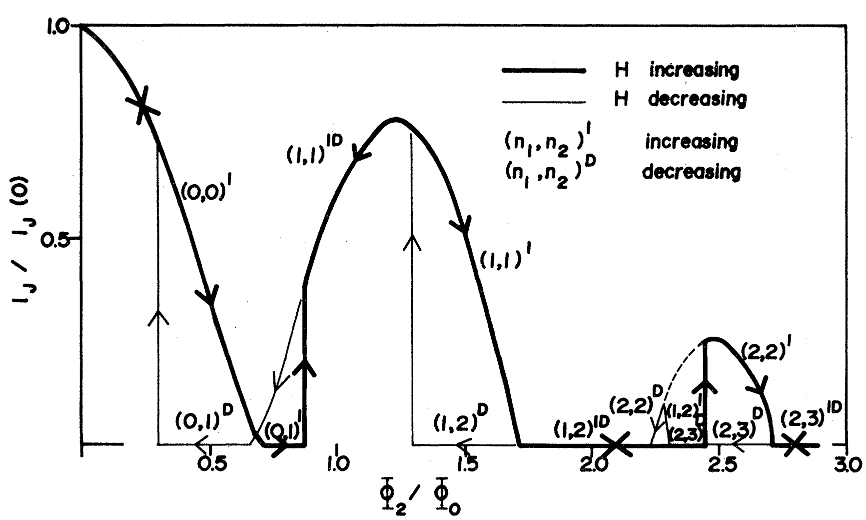

FIG. 5. Josephson current between cylinders for case b if there is no auxiliary mechanism for inducing transitions from meta-stable states.

means, therefore, of checking the predictions of this theory.

If there is an electric potential difference V between the two cylinders then the Josephson current density becomes

$$J_J = J_0(H)\sin(n_1 - n_2)\theta + \delta_0 + (2eV/\hbar)t \quad .$$

Therefore, there is an ac component when $n_1 = n_2$ and no net ac current between the cylinders for $n_1 \neq n_2$. There is, however, a sinusoidal ac current density which rotates with angular velocity $2eV/\hbar$.

ACKNOWLEDGMENTS

We are grateful to W. E. Gettys, M. J. Skove, E. P. Stillwell, and other colleagues for helpful discussions and comments.

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