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Scattering and absorption of electromagnetic waves by a gyrotropic sphere

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The problem of the scattering and absorption of a plane electromagnetic wave by a gyrotropic sphere is solved in this paper. This is a generalization of the classic Mie scattering problem to the case where the dielectric constant is a tensor having axial symmetry. For this problem, Maxwell's equations are not separable in spherical coordinates. The method of solution involves the expansion of the electromagnetic field inside the sphere in a complete set of vector spherical waves, which are solutions of the ordinary vector wave equation. The amplitudes of the scattered spherical waves are found to be expressible in the form of a series of ratios of determinants dependent upon the components of the dielectric tensor, the wavelength of the incident plane wave, and the sphere radius. These scattering amplitudes are examined in various limits. In the limit when the dielectric tensor is a scalar, the Mie results are recovered. When the wavelength of the incident plane wave is large in comparison to the sphere radius, our previous results for helicon oscillations are obtained in addition to new resonant structure induced by the incident electric field. Under conditions when the wavelength inside the sphere is also large compared to the sphere radius (but large compared to the incident wavelength), previous results in the Rayleigh limit are obtained. Selected applications of the results of this paper have been made by Dixon and Furdyna (helicon oscillations, electric dimensional resonances and cyclotron resonance in metals and semiconductors, Alfvén resonances in semimetals), and by Markiewicz (Alfvén oscillations in electron-hole droplets).

I. INTRODUCTION

We consider the problem of the scattering and absorption of a plane electromagnetic wave by a gyrotropic sphere. This is a generalization of the problem of the scattering by a dielectric sphere, discussed in the classic paper by Mie.[1] The difference lies in the assumed form of the dielectric relation between the displacement vector $\vec{D}$ and the electric vector $\vec{E}$:

$$\vec{D}=\overleftrightarrow{\epsilon}\cdot\vec{E}. \tag{1.1}$$

Within the sphere we take the most general form of the dielectric tensor consistent with axial symmetry:

$$\overleftrightarrow{\epsilon}=\begin{pmatrix}\epsilon_{xx} & \epsilon_{xy} & 0\\ -\epsilon_{xy} & \epsilon_{xx} & 0\\ 0 & 0 & \epsilon_{zz}\end{pmatrix}. \tag{1.2}$$

In the classic Mie problem the dielectric tensor is taken to be isotropic ($\epsilon_{xy}=0,\ \epsilon_{zz}=\epsilon_{xx}\equiv\epsilon$). The basic difficulty of our problem is that there is no coordinate system in which the Maxwell equations are separable and which has both axial and spherical symmetry. In an earlier paper on the helicon oscillations of a sphere we showed how to surmount this difficulty and we apply here an equivalent technique to the more general problem.[2]

The basic equations are the Maxwell equations for fields varying in time $\propto e^{-i\omega t}$:

$$\mathrm{curl}\,\vec{E}-i(\omega/c)\vec{B}=0,\quad \mathrm{curl}\,\vec{B}+i(\omega/c)\vec{D}=0, \tag{1.3}$$

where $\vec{B}$ is the magnetic field, c is the velocity of light, and $\vec{D}$ and $\vec{E}$ are related by the dielectric relation (1.1). The boundary conditions at the surface of the sphere follow from these equations by standard arguments.[3] The normal components of $\vec{B}$ and $\vec{D}$ and the tangential components of $\vec{E}$ and $\vec{B}$ are all continuous at the surface. These boundary conditions are not all independent, as we shall see explicitly in our later discussion.

The problem we consider then is to find the solution of these equations corresponding to an incident plane wave plus outgoing spherical waves (scattered waves) outside the sphere (where we take the dielectric relation to be isotropic). Inside the sphere we take the dielectric tensor to have the general axially symmetric form (1.2). That is, inside the sphere $\vec{D}$ must satisfy the equation obtained by eliminating $\vec{B}$ from (1.3) and using (1.1)

$$\mathrm{curl}\,\mathrm{curl}\,(\overleftrightarrow{\epsilon}^{-1}\cdot\vec{D})-(\omega^2/c^2)\vec{D}=0. \tag{1.4}$$

Here

$$\overleftrightarrow{\epsilon}^{-1}=\begin{bmatrix} \dfrac{\epsilon_{xx}}{\epsilon_{xx}^2+\epsilon_{xy}^2} & \dfrac{-\epsilon_{xy}}{\epsilon_{xx}^2+\epsilon_{xy}^2} & 0 \\ \dfrac{\epsilon_{xy}}{\epsilon_{xx}^2+\epsilon_{xy}^2} & \dfrac{\epsilon_{xx}}{\epsilon_{xx}^2+\epsilon_{xy}^2} & 0 \\ 0 & 0 & \dfrac{1}{\epsilon_{zz}} \end{bmatrix} \tag{1.5}$$

is the inverse of the dielectric tensor (1.2). We call (1.4) the gyrotropic wave equation.

In Sec. II we discuss the classic Mie problem, since much of the strategy is the same as in our more general problem. There we introduce vector spherical waves, expand the incident plane wave in terms of these waves, and apply the boundary conditions to obtain the solution. We then construct the cross sections. These items are preludes to corresponding items in the discussion of the more general problem.

In Sec. III we construct the solution for the most general axially symmetric dielectric relation. The first task there is to construct the general solution of the gyrotropic wave equation (1.4) inside the sphere; outside the sphere the fields have the same form as in the Mie problem. We then apply the boundary conditions to obtain the solution. In Sec. IV we discuss various limiting cases of the solution.

The applications of the theory depends upon the particular form of the dielectric coefficients ϵ_{ij}, their dependence upon frequency and upon various material parameters as well as external parameters such as magnetic field. This is generally determined from calculations based upon more or less simplified models of the material medium. In the following paper by Dixon and Furdyna selected applications based on the single relaxation-time magnetoplasma are given.[23] These include helicon and electric dimensional resonances in metals and semiconductors, cyclotron resonances, and Alfvén resonances. These applications, and many others, are of considerable experimental interest and are worthy of much greater development. If the history of the classic Mie solution is any indication, the scope for further discussion of our solution is enormous. In Sec. V we discuss the possibilities for further work.

II. SOLUTION FOR AN ISOTROPIC SPHERE

In Sec. II A we introduce the vector spherical waves, giving the essential formulas we need. Then in Sec. II B we expand the plane wave solution of Maxwell's equations in terms of these waves. With these results it is a simple matter to construct the Mie solution; this we do in Sec. II C. Finally, in Sec. II D we give expressions for the cross sections and discuss the solution in various limits.

A. Vector spherical waves

The vector spherical waves are solutions in spherical coordinates of the vector wave equation[4]:

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{U})-\vec{\nabla}\times(\vec{\nabla}\times\vec{U})+q^2\vec{U}=0\,. \tag{2.1}$$

They can be expressed in terms of simple vector analytical operations on the scalar spherical waves, which are solutions in spherical coordinates of the scalar wave equation

$$\nabla^2 u+q^2 u=0\,. \tag{2.2}$$

The scalar spherical waves regular at the origin are

$$u_{lm}(qr)=j_l(qr)\,Y_{lm}(\hat{r})\,,\qquad l=0,1,2,\ldots;\quad m=0,\pm1,\ldots,\pm l \tag{2.3}$$

where j_l is the spherical Bessel function,[5] and Y_{lm} is the (scalar) spherical harmonic.[6] We shall also want the outgoing spherical waves

$$u_{lm}^{(1)}(qr)=h_l^{(1)}(qr)\,Y_{lm}(\hat{r})\,, \tag{2.4}$$

where $h_l^{(1)}$ is the spherical Hankel function.[5]

The vector spherical waves regular at the origin are of three kinds:

$$\begin{aligned}\vec{B}_{lm}(q\vec{r})&=(1/q)\,\vec{\nabla}u_{lm}\,,\\ \vec{C}_{lm}(q\vec{r})&=[l(l+1)]^{-1/2}\,\vec{L}u_{lm}\,,\\ \vec{A}_{lm}(q\vec{r})&=(i/q)\,\vec{\nabla}\times\vec{C}_{lm}\,,\end{aligned} \tag{2.5}$$

where

$$\vec{L}=-i\,\vec{r}\times\vec{\nabla}\,. \tag{2.6}$$

($\vec{L}$ is the infinitesimal generator of rotations for a scalar field.) With the help of well-known formulas from vector analysis,[7] one can readily verify that these vector fields satisfy the vector wave equation (2.1). In the same way one can also demonstrate the following identities:

$$\vec{\nabla}\cdot\vec{A}_{lm}=0\,,\quad \vec{\nabla}\cdot\vec{B}_{lm}=-qu_{lm}\,,\quad \vec{\nabla}\cdot\vec{C}_{lm}=0 \tag{2.7}$$

and

$$\vec{\nabla}\times\vec{A}_{lm}=iq\vec{C}_{lm}\,,\quad \vec{\nabla}\times\vec{B}_{lm}=0\,,\quad \vec{\nabla}\times\vec{C}_{lm}=-iq\vec{A}_{lm}\,. \tag{2.8}$$

The vector spherical waves can be expressed explicitly in terms of vector spherical harmonics[2]:

$$\vec{A}_{lm}=\left(\frac{l}{2l+1}\right)^{1/2} j_{l+1}(qr)\,\vec{Y}^m_{l,\,l+1}(\hat{r})$$
$$-\left(\frac{l+1}{2l+1}\right)^{1/2} j_{l-1}(qr)\,\vec{Y}^m_{l,\,l-1}(\hat{r}),$$
$$\vec{B}_{lm}=\left(\frac{l+1}{2l+1}\right)^{1/2} j_{l+1}(qr)\,\vec{Y}^m_{l,\,l+1}(\hat{r}) \tag{2.9}$$
$$+\left(\frac{l}{2l+1}\right)^{1/2} j_{l-1}(qr)\,\vec{Y}^m_{l,\,l-1}(\hat{r}),$$
$$\vec{C}_{lm}=j_l(qr)\,\vec{Y}^m_{l,\,l}(\hat{r}).$$

The vector spherical harmonics $\vec{Y}^m_{L,\,l}$ in turn can be expressed in terms of simple operations on the usual scalar spherical harmonics:

$$\vec{Y}^m_{l,\,l+1}(\hat{r})=[(l+1)(2l+1)]^{-1/2}r^{l+2}\vec{\nabla}r^{-l-1}Y_{lm}(\hat{r}),$$
$$\vec{Y}^m_{l,\,l}(\hat{r})=[l(l+1)]^{-1/2}\vec{L}Y_{lm}(\hat{r}), \tag{2.10}$$
$$\vec{Y}^m_{l,\,l-1}(\hat{r})=[l(2l+1)]^{-1/2}r^{-l+1}\vec{\nabla}r^{l}Y_{lm}(\hat{r}).$$

In the application of the boundary conditions we shall also need the formulas

$$\hat{r}\cdot\vec{A}_{lm}=-[l(l+1)]^{1/2}[j_l(qr)/qr]Y_{lm}(\hat{r}),$$
$$\hat{r}\cdot\vec{B}_{lm}=[dj_l(qr)/d(qr)]Y_{lm}(\hat{r}), \tag{2.11}$$
$$\hat{r}\cdot\vec{C}_{lm}=0,$$

and

$$\hat{r}\times\vec{A}_{lm}=-i[j_l(qr)/qr+dj_l(qr)/d(qr)]\,\vec{Y}^m_{l,\,l}(\hat{r})$$
$$\equiv -i\alpha_l(qr)\,\vec{Y}^m_{l,\,l}(\hat{r}),$$
$$\hat{r}\times\vec{B}_{lm}=i[l(l+1)]^{1/2}[j_l(qr)/qr]\,\vec{Y}^m_{l,\,l}(\hat{r}), \tag{2.12}$$
$$\hat{r}\times\vec{C}_{lm}=ij_l(qr)\,\{(l/2l+1)^{1/2}\vec{Y}^m_{l,\,l+1}(\hat{r})$$
$$+[(l+1)/(2l+1)]^{1/2}\vec{Y}^m_{l,\,l-1}(\hat{r})\}.$$

These all follow directly from the definitions (2.5) with the use of (2.9), (2.10), and, in the case of the first formula in (2.11), the identity

$$L^2Y_{lm}=l(l+1)Y_{lm}. \tag{2.13}$$

Finally, we shall want the outgoing vector spherical waves $\vec{A}^{(1)}_{lm}, \vec{B}^{(1)}_{lm}, \vec{C}^{(1)}_{lm}$, which are obtained by replacing u_{lm} with $u^{(1)}_{lm}$ in (2.5). If the spherical Bessel functions j_l are replaced by spherical Hankel functions $h^{(1)}_l$, the formulas of this section hold as well for the outgoing vector spherical waves.

B. Expansion of a plane wave

The plane-wave solution of Maxwell's equations (1.3) in an isotropic medium with dielectric constant ϵ is

$$\vec{E}(\vec{r})=\vec{E}_1e^{i\vec{k}\cdot\vec{r}},\quad \vec{B}(\vec{r})=\vec{B}_1e^{i\vec{k}\cdot\vec{r}}, \tag{2.14}$$

where

$$k^2=\epsilon\omega^2/c^2, \tag{2.15}$$

and the amplitudes $\vec{E}_1$ and $\vec{B}_1$ are constant vectors with the relations

$$\vec{B}_1=\frac{ck}{\omega}\hat{k}\times\vec{E}_1,\quad \vec{E}_1=-\frac{\omega}{ck}\hat{k}\times\vec{B}_1. \tag{2.16}$$

Consider first the expansion of the electric field $\vec{E}(\vec{r})$ in (2.14). We begin by using the spherical unit vectors

$$\hat{e}_1=-(\hat{x}+i\hat{y})/\sqrt{2},\quad \hat{e}_0=\hat{z},\quad \hat{e}_{-1}=(\hat{x}-i\hat{y})/\sqrt{2} \tag{2.17}$$

to expand the amplitude

$$\vec{E}_1=\sum_{m=-1}^{1}\hat{e}^*_m\cdot\vec{E}_1\hat{e}_m. \tag{2.18}$$

Next we use the well-known expansion

$$e^{i\vec{k}\cdot\vec{r}}=\sum_{l=0}^{\infty}(i)^l(2l+1)j_l(kr)P_l(\hat{k}\cdot\hat{r}), \tag{2.19}$$

and the addition theorem for spherical harmonics[8]:

$$\frac{2l+1}{4\pi}P_l(\hat{k}\cdot\hat{r})=\sum_{m=-l}^{l}Y_{lm}(\hat{k})^*Y_{lm}(\hat{r}), \tag{2.20}$$

to obtain the expansion

$$\vec{E}(\vec{r})=4\pi\sum_{l=0}^{\infty}(i)^lj_l(kr)$$
$$\times\sum_{m=-l}^{+l}\sum_{m'=-1}^{+1}[\hat{e}_{m'}Y_{lm}(\hat{k})]^*\cdot\vec{E}_1[\hat{e}_{m'}Y_{lm}(\hat{r})]. \tag{2.21}$$

The products of spherical unit vectors and spherical harmonics can be expressed in terms of vector spherical harmonics[6]:

$$\hat{e}_{m'}Y_{lm}(r)=\sum_{L,M}\vec{Y}^M_{L,\,l}(\hat{r})(l1LM\,|\,lm1m'), \tag{2.22}$$

where $(l1LM\,|\,lm1m')$ is the Clebsch-Gordan coefficient. Inserting this expression for each product in (2.21) and using the orthogonality relation for the Clebsch-Gordan coefficients,

$$\sum_{m,m'}(l1LM\,|\,lm1m')(lm1m'\,|\,l1L'M')=\delta_{LL'}\delta_{MM'}, \tag{2.23}$$

we get the result

$$\vec{E}(\vec{r})=4\pi\sum_{L=1}^{\infty}\sum_{M=-L}^{L}\sum_{l=L-1}^{L+1}(i)^l\,\vec{Y}^M_{L,\,l}(\hat{k})^*$$
$$\cdot\vec{E}_1j_l(kr)\,\vec{Y}^M_{L,\,l}(\hat{r}). \tag{2.24}$$

The products of spherical Bessel functions and vector spherical harmonics in this expression can be expressed in terms of vector spherical waves using (2.9). This allows us to rewrite (2.24) in the form

$$\vec{E}(\vec{r})=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(i)^l[\vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{E}_1\vec{A}_{lm}(k\vec{r}) - iY_{lm}(\hat{k})^*\hat{k}\cdot\vec{E}_1\vec{B}_{lm}(k\vec{r}) + \vec{Y}_{ll}^m(\hat{k})^*\cdot\vec{E}_1\vec{C}_{lm}(k\vec{r})], \tag{2.25}$$

where we have used the relations

$$\left(\frac{l+1}{2l+1}\right)^{1/2}\vec{Y}_{l,l-1}^m(\hat{k})+\left(\frac{l}{2l+1}\right)^{1/2}\vec{Y}_{l,l+1}^m(\hat{k}) = -i\hat{k}\times\vec{Y}_{l,l}^m(\hat{k}),$$

$$\left(\frac{l}{2l+1}\right)^{1/2}\vec{Y}_{l,l-1}^m(\hat{k})-\left(\frac{l+1}{2l+1}\right)^{1/2}\vec{Y}_{l,l+1}^m(\hat{k}) = Y_{lm}(\hat{k})\hat{k}, \tag{2.26}$$

which follow from (2.10).

Using now (2.16), we see that the coefficient of $\vec{B}_{lm}$ vanishes, as it must since $\vec{\nabla}\cdot\vec{E}=0$, and we can write (2.25) in the final form:

$$\vec{E}(\vec{r})=4\pi\sum_{l=1}^{\infty}\sum_{m=-l}^{l}(i)^l[\vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{E}_1\vec{A}_{lm}(k\vec{r}) - \frac{\omega}{ck}\vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{B}_1\vec{C}_{lm}(k\vec{r})]. \tag{2.27}$$

In the same way we obtain the corresponding expression for the magnetic field:

$$\vec{B}(\vec{r})=4\pi\sum_{l=1}^{\infty}\sum_{m=-l}^{l}(i)^l[\vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{B}_1\vec{A}_{lm}(k\vec{r}) + \frac{ck}{\omega}\vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{E}_1\vec{C}_{lm}(k\vec{r})] \tag{2.28}$$

The expressions (2.27) and (2.28) are our desired expansion of the plane-wave solution of Maxwell's equations in vector spherical waves. They have been arranged so that the terms proportional to the electric amplitude $\vec{E}_1$ correspond to *electric* or TM waves, while the terms proportional to the magnetic amplitude $\vec{B}_1$ correspond to *magnetic* or TE waves.[9] There being no natural axis of symmetry for an isotropic sphere, the most obvious choice of the polar axis in the Mie problem is the direction of propagation of the incident plane wave, i.e., $\hat{z}=\hat{k}$. In this case

$$\vec{Y}_{ll}^m(\hat{z})\times\hat{z}=i[(2l+1)/8\pi]^{1/2}\hat{e}_m,\quad m=\pm 1 \tag{2.29}$$

and vanishes for other values of m. For this choice of axis the expansion is well known.[10] We will need the more general case when we attack the problem of a gyrotropic sphere, where we must choose the axis of cylindrical symmetry as the polar axis.

C. Mie solution

The classic Mie problem is the scattering of a plane electromagnetic wave by a conducting dielectric sphere. The medium outside the sphere has a real isotropic dielectric constant ϵ_2, while inside the sphere the dielectric constant is isotropic and of the form:

$$\epsilon_1=\epsilon_l+i(4\pi\sigma/\omega), \tag{2.30}$$

where ϵ_l is the background (or lattice) dielectric constant, and σ is the conductivity. The fields outside the sphere correspond to the incident plane wave plus an outgoing scattered wave:

$$\vec{E}_{\rm out}=\vec{E}_1e^{i\vec{k}\cdot\vec{r}}+\vec{E}_{\rm scat},\quad \vec{B}_{\rm out}=\vec{B}_1e^{i\vec{k}\cdot\vec{r}}+\vec{B}_{\rm scat}, \tag{2.31}$$

where

$$k^2=\epsilon_2\omega^2/c^2. \tag{2.32}$$

The expansion of the plane wave in terms of vector spherical waves is given in (2.27) and (2.28); the scattered fields can be expanded in terms of outgoing spherical waves

$$\vec{E}_{\rm scat}=4\pi\sum_{l,m}(i)^l\left(f_{lm}\vec{A}_{lm}^{(1)}(k\vec{r}) - \frac{\omega}{ck}g_{lm}\vec{C}_{lm}^{(1)}(k\vec{r})\right),$$

$$\vec{B}_{\rm scat}=4\pi\sum_{l,m}(i)^l\left(g_{lm}\vec{A}_{lm}^{(1)}(k\vec{r}) + \frac{ck}{\omega}f_{lm}\vec{C}_{lm}^{(1)}(k\vec{r})\right). \tag{2.33}$$

From formulas (2.7) and (2.8) it is obvious that these scattered fields fulfill Maxwell's equations (1.3). In the same way it is clear that the fields inside the sphere can be expanded in regular vector spherical waves

$$\vec{E}_{\rm in}=4\pi\sum_{l,m}(i)^l\left(a_{lm}\vec{A}_{lm}(q\vec{r}) - \frac{\omega}{cq}c_{lm}\vec{C}_{lm}(q\vec{r})\right),$$

$$\vec{B}_{\rm in}=4\pi\sum_{l,m}(i)^l\left(c_{lm}\vec{A}_{lm}(q\vec{r}) + \frac{cq}{\omega}a_{lm}\vec{C}_{lm}(q\vec{r})\right), \tag{2.34}$$

where

$$q^2=\epsilon_1\omega^2/c^2. \tag{2.35}$$

The coefficients f_{lm}, g_{lm}, a_{lm}, and c_{lm} are determined from the boundary conditions at the surface of the sphere. The continuity of the tangential components of the electric field requires $\hat{r}\times\vec{E}_{\rm out}$

$=\hat{r}\times\vec{E}_{\text{in}}$ at the surface of the sphere. Using (2.12) and the orthogonality of the vector spherical harmonics, this gives

$$a_{lm}\alpha_l(qa)=f_{lm}\alpha_l^{(1)}(ka)+\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1\alpha_l(ka), \tag{2.36}$$

and

$$c_{lm}\frac{j_l(qa)}{qa}=g_{lm}\frac{h_l^{(1)}(ka)}{ka}+\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{B}_1\frac{j_l(ka)}{ka}, \tag{2.37}$$

where

$$\alpha_l(x)\equiv\frac{1}{x}\frac{d}{dx}[xj_l(x)],\qquad \alpha_l^{(1)}(x)\equiv\frac{1}{x}\frac{d}{dx}[xh_l^{(1)}(x)]. \tag{2.38}$$

The continuity of the normal component of the displacement vector requires $\epsilon_2\hat{r}\cdot\vec{E}_{\text{out}}=\epsilon_1\hat{r}\cdot\vec{E}_{\text{in}}$. Using (2.11) and the orthogonality of the spherical harmonics, this gives

$$a_{lm}\epsilon_1\frac{j_l(qa)}{qa}=f_{lm}\epsilon_2\frac{h_l^{(1)}(ka)}{ka}+\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1\epsilon_2\frac{j_l(ka)}{ka}. \tag{2.39}$$

In the same way from the continuity of the tangential components of the magnetic field we get

$$c_{lm}\alpha_l(qa)=g_{lm}\alpha_l^{(1)}(ka)+\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{B}_1\alpha_l(ka), \tag{2.40}$$

and

$$a_{lm}qaj_l(qa)=f_{lm}kah_l^{(1)}(ka)+\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1kaj_l(ka). \tag{2.41}$$

Continuity of the normal component of the magnetic field gives

$$c_{lm}\frac{j_l(qa)}{qa}=g_{lm}\frac{h_l^{(1)}(ka)}{ka}+\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{B}_1\frac{j_l(ka)}{ka}. \tag{2.42}$$

The six equations (2.36), (2.37), and (2.39)–(2.42) are not independent since (2.42) is identical with (2.37) and, using (2.32) and (2.35), it is seen that (2.39) is equivalent to (2.41). We discard equation (2.37) so that the *electric* coefficients a_{lm} and f_{lm} are determined from (2.36) and (2.39), which come from the electric boundary conditions. Similarly we discard (2.41) so that the *magnetic* coefficients c_{lm} and g_{lm} are determined from (2.40) and (2.42), which come from the magnetic boundary conditions. This differs from the universal practice of applying boundary conditions only to the tangential components of the fields.[11] Our choice, however, is more convenient in discussing the quasistatic limit, where the wavelength of the incident field is large compared with the sphere radius, but the wavelength inside the sphere is not necessarily small.

The solution of these equations is straightforward. To simplify the notation we introduce

$$x\equiv ka,\quad y\equiv qa. \tag{2.43}$$

Then, from (2.36) and (2.39) we find

$$\begin{aligned}a_{lm}&=\left[i\epsilon_2x^{-3}\Big/\left(\epsilon_1\frac{j_l(y)}{y}\alpha_l^{(1)}(x)-\epsilon_2\frac{h_l^{(1)}(x)}{x}\alpha_l(y)\right)\right]\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1,\\ f_{lm}&=\left[\left(\epsilon_2\frac{j_l(x)}{x}\alpha_l(y)-\epsilon_1\frac{j_l(y)}{y}\alpha_l(x)\right)\Big/\left(\epsilon_1\frac{j_l(y)}{y}\alpha_l^{(1)}(x)-\epsilon_2\frac{h_l^{(1)}(x)}{x}\alpha_l(y)\right)\right]\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1,\end{aligned} \tag{2.44}$$

where we have used the identity[5]:

$$\alpha_l^{(1)}(x)j_l(x)-\alpha_l(x)h_l^{(1)}(x)=j_l(x)\frac{d}{dx}h_l^{(1)}(x)-h_l^{(1)}(x)\frac{d}{dx}j_l(x)=ix^{-2}. \tag{2.45}$$

Similarly, from Eqs. (2.40) and (2.42) we find

$$\begin{aligned}c_{lm}&=\left[ix^{-3}\Big/\left(\frac{j_l(y)}{y}\alpha_l^{(1)}(x)-\frac{h_l^{(1)}(x)}{x}\alpha_l(y)\right)\right]\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{B}_1,\\ g_{lm}&=\left[\left(\frac{j_l(x)}{x}\alpha_l(y)-\frac{j_l(y)}{y}\alpha_l(x)\right)\Big/\left(\frac{j_l(y)}{y}\alpha_l^{(1)}(x)-\frac{h_l^{(1)}(x)}{x}\alpha_l(y)\right)\right]\vec{Y}^m_{l,l}(\hat{k})^*\times\hat{k}\cdot\vec{B}_1.\end{aligned} \tag{2.46}$$

These expressions for f_{lm} and g_{lm} are well known. Thus, in terms of the coefficients a_l and b_l used by van de Hulst and by Kerker,[1]

$$f_{lm} = -a_l \vec{Y}^m_{l,l}(\hat{k})^* \times \hat{k}\cdot \vec{E}_1 , \quad g_{lm} = -b_l \vec{Y}^m_{l,l}(\hat{k}) \times \hat{k}\cdot \vec{B}_1 . \tag{2.47}$$

For a summary of notations used by various authors see Kerker's Table 3.1, pp. 60–63.

D. Cross sections

The intensity (energy per unit time per unit area) of an electromagnetic wave varying in time $\propto e^{-i\omega t}$ is given by the time average Poynting vector[12]

$$\vec{S} = (c/8\pi)\,\mathrm{Re}(\vec{E}\times\vec{B}^*) . \tag{2.48}$$

The *differential scattering cross section* is the ratio of the scattered energy per unit time per unit solid angle far from the sphere to the intensity of the incident plane wave, that is,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{scat}} = \lim_{r\to\infty} r^2 \hat{r}\cdot\vec{S}_{\mathrm{scat}} \Big/ \hat{k}\cdot\vec{S}_{\mathrm{inc}} . \tag{2.49}$$

The fields far from the sphere have the asymptotic form

$$\begin{aligned} \vec{E}_{\mathrm{out}} &\sim \vec{E}_1 e^{i\vec{k}\cdot\vec{r}} + \vec{F} e^{ikr}/r , \\ \vec{B}_{\mathrm{out}} &\sim \vec{B}_1 e^{i\vec{k}\cdot\vec{r}} + (ck/\omega)\,\hat{r}\times\vec{F}e^{ikr}/r , \end{aligned} \tag{2.50}$$

where

$$\vec{F}(\hat{k},\hat{r}) = \frac{4\pi}{ik} \sum_{l,m} \left(f_{lm}\vec{Y}^m_{ll}(\hat{r})\times\hat{r} - \frac{\omega}{ck} g_{lm}\vec{Y}^m_{ll}(\hat{r}) \right) \tag{2.51}$$

is the *vector scattering amplitude*. This result is easily obtained from (2.33) and (2.5), using the asymptotic form of the spherical Hankel function[5]:

$$h_l^{(1)}(x) \sim (-i)^{l+1} x^{-1} e^{ix} , \quad x \gg 1 . \tag{2.52}$$

In forming $\vec{S}_{\mathrm{inc}}$ and $\vec{S}_{\mathrm{scat}}$ in (2.49) we use, respectively, the first and second terms in (2.50). The result is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{scat}} = \frac{|\vec{F}|^2}{|\vec{E}_1|^2} . \tag{2.53}$$

The total scattering cross section is

$$\begin{aligned} \sigma_{\mathrm{scat}} &= \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{scat}} \\ &= \left(\frac{4\pi}{k}\right)^2 \sum_{l,m} \frac{|f_{lm}|^2 + (\omega/ck)^2 |g_{lm}|^2}{|\vec{E}_1|^2} , \end{aligned} \tag{2.54}$$

where we have used the orthonormality of the vector spherical harmonics[6]

$$\int d\Omega\, \vec{Y}^{M'}_{L',l'}(\hat{r})\cdot\vec{Y}^{M*}_{L,l}(\hat{r}) = \delta_{L',L}\delta_{l',l}\delta_{M',M} . \tag{2.55}$$

The *absorption cross section* is the ratio of the power absorbed by the sphere to the incident intensity, that is

$$\sigma_{\mathrm{abs}} = -\left(\int d\Omega \hat{r}\cdot\vec{S}\right)_{r=a} \Big/ \hat{k}\cdot\vec{S}_{\mathrm{inc}} . \tag{2.56}$$

In the integral we form $\vec{S}$ from the fields (2.31) just outside the sphere. The integral itself can be performed using (2.9) and the orthonormality of the vector spherical harmonics. After a fair amount of algebra the result can be expressed as

$$\sigma_{\mathrm{abs}} = -\left(\frac{4\pi}{k}\right)^2 \sum_{l,m} \left[\frac{|f_{lm}|^2 + (\omega/ck)^2 |g_{lm}|^2}{|\vec{E}_1|^2} + \mathrm{Re}\left(\frac{f_{lm}\vec{Y}^m_{ll}(\hat{k})\times\hat{k}\cdot\vec{E}_1^* - (\omega/ck) g_{lm}\vec{Y}^m_{ll}(\hat{k})\cdot\vec{E}_1^*}{|\vec{E}_1|^2} \right) \right] . \tag{2.57}$$

The *total cross section* (extinction cross section) is the sum of the scattering and absorption cross sections

$$\sigma_{\mathrm{tot}} = \sigma_{\mathrm{scat}} + \sigma_{\mathrm{abs}} = -\left(\frac{4\pi}{k}\right)^2 \sum_{l,m} \mathrm{Re}\left(\frac{f_{lm}\vec{Y}^m_{ll}(\hat{k})\times\hat{k}\cdot\vec{E}_1^* - (\omega/ck) g_{lm}\vec{Y}^m_{ll}(\hat{k})\cdot\vec{E}_1^*}{|\vec{E}_1|^2} \right) . \tag{2.58}$$

Comparing this with (2.51) we see the well-known relation between the total cross section and the imaginary part of the forward scattering amplitude[13]

$$\sigma_{\mathrm{tot}} = (4\pi/k)\,\mathrm{Im}[\vec{F}(\hat{k},\hat{k})\cdot\vec{E}_1^*/|\vec{E}_1|^2] . \tag{2.59}$$

Note that the various cross sections are in fact independent of the incident field amplitude $|\vec{E}_1| = (\omega/ck)|\vec{B}_1|$, since the vector scattering amplitude $\vec{F}$ and the coefficients f_{lm} and g_{lm} are all proportional to $|\vec{E}_1|$.

From the point of view of the (isotropic) Mie problem the above results are unnecessarily complicated, since the direction of the polar axis is left arbitrary. However, in the discussion of the scattering by a gyrotropic sphere we must choose the polar axis along the axis of symmetry of the dielectric tensor, so this generality will be required.

III. SOLUTION FOR A GYROTROPIC SPHERE

In Sec. III A we construct the general regular solution in spherical coordinates of Maxwell's equations for a gyrotropic medium. In Sec. III B we expand the fields inside the sphere in terms of this solution; the fields outside have the same form as in the isotropic case. We then fit the boundary conditions to obtain an infinite set of coupled equations for the expansion coefficients. In Sec. III C we discuss the numerical solution of these equations, obtaining expressions for the coefficients f_{lm} and g_{lm} which characterize the scattered fields. Finally, in Sec. III D we discuss the auxiliary eigenvalue problem which occurs in the general solution.

A. General solution in a gyrotropic medium

We want to find the general solution of the gyrotropic wave equation (1.4). It will be convenient to express the inverse dielectric relation occurring in this equation in vector notation

$$\vec{E}=\tilde{\epsilon}^{-1}\cdot\vec{D}=(\vec{D}+\tilde{\gamma}\hat{z}\cdot\vec{D}\hat{z}+\tilde{W}\hat{z}\times\vec{D})/\tilde{\epsilon}\ , \tag{3.1}$$

where

$$\tilde{\epsilon}=\frac{\epsilon_{xx}^2+\epsilon_{xy}^2}{\epsilon_{xx}}\ ,\quad \tilde{\gamma}=\frac{\epsilon_{xx}^2+\epsilon_{xy}^2}{\epsilon_{xx}\epsilon_{zz}}-1,\quad \tilde{W}=\frac{\epsilon_{xy}}{\epsilon_{xx}}\ . \tag{3.2}$$

We will seek a solution of (1.4) in the form

$$\vec{D}=\sum_l\,[a_{lm}\vec{A}_{lm}(q\vec{r})+c_{lm}\vec{C}_{lm}(q\vec{r})]\,, \tag{3.3}$$

where q is as yet undetermined. The irrotational vector spherical waves $\vec{B}_{lm}$ do not occur in this expansion since $\operatorname{div}\vec{D}$ is zero. In forming (3.1) we need the formulas[14]

$$\begin{aligned}
i\hat{z}\times\vec{A}_{lm}&=\sum_{l'}\left(\frac{m}{l(l+1)}\,\delta_{ll'}\vec{A}_{l'm}-\frac{m}{[l(l+1)]^{1/2}}\,\delta_{ll'}\vec{B}_{l'm}+M^m_{ll'}\vec{C}_{l'm}\right),\\
i\hat{z}\times\vec{B}_{lm}&=\sum_{l'}\left(-\frac{m}{[l(l+1)]^{1/2}}\,\delta_{ll'}A_{l'm}-N^m_{ll'}\vec{C}_{l'm}\right),\\
i\hat{z}\times\vec{C}_{lm}&=\sum_{l'}\left(-M^m_{ll'}\vec{A}_{l'm}-L^m_{ll'}\vec{B}_{l'm}+\frac{m}{l(l+1)}\delta_{ll'}\vec{C}_{l'm}\right),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
M^m_{ll'}&=-H(l+1,m)\delta_{l',l+1}+H(l,m)\delta_{l',l-1}=-M^m_{l'l}\ ,\\
N^m_{ll'}&=[(l+1)/l]^{1/2}H(l+1,m)\delta_{l',l+1}+[l/(l+1)]^{1/2}H(l,m)\delta_{l',l-1}=L^m_{l'l}\ ,\\
L^m_{ll'}&=[(l+1)/(l+2)]^{1/2}H(l+1,m)\delta_{l',l+1}+[l/(l-1)]^{1/2}H(l,m)\delta_{l',l-1}=N^m_{l'l}\ ,
\end{aligned} \tag{3.5}$$

with

$$H(l,m)=[(l^2-1)(l^2-m^2)/l^2(4l^2-1)]^{1/2}\ . \tag{3.6}$$

Using the identity

$$\hat{z}\cdot\vec{V}\hat{z}=\vec{V}+\hat{z}\times(\hat{z}\times\vec{V}) \tag{3.7}$$

and (3.4), the remaining formulas which we need are easily obtained;

$$\begin{aligned}
\hat{z}\cdot\vec{A}_{lm}\hat{z}&=\sum_{l'}\,(R^m_{ll'}\vec{A}_{l'm}+S^m_{ll'}\vec{B}_{l'm}+T^m_{ll'}\vec{C}_{l'm})\ ,\\
\hat{z}\cdot\vec{C}_{lm}\hat{z}&=\sum_{l'}\left(P^m_{ll'}\vec{A}_{l'm}+Q^m_{ll'}\vec{B}_{l'm}+\frac{m^2}{l(l+1)}\delta_{ll'}\vec{C}_{l'm}\right),
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
R^m_{ll'}&=H(l+2,m)H(l+1,m)\delta_{l',l+2}+\left(\frac{l+1}{l+2}H^2(l+1,m)+\frac{l}{l-1}H^2(l,m)\right)\delta_{l',l}+H(l,m)H(l-1,m)\delta_{l',l-2}=R^m_{l'l}\ ,\\
S^m_{ll'}&=-\left(\frac{l+2}{l+3}\right)^{1/2}H(l+1,m)H(l+2,m)\delta_{l',l+2}+[l(l+1)]^{-1/2}\left(\frac{m^2}{l(l+1)}-(l+1)H^2(l+1,m)+lH^2(l,m)\right)\delta_{l',l}\\
&\quad+\left(\frac{l-1}{l-2}\right)^{1/2}H(l,m)H(l-1,m)\delta_{l',l-2}\ ,\\
T^m_{ll'}&=-[m/(l+2)]H(l+1,m)\delta_{l',l+1}-[m/(l-1)]H(l,m)\delta_{l',l-1}=P^m_{l'l}\ ,\\
Q^m_{ll'}&=(m/l)[(l+1)/(l+2)]^{1/2}H(l+1,m)\delta_{l',l+1}-[m/(l+1)][l/(l-1)]^{1/2}H(l,m)\delta_{l',l-1}\ .
\end{aligned} \tag{3.9}$$

Putting expansion (3.3) in (3.1) and using formulas (3.4) and (3.8), we get the following expansion for the electric field vector:

$$\tilde{\epsilon}\vec{E}=\sum_{l,l'}\left\{\left[a_{lm}\left(\delta_{ll'}+\tilde{\gamma}R^m_{ll'}-i\tilde{W}\frac{m}{l(l+1)}\delta_{ll'}\right)+c_{lm}(\tilde{\gamma}P^m_{ll'}+i\tilde{W}M^m_{ll'})\right]\vec{A}_{l'm}\right.$$
$$+\left[a_{lm}\left(\tilde{\gamma}S^m_{ll'}+i\tilde{W}\frac{m}{[l(l+1)]^{1/2}}\delta_{ll'}\right)+c_{lm}(\tilde{\gamma}Q^m_{ll'}+i\tilde{W}L^m_{ll'})\right]\vec{B}_{l'm}$$
$$\left.+\left[a_{lm}(\tilde{\gamma}T^m_{ll'}-i\tilde{W}M^m_{ll'})+c_{lm}\left(1-\frac{m(m\tilde{\gamma}-i\tilde{W})}{l(l+1)}\right)\delta_{ll'}\right]\vec{C}_{l'm}\right\}. \tag{3.10}$$

Remembering that $\vec{E}=\tilde{\epsilon}^{-1}\vec{D}$, we put this expression in the gyrotropic wave equation (1.4). We then use the curl formulas (2.8). Finally, equating separately the coefficients of $\vec{A}_{lm}$ and $\vec{C}_{lm}$ to zero, we get an infinite set of equations for the coefficients a_{lm} and c_{lm},

$$\sum_{l'}\left\{a_{l'm}\left[\left(\lambda-\frac{m}{l(l+1)}\right)\delta_{l'l}-i\frac{\tilde{\gamma}}{\tilde{W}}R^m_{l'l}\right]+c_{l'm}\left(M^m_{l'l}-i\frac{\tilde{\gamma}}{\tilde{W}}P^m_{l'l}\right)\right\}=0\ ,$$
$$\sum_{l'}\left[c_{l'm}\left(\lambda-\frac{m}{l(l+1)}-i\frac{\tilde{\gamma}}{\tilde{W}}\frac{m^2}{l(l+1)}\right)\delta_{l'l}-a_{l'm}\left(M^m_{l'l}+i\frac{\tilde{\gamma}}{\tilde{W}}T^m_{l'l}\right)\right]=0\ , \tag{3.11}$$

where we have put

$$q^2\equiv[\tilde{\epsilon}/(1-i\lambda\tilde{W})]\omega^2/c^2\ . \tag{3.12}$$

For every value of λ for which there is a solution of these equations, there will be a corresponding solution given by (3.3) of the gyrotropic wave equation.

The solution of Eqs. (3.11) are of two types, corresponding to odd and even parity for the corresponding $\vec{D}$. For the *odd* solutions we write

$$d^-_{lm}(\lambda)\equiv\begin{cases}c_{lm}, & l\text{ odd},\\ a_{lm}, & l\text{ even},\end{cases} \tag{3.13}$$

while for the *even* solutions we write

$$d^+_{lm}(\lambda)\equiv\begin{cases}a_{lm}, & l\text{ odd},\\ c_{lm}, & l\text{ even}.\end{cases} \tag{3.14}$$

Then Eqs. (3.11) can be written in the form of an eigenvalue problem

$$\sum_{l'=1}^{\infty}\left(\mathfrak{M}^{m\sigma}_{ll'}+i\frac{\tilde{\gamma}}{\tilde{W}}\mathfrak{N}^{m\sigma}_{ll'}-\delta_{ll'}\lambda\right)d^\sigma_{l'm}=0,\quad l=1,2,\cdots, \tag{3.15}$$

where $\sigma=\pm$, and

$$\mathfrak{M}^{m\sigma}_{ll'}=(-)^l\sigma H(l+1,m)\delta_{l',l+1}+[m/l(l+1)]\delta_{l'l}+(-)^{l+1}\sigma H(l,m)\delta_{l',l-1} \tag{3.16}$$

and

$$\mathfrak{N}^{m\sigma}_{ll'}=\begin{cases}-(m/l)H(l+1,m)\delta_{l',l+1}+[m^2/l(l+1)]\delta_{l'l}\\ \quad-[m/(l+1)]H(l,m)\delta_{l',l-1},\quad \sigma=(-)^l,\\ H(l+2,m)H(l+1,m)\delta_{l',l+2}-[m/(l+2)]H(l+1,m)\delta_{l',l+1},\\ \quad+\left(\frac{l+1}{l+2}H^2(l+1,m)+\frac{l}{l-1}H^2(l,m)\right)\delta_{l',l},\\ \quad-[m/(l-1)]H(l,m)\delta_{l',l-1}+H(l,m)H(l-1,m)\delta_{l',l-2},\\ \quad\sigma=(-)^{l+1}.\end{cases} \tag{3.17}$$

The infinite matrices $\mathfrak{M}^{m\sigma}_{ll'}$ and $\mathfrak{N}^{m\sigma}_{ll'}$ are real and symmetric so that when $i\tilde{\gamma}/\tilde{W}$ is real the eigenvalues λ are real. Further discussion of this eigenvalue problem is given in Sec. III C.

For each m, σ, and eigenvalue λ there will be a corresponding regular solution $\vec{D}^\sigma_{\lambda m}$ of the gyrotropic wave equation. Thus

$$\vec{D}^{m-}_{\lambda}=\sum_{l\text{odd}}d^-_{lm}(\lambda)\vec{C}_{lm}(q\vec{r})+\sum_{l\text{even}}d^-_{lm}(\lambda)\vec{A}_{lm}(q\vec{r}), \tag{3.18}$$

and

$$\vec{D}^{m+}_{\lambda}=\sum_{l\text{odd}}d^+_{lm}(\lambda)\vec{A}_{lm}(q\vec{r})+\sum_{l\text{even}}d^+_{lm}(\lambda)\vec{C}_{lm}(q\vec{r}), \tag{3.19}$$

where q is given by (3.12). The corresponding electric field is found from (3.10). Using (3.11) we can write

$$\vec{E}_\lambda^{m\sigma} = \frac{1 - i\lambda \tilde{W}}{\tilde{\epsilon}} \vec{D}_\lambda^{m\sigma} + \sum_l \frac{1}{\tilde{\epsilon}[l(l+1)]^{1/2}} \Delta_{lm}^\sigma(\lambda) \vec{B}_{lm}(q\vec{r}), \tag{3.20}$$

where

$$\Delta_{lm}^\sigma(\lambda) = \begin{cases} [l(l+1)]^{1/2} \sum_{l'} d_{l'm}^\sigma \left[\tilde{\gamma}(S_{l'l}^m + Q_{l'l}^m) + i\tilde{W}\left(\frac{m}{[l(l+1)]^{1/2}} \delta_{l'l} + L_{l'l}^m\right)\right], & \sigma = (-)^{l+1} \\ 0, \quad \sigma = (-)^l. \end{cases} \tag{3.21}$$

Finally, the corresponding magnetic field is obtained from the first of Eqs. (1.3), using the formulas (2.8) for curl

$$\vec{B}_\lambda^{m-} = \frac{\omega}{cq}\left(-\sum_{l\,\mathrm{odd}} d_{lm}^-(\lambda)\vec{A}_{lm}(q\vec{r}) + \sum_{l\,\mathrm{even}} d_{lm}^-(\lambda)\vec{C}_{lm}(q\vec{r})\right), \tag{3.22}$$

and

$$\vec{B}_\lambda^{m+} = \frac{\omega}{cq}\left(\sum_{l\,\mathrm{odd}} d_{lm}^+(\lambda)\vec{C}_{lm}(q\vec{r}) - \sum_{l\,\mathrm{even}} d_{lm}^+(\lambda)\vec{A}_{lm}(q\vec{r})\right). \tag{3.23}$$

The symbol used here $\vec{B}_\lambda^{m\sigma}$ for the magnetic field should not be confused with the symbol $\vec{B}_{lm}$ for the irrotational solutions of the vector wave equation.

B. Satisfying boundary conditions

The electric displacement vector $\vec{D}$ inside the sphere is a linear combination of solutions (3.18) and (3.19), which we write in the form

$$\vec{D}_{\rm ins} = \sum_{\lambda,m,\sigma} i\frac{cq}{\omega} G^{m\sigma}(\lambda)\vec{D}_\lambda^{m\sigma}(\vec{r}). \tag{3.24}$$

The corresponding electric and magnetic fields are

$$\vec{E}_{\rm ins} = \sum_{\lambda,m,\sigma} i\frac{cq}{\omega} G^{m\sigma}(\lambda)\vec{E}_\lambda^{m\sigma}(\vec{r}) \tag{3.25}$$

and

$$\vec{B}_{\rm ins} = \sum_{\lambda,m,\sigma} i\frac{cq}{\omega} G^{m\sigma}(\lambda)\vec{B}_\lambda^{m\sigma}(\vec{r}). \tag{3.26}$$

These expansions take the place of expansions (2.34) of the fields inside the sphere in the Mie solution. The fields outside the sphere are of the same form as in the Mie solution. They are given by Eq. (2.31), where the incident plane wave fields are given by (2.27) and (2.28), and the scattered fields are given by (2.33). The coefficients f_{lm}, g_{lm}, and $G^{m\sigma}(\lambda)$ are determined by the boundary conditions at the surface of the sphere.

The continuity of the tangential components of the electric field requires $\hat{r}\times\vec{E}_{\rm out} = \hat{r}\times\vec{E}_{\rm ins}$ at the surface of the sphere. Using (2.12) and the orthogonality of the vector spherical harmonics gives

$$\frac{(-1)^{l-1}}{4\pi}\left[\sum_\lambda \left(G^{m\sigma}(\lambda)\frac{\omega}{cq} d_{lm}^\sigma(\lambda)\alpha_l(qa) - \frac{cq}{\omega\tilde{\epsilon}}\Delta_{lm}^\sigma(\lambda)\frac{j_l(qa)}{qa}\right)\right]_{\sigma=(-)^{l+1}} = f_{lm}\alpha_l^{(1)}(ka) + \vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{E}_1\alpha_l(ka) \tag{3.27}$$

and

$$\frac{(-i)^{l+1}}{4\pi}\sum_\lambda\left(G^{m\sigma}(\lambda)d_{lm}^\sigma(\lambda)\frac{j_l(qa)}{qa}\right)_{\sigma=(-)^l} = g_{lm}\frac{h_l^{(1)}(ka)}{ka} + \vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{B}_1\frac{j_l(ka)}{ka}, \tag{3.28}$$

where we have used the definitions (2.38). The subscript $\sigma=(-)^{l+1}$ on the left-hand side of (3.27) means that the parity σ is odd if l is even and even if l is odd; the meaning of the subscript $\sigma=(-)^l$ in (3.28) is just the reverse. The continuity of the normal component of the displacement vector requires $\epsilon_2\hat{r}\cdot\vec{E}_{\rm out} = \hat{r}\cdot\vec{D}_{\rm in}$. Using (2.11) and the orthogonality of the spherical harmonics, this gives

$$\frac{(-i)^{l-1}}{4\pi}\sum_\lambda\left(\frac{cq}{\omega}G^{m\sigma}(\lambda)d_{lm}^\sigma(\lambda)\frac{j_l(qa)}{qa}\right)_{\sigma=(-)^{l+1}} = f_{lm}\epsilon_2\frac{h_l^{(1)}(ka)}{ka} + \vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{E}_1\epsilon_2\frac{j_l(ka)}{ka}. \tag{3.29}$$

In the same way from the continuity of the tangential components of the magnetic field we get

$$\frac{(-i)^{l+1}}{4\pi}\sum_\lambda\left(G^{m\sigma}(\lambda)d_{lm}^\sigma(\lambda)\alpha_l(qa)\right)_{\sigma=(-)^l} = g_{lm}\alpha_l^{(1)}(ka) + \vec{Y}_{ll}^m(\hat{k})^*\times\hat{k}\cdot\vec{B}_1\alpha_l(ka) \tag{3.30}$$

and

$$\frac{(-i)^{l-1}}{4\pi}\frac{\omega a}{c}\sum_\lambda\left(G^{m\sigma}(\lambda)d^\sigma_{lm}(\lambda)j_l(qa)\right)_{\sigma=(-)^{l+1}} = f_{lm}kah_l^{(1)}(ka)+\vec{Y}^m_{ll}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1kaj_l(ka). \quad (3.31)$$

Continuity of the normal component of the magnetic field gives

$$\frac{(-i)^{l+1}}{4\pi}\sum_\lambda\left(G^{m\sigma}(\lambda)d^\sigma_{lm}(\lambda)\frac{j_l(qa)}{qa}\right)_{\sigma=(-)^l} = g_{lm}\frac{h_l^{(1)}(ka)}{ka}+\vec{Y}^m_{ll}(\hat{k})^*\times\hat{k}\cdot\vec{B}_1\frac{j_l(ka)}{ka}. \quad (3.32)$$

The six equations (3.27)–(3.32) are not independent. This is obvious since (3.32) and (3.28) are identical, while using (2.32) we see that (3.31) and (3.29) are also equivalent. Just as in the Mie solution we discard equations (3.28) and (3.31), so that the electric coefficients are determined from the electric boundary conditions and the magnetic coefficients from the magnetic boundary conditions.

We now arrange these equations in a form more suitable for their solution. To simplify the notation we introduce

$$x\equiv ka, \quad y\equiv qa, \quad (3.33)$$

where here q is given by (3.12). Eliminating f_{lm} between (3.27) and (3.29), and g_{lm} between (3.30) and (3.32), we get equations for $G^{m\sigma}(\lambda)$, which can be written in the form

$$\sum_\lambda X_l^{m\sigma}(\lambda)G^{m\sigma}(\lambda)=r_l^{m\sigma}, \quad l=1,2,\ldots, \quad (3.34)$$

where

$$X_l^{m\sigma}(\lambda)=\begin{cases}\left(\alpha_l(y)-\dfrac{j_l(y)}{y}\dfrac{x\alpha_l^{(1)}(x)}{h_l^{(1)}(x)}\right)d^\sigma_{lm}(\lambda), & \sigma=(-)^l,\\[2ex] \left(\dfrac{\omega}{cq}\alpha_l(y)-\dfrac{cq}{\omega\epsilon_2}\dfrac{j_l(y)}{y}\dfrac{x\alpha_l^{(1)}(x)}{h_l^{(1)}(x)}\right)d^\sigma_{lm}(\lambda) \\ \qquad -\dfrac{cq}{\omega\bar{\epsilon}}\dfrac{j_l(y)}{y}\Delta^\sigma_{lm}(\lambda), & \sigma=(-)^{l+1}\end{cases} \quad (3.35)$$

and

$$r_l^{m\sigma}=\frac{4\pi(i)^l}{x^2h_l^{(1)}(x)}\begin{cases}\vec{Y}^m_{ll}(\hat{k})^*\times\hat{k}\cdot\vec{B}_1, & \sigma=(-)^l,\\ -\vec{Y}^m_{ll}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1, & \sigma=(-)^{l+1}.\end{cases} \quad (3.36)$$

In obtaining these expressions we have used (2.45). Equations (3.34) determine $G^{m\sigma}(\lambda)$. The coefficients f_{lm} and g_{lm} can be expressed in terms of this quantity by eliminating the terms involving $\vec{E}_1$ between (3.27) and (3.29), and the terms involving $\vec{B}_1$ between (3.30) and (3.32). The results can be expressed in the form:

$$f_{lm}=\frac{x^2j_l(x)}{4\pi(i)^l}\sum_\lambda[Y_l^{m\sigma}(\lambda)G^{m\sigma}(\lambda)]_{\sigma=(-)^{l+1}}, \quad (3.37)$$

$$g_{lm}=\frac{x^2j_l(x)}{4\pi(i)^l}\sum_\lambda[Y_l^{m\sigma}(\lambda)G^{m\sigma}(\lambda)]_{\sigma=(-)^l}, \quad (3.38)$$

where

$$Y_l^{m\sigma}(\lambda)=\begin{cases}\left(\dfrac{j_l(y)}{y}\dfrac{x\alpha_l(x)}{j_l(x)}-\alpha_l(y)\right)d^\sigma_{lm}(\lambda), & \sigma=(-)^l,\\[2ex] \left(\dfrac{\omega}{cq}\alpha_l(y)-\dfrac{cq}{\omega\epsilon_2}\dfrac{j_l(y)}{y}\dfrac{x\alpha_l(x)}{j_l(x)}\right)d^\sigma_{lm}(\lambda) \\ \qquad -\dfrac{cq}{\omega\bar{\epsilon}}\dfrac{j_l(y)}{y}\Delta^\sigma_{lm}(\lambda), & \sigma=(-)^{l+1}.\end{cases} \quad (3.39)$$

Again we have used (2.45).

For each m and σ equations (3.34) are an infinite set of inhomogeneous linear equations for the quantity $G^{m\sigma}(\lambda)$.

C. Numerical solution

Our method of numerical solution is the same as that in Ref. 2. We truncate the eigenvalue problem (3.15) by replacing the matrices by their $N\times N$ upper-left-hand corner (i.e., $l=1,2,\ldots,N$), There will then be N discrete eigenvalues $\lambda_k(k=1,2,\ldots,N)$, which in general will be complex, since the quantity $i\tilde{\gamma}/\tilde{W}$ is in general complex. We accordingly approximate the infinite set of equations (3.34) by the finite set of equations

$$\sum_{k=1}^N X_{lk}^{m\sigma}G_k^{m\sigma}=r_l^{m\sigma}, \quad l=1,2,\ldots,N, \quad (3.40)$$

where

$$X_{lk}^{m\sigma}\equiv X_l^{m\sigma}(\lambda_k), \quad G_k^{m\sigma}=G^{m\sigma}(\lambda_k). \quad (3.41)$$

Similarly we approximate the expressions (3.37) and (3.38) by

$$f_{lm}=\frac{x^2 j_l(x)}{4\pi(i)^l}\sum_{k=1}^{N}\{Y_{lk}^{m\sigma}G_k^{m\sigma}\}_{\sigma=(-)^{l+1}},\quad l=1,2,\ldots,N,$$

$$g_{lm}=\frac{x^2 j_l(x)}{4\pi(i)^l}\sum_{k=1}^{N}\{Y_{lk}^{m\sigma}G_k^{m\sigma}\}_{\sigma=(-)^{l}},\quad l=1,2,\ldots,N, \tag{3.42}$$

where

$$Y_{lk}^{m\sigma}\equiv Y_l^{m\sigma}(\lambda_k). \tag{3.43}$$

The approximate equations (3.40) are, for each m and σ, a finite set of inhomogeneous linear equations for the quantities $G_k^{m\sigma}$. By Cramer's rule the solution is[15]

$$G_k^{m\sigma}=\sum_{l=1}^{N} r_l^{m\sigma}\operatorname{cof}(X_{lk}^{m\sigma})/\det(X^{m\sigma}), \tag{3.44}$$

where the denominator is the determinant of the $N\times N$ matrix $X^{m\sigma}$ whose elements are the $X_{lk}^{m\sigma}$. Inserting this in expressions (3.42), we can write them in the form

$$f_{lm}=\frac{x^2 j_l(x)}{4\pi(i)^l}\sum_{l'=1}^{N}(Z_{ll'}^{m\sigma}r_{l'}^{m\sigma})_{\sigma=(-)^{l+1}},$$

$$g_{lm}=\frac{x^2 j_l(x)}{4\pi(i)^l}\sum_{l'=1}^{N}(Z_{ll'}^{m\sigma}r_{l'}^{m\sigma})_{\sigma=(-)^{l}}, \tag{3.45}$$

where

$$Z_{ll'}^{m\sigma}=\frac{\sum_{k=1}^{N}Y_{lk}^{m\sigma}\operatorname{cof}(X_{l'k}^{m\sigma})}{\det(X^{m\sigma})}. \tag{3.46}$$

But, recalling the familiar rule for the expansion of a determinant in terms of the elements of a row, the numerator in this expression for $Z_{ll'}^{m\sigma}$ is just the determinant of the matrix obtained by replacing the (l')'th row of $X^{m\sigma}$ by the l'th row of $Y^{m\sigma}$, the matrix whose elements are the $Y_{lk}^{m\sigma}$. Thus, suppressing the indices m and σ,

$$Z_{l1}=\begin{vmatrix} Y_{l1} & Y_{l2} & Y_{l3} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ X_{31} & X_{32} & X_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}\Bigg/\begin{vmatrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ X_{31} & X_{32} & X_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} \tag{3.47}$$

and

$$Z_{l2}=\begin{vmatrix} X_{11} & X_{12} & X_{13} & \cdots \\ Y_{l1} & Y_{l2} & Y_{l3} & \cdots \\ X_{31} & X_{32} & X_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}\Bigg/\begin{vmatrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ X_{31} & X_{32} & X_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} \tag{3.48}$$

and so on.

With this interpretation of $Z_{ll'}^{m\sigma}$ as the ratio of two determinants, the expressions (3.45) represent, for N large, a solution of the Mie problem for a gyrotropic sphere. With these expressions the various cross sections can be calculated using the formulas given in Sec. II D. Rather than exhibit all the resulting expressions we will here write down only the expression for the forward-scattering amplitude. Thus, from (2.51) we get

$$\frac{\vec{F}(\hat{k},\hat{k})\cdot\vec{E}_1^*}{|\vec{E}_1|^2}=a\sum_{l=1}^{\infty}\sum_{l'=1}^{\infty}\sum_{m=-\min(l,l')}^{\min(l,l')}(-i)^{l+1}x j_l(x)\sum_{\sigma=\pm}Z_{ll'}^{m\sigma}r_{l'}^{m\sigma}\times\begin{Bmatrix}\dfrac{\vec{Y}_{l'l}^{m}(\hat{k})\times\hat{k}\cdot\vec{E}_1^*}{|\vec{E}_1|^2}, & \sigma=(-)^{l+1}\\[2ex] \dfrac{\vec{Y}_{l'l}^{m}(\hat{k})\times\hat{k}\cdot\vec{B}_1^*}{|\vec{B}_1|^2}, & \sigma=(-)^{l},\end{Bmatrix} \tag{3.49}$$

where we have used (2.16) and (3.33). The total scattering cross section is related to the imaginary part of the forward scattering amplitude through (2.59).

It is not our intention to claim that the method of numerical solution outlined in this section is unique. For the applications we and others have made, it has been found to be computationally efficient and accurate.

D. Auxiliary Eigenvalue problem

The auxiliary eigenvalue problem (3.15) can be solved exactly. The solution is[16]

$$d_{lm}^{\sigma}=\frac{(i)^l(-)^m}{\sin\theta}\left(\frac{(2l+1)(l-m)!}{4\pi l(l+1)(l+m)!}\right)^{1/2}\times\begin{cases} m\lambda P_l^m-\cos\theta\sin^2\theta\dfrac{dP_l^m}{d\cos\theta}, & \sigma=(-)^l\\[2ex] -i\left(m\cos\theta P_l^m-\lambda\sin^2\theta\dfrac{dP_l^m}{d\cos\theta}\right), & \sigma=(-)^{l+1},\end{cases} \tag{3.50}$$

where $P_l^m \equiv P_l^m(\cos\theta)$ is the Legendre polynomial and the eigenvalue λ is given by

$$\lambda = \frac{i\tilde{\gamma}}{2W}\sin^2\theta + \cos\theta\left[1 - \left(\frac{\tilde{\gamma}\sin^2\theta}{2W\cos\theta}\right)^2\right]^{1/2}. \qquad (3.51)$$

Here we choose, say, the branch of the square root with positive real part. Then for each θ with $0 \leq \theta \leq \pi$ there is a unique eigenvalue given by (3.51) and a corresponding eigenvector given by (3.50). The spectrum of eigenvalues is therefore continuous, lying along an arc in the complex λ plane joining the two points $\lambda = \pm 1$.

This solution can be checked directly by inserting (3.50) in (3.15) and using the recursion relations for the Legendre polynomials,[17] although we should warn that this is quite laborious. We in fact discovered this solution by expanding the fields inside the sphere in terms of plane-wave solutions of the gyrotropic wave equation (1.4), expanding these plane waves in terms of vector spherical waves, fitting boundary conditions, and finally comparing with the equations obtained in Sec. III B.

The spectrum of eigenvalues being continuous, the sums over λ appearing in Sec. III B should be interpreted as integrals, a convenient choice being

$$\sum_\lambda \to \int_0^\pi d\Theta\,. \qquad (3.52)$$

The solutions (3.50) are not normalized, but this is not important since a change in normalization only results in a multiplicative factor in the quantities $X_l^{m\sigma}(\lambda)$ and $Y_l^{m\sigma}(\lambda)$, which cancels in the expressions for f_{lm} and g_{lm}. This is seen most explicitly in the (3.47) and (3.48) where a common factor in each *column* obviously cancels between the numerator and denominator determinant.

One might expect that a knowledge of the exact solution of the eigenvector problem would be an advantage in numerical computations. It seems, however, that the numerical solution of the truncated eigenvalue problem, described in Sec. III C, gives a "best fit" to the eigenvalues and eigenvectors in each order, with corresponding rapid convergence. An attempt at numerical solution using the exact eigenvalues and eigenvectors has been made by Dixon, using the Purdue University CDC 6500 computer.[18] He used the exact eigenfunctions in the expressions (3.41) and (3.43) for $X_{lk}^{m\sigma}$ and $Y_{lk}^{m\sigma}$ with λ_k given by (3.51) with

$$\theta = k\pi/(N+1), \quad k = 1, 2, \ldots, N\,. \qquad (3.53)$$

He found that the convergence was much slower than with the method described in Sec. III C. In fact, before satisfactory accuracy was obtained the matrix size N became so large that round-off error in the matrix elements in the determinants produced spurious results.

IV. RESULTS IN VARIOUS LIMITS

Here we discuss the form of our result in various limits. The first is that of an isotropic dielectric relation, which we call the Mie limit. There we show that we recover the results of Sec. II C. The next is the limit where the wavelength outside the sphere is long compared with the sphere radius, although no restriction is made on the wavelength inside the sphere. This is the limit appropriate to most of the applications discussed in the following paper by Dixon and Furdyna.[23] When we further specialize to the case where the wavelength inside is long compared to the sphere radius, we speak of the Rayleigh limit. The geometric-optics limit, where the sphere radius is large compared with the wavelength, is of great and enduring importance in the classic Mie problem. However, we are unaware of any applications of importance in this limit for gyrotropic spheres, and therefore do not pursue it here.

A. Mie limit

To recover the classical Mie solution for an isotropic dielectric relation from the solution of Sec. III, we must set $\epsilon_{xx} = \epsilon_{zz} = \epsilon_1$ and $\epsilon_{xy} = 0$. That is, the parameters in the inverse dielectric relation (3.1) must take the limiting values

$$\tilde{\gamma} \to 0, \quad \tilde{W} \to 0, \quad \tilde{\epsilon} \to \epsilon_1\,. \qquad (4.1)$$

From (3.21) we see that $\Delta_{lm}^\sigma \to 0$, and from (3.12) that

$$cq/\omega \to \sqrt{\epsilon_1}\,, \qquad (4.2)$$

independent of λ. Then the expression (3.35) for $X_l^{m\sigma}(\lambda)$ becomes

$$X_l^{m\sigma}(\lambda) \to \begin{cases} \left(\alpha_l(y) - \dfrac{j_l(y)}{y}\,\dfrac{x\alpha_l^{(1)}(x)}{h_l^{(1)}(x)}\right)d_{lm}^\sigma(\lambda), & \sigma = (-)^l \\ \left(\dfrac{\omega}{cq}\,\alpha_l(y) - \dfrac{cq}{\omega\epsilon_2}\,\dfrac{j_l(y)}{y}\,\dfrac{x\alpha_l^{(1)}(x)}{h_l^{(1)}(x)}\right)d_{lm}^\sigma(\lambda), & \\ \sigma = (-)^{l+1}\,, & \end{cases} \qquad (4.3)$$

and the expression (3.42) for $Y_l^{m\rho}(\mu)$ becomes

$$Y_l^{m\sigma}(\lambda) \to \begin{cases} \left(\dfrac{j_l(y)}{y}\,\dfrac{x\alpha_l(x)}{j_l(x)} - \alpha_l(y)\right)d_{lm}^\sigma(\lambda), & \sigma = (-)^l \\ \left(\dfrac{\omega}{cq}\,\alpha_l(y) - \dfrac{cq}{\omega\epsilon_2}\,\dfrac{j_l(y)}{y}\,\dfrac{x\alpha_l(x)}{j_l(x)}\right)d_{lm}^\sigma(\lambda), & \\ \sigma = (-)^{l+1}\,. & \end{cases} \qquad (4.4)$$

Recalling that $y = qa = (\epsilon_1/\epsilon_2)^{1/2}x$ we see that the quantities in large parentheses in these expressions are independent of λ; the only place λ appears is in the quantities $d^\sigma_{lm}(\lambda)$. This has the consequence that in forming the ratio of determinants $Z^{m\sigma}_{ll'}$ [illustrated in (3.56) and (3.57)] the quantities factor from each row in both numerator and denominator determinants. Hence, $Z^{m\sigma}_{ll'}$ will be proportional to a ratio of determinants involving only the $d^\sigma_{lm}(\lambda)$. But in this ratio when $l'\neq l$ two of the rows in the numerator determinant are identical and the determinant vanishes. When $l'=l$ this ratio is unity and, since the common factors from all the other rows cancel, we are left with

$$Z^{m\sigma}_{ll}=\begin{cases}\left(+\dfrac{j_l(y)}{y}\dfrac{x\alpha_l(x)}{j_l(x)}-\alpha_l(y)\right)\Big/\left(\alpha_l(y)-\dfrac{j_l(y)}{y}\dfrac{x\alpha_l^{(1)}(x)}{h_l^{(1)}(x)}\right), & \sigma=(-)^l\\[2ex] \left(\dfrac{\omega}{cq}\alpha_l(y)-\dfrac{cq}{\omega\epsilon_2}\dfrac{j_l(y)}{y}\dfrac{x\alpha_l(x)}{j_l(x)}\right)\Big/\left(\dfrac{\omega}{cq}\alpha_l(y)-\dfrac{cq}{\omega\epsilon_2}\dfrac{j_l(y)}{y}\dfrac{x\alpha_l^{(1)}(x)}{h_l^{(1)}(x)}\right), & \sigma=(-)^{l+1}.\end{cases} \tag{4.5}$$

Using this in (3.45) with $r^{m\sigma}_l$ given by (3.36), and recalling (4.2), we can write

$$f_{lm}=\left(\epsilon_2\frac{j_l(x)}{x}\alpha_l(y)-\epsilon_1\frac{j_l(y)}{y}\alpha_l(x)\right)\Big/\left(\epsilon_1\frac{j_l(y)}{y}\alpha_l^{(1)}(x)-\epsilon_2\frac{h_l^{(1)}(x)}{x}\alpha_l(y)\right)\vec{Y}^m_{ll}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1, \tag{4.6}$$

which is identical to the corresponding expression (2.44) for the Mie solution. In the same way we recover the expression (2.46) for g_{lm}.

B. Small-*ka* limit

We now consider the case in which the wavelength of the incident wave is long compared with the sphere radius. In our formulas this corresponds to the limit

$$x\equiv ka\to 0, \tag{4.7}$$

while $y=qa$ is fixed. For small x the spherical Bessel functions can be approximated by[5]

$$j_l(x)\approx x^l/(2l+1)!!,\quad h_l^{(1)}(x)\approx -i(2l-1)!!/x^{l+1}, \tag{4.8}$$

where $(2l+1)!!=1\times3\times5\cdots(2l+1)$ and from which, using the definitions (2.38), we find

$$\begin{aligned}&\alpha_l(x)\approx(l+1)x^{l-1}/(2l+1)!!,\\ &\alpha_l^{(1)}(x)\approx il(2l-1)!!/x^{l+2}.\end{aligned} \tag{4.9}$$

We will also use the recursion relations for spherical Bessel functions[5]

$$\begin{aligned}&(2l+1)j_l(y)/y=j_{l-1}(y)+j_{l+1}(y),\\ &(2l+1)\,dj_l(y)/dy=lj_{l-1}(y)-(l+1)j_{l+1}(y).\end{aligned} \tag{4.10}$$

The expressions (3.35) and (3.39) for $X^{m\sigma}_l(\lambda)$ and $Y^{m\sigma}_l(\lambda)$ become

$$X^{m\sigma}_l(\lambda)\to\begin{cases}j_{l-1}(y)\,d^\sigma_{lm}(\lambda), & \sigma=(-)^l\\ \left(\dfrac{\omega}{cq}\alpha_l(y)+\dfrac{cq}{\omega\epsilon_2}l\dfrac{j_l(y)}{y}\right)d^\sigma_{lm}(\lambda) & \\ -\dfrac{cq}{\omega\bar{\epsilon}}\dfrac{j_l(y)}{y}\Delta^\sigma_{lm}(\lambda), & \sigma=(-)^{l+1},\end{cases} \tag{4.11}$$

and

$$Y^{m\sigma}_l(\lambda)\to\begin{cases}j_{l+1}(y)\,d^\sigma_{lm}(\lambda), & \sigma=(-)^l\\ \left(\dfrac{\omega}{cq}\alpha_l(y)-\dfrac{cq}{\omega\epsilon_2}(l+1)\dfrac{j_l(y)}{y}\right)d^\sigma_{lm}(\lambda) & \\ -\dfrac{cq}{\omega\epsilon}\dfrac{j_l(y)}{y}\Delta^\sigma_{lm}(\lambda), & \sigma=(-)^{l+1}.\end{cases} \tag{4.12}$$

Thus these quantities, and therefore also the quantities $Z^{m\sigma}_{ll'}$ defined by (3.46), approach a limit independent of x. On the other hand $r^{m\sigma}_l$ as given by (3.36) is

$$r^{m\sigma}_l\to\frac{4\pi i^{l+1}x^{l-1}}{(2l-1)!!}\begin{cases}\vec{Y}^m_{ll}(\hat{k})^*\times\hat{k}\cdot\vec{B}_1, & \sigma=(-)^l\\ \vec{Y}^m_{ll}(\hat{k})^*\times\hat{k}\cdot\vec{E}_1, & \sigma=(-)^{l+1}.\end{cases} \tag{4.13}$$

Therefore in the sum in the expressions (3.45) the dominant term comes from $l'=1$. In these same expressions the prefactor is

$$\frac{x^2j_l(x)}{4\pi i^l}\to\frac{x^{l+2}}{4\pi i^l(2l+1)!!}. \tag{4.14}$$

Hence, we only keep f_{1m} and g_{1m}, the f_{lm} and g_{lm}

for higher l being negligibly smaller. Putting these results together the expressions (3.45) give

$$f_{1m} \rightarrow \frac{x^3}{(24\pi)^{1/2}} Z_{11}^{m+} \hat{e}_m^* \cdot \vec{E}_1 ,$$
$$g_{1m} \rightarrow \frac{-x^3}{(24\pi)^{1/2}} Z_{11}^{m-} \hat{e}_m^* \cdot \vec{B}_1 , \qquad (4.15)$$

where we have used the formula

$$\vec{Y}_{11}^m(\hat{k}) = -i\left(\frac{3}{8\pi}\right)^{1/2} \hat{k}\times\hat{e}_m , \qquad (4.16)$$

and the fact, which follows from (2.16), that $\hat{k}\cdot\vec{E}_1 = \hat{k}\cdot\vec{B}_1 = 0$. Note that the direction of propagation $\hat{k}$ no longer appears in these expressions for f_{lm} and g_{lm}, which is to be expected in this limit where the wavelength is so long that the incident fields are uniform across the sphere.

With the expressions (4.15) we can form the various cross sections introduced in Sec. II D (the absorption cross section dominates). However, in this long-wavelength limit one usually views the incident fields as uniform ac fields which induce ac dipole moments in the sphere, and it is these which are the quantities of interest. To obtain expressions for the induced electric dipole moment $\vec{P}$ and the induced magnetic dipole moment $\vec{M}$, we make a small-kr expansion of the expressions (2.33) for the scattered fields. Using (4.8) in (2.9) in which j_l is replaced by $h_l^{(1)}$, we see that in the sums in (2.33) the terms with $l=1$ dominate and in these terms the dominant contribution comes from $\vec{A}_{1m}^{(1)}$, which becomes

$$\vec{A}_{1m}^{(1)}(k\vec{r}) \approx (1/i\sqrt{2}\, k^3)\, \vec{\nabla}\, [\, Y_{1m}(\hat{r})/r^2] \ . \qquad (4.17)$$

Hence, using the formula

$$Y_{1m}(\hat{r}) = (3/4\pi)^{1/2}\, \hat{e}_m\cdot\hat{r} , \qquad (4.18)$$

we see that the scattered fields take the form of dipole fields

$$\vec{E}_{\rm scatt} \approx \vec{\nabla}(\vec{r}\cdot\vec{P}/r^3), \quad \vec{B}_{\rm scatt} \approx -\vec{\nabla}(\vec{r}\cdot\vec{M}/r^3) , \qquad (4.19)$$

where, using (4.15) and $x=ka$, the induced dipole moments are given by

$$\vec{P} = -\tfrac{1}{2}a^3 \sum_{m=-1}^{1} Z_{11}^{m+}\, \hat{e}_m^* \cdot \vec{E}_1 \hat{e}_m \qquad (4.20)$$

and

$$\vec{M} = \tfrac{1}{2}a^3 \sum_{m=-1}^{1} Z_{11}^{m-}\, \hat{e}_m^* \cdot \vec{B}_1 \hat{e}_m \ . \qquad (4.21)$$

In these expressions the quantities $Z_{11}^{m\sigma}$ are ratios of determinants given by (3.47) with $l=1$ and with $X_1^{m\sigma}(\lambda)$ and $Y_1^{m\sigma}(\lambda)$ given by (4.11) and (4.12). Explicitly, for $\sigma=+$ (even parity or electric) we get

$$Z_{11}^{m+} = \frac{\begin{vmatrix} \cdots\left(\dfrac{\omega}{cq_k}\alpha_1(y_k) - 2\dfrac{cq_k}{\omega\epsilon_2}\dfrac{j_1(y_k)}{y_k}\right) d_{1m}^+(\lambda_k) - \dfrac{cq_k}{\omega\bar{\epsilon}}\dfrac{j_1(y_k)}{y_k}\Delta_{1m}^+(\lambda_k)\cdots \\ \cdots j_1(y_k)\, d_{2m}^+(\lambda_k)\cdots \\ \cdots\left(\dfrac{\omega}{cq_k}\alpha_3(y_k) + 3\dfrac{cq_k}{\omega\epsilon_2}\dfrac{j_3(y_k)}{y_k}\right) d_{3m}^+(\lambda_k) - \dfrac{cq_k}{\omega\bar{\epsilon}}\dfrac{j_3(y_k)}{y_k}\Delta_{3m}^+(\lambda_k)\cdots \\ \vdots \end{vmatrix}}{\begin{vmatrix} \cdots\left(\dfrac{\omega}{cq_k}\alpha_1(y_k) + \dfrac{cq_k}{\omega\epsilon_2}\dfrac{j_1(y_k)}{y_k}\right) d_{1m}^+(\lambda_k) - \dfrac{cq_k}{\omega\bar{\epsilon}}\dfrac{j_1(y_k)}{y_k}\Delta_{1m}^+(\lambda_k)\cdots \\ \cdots j_1(y_k)\, d_{2m}^+(\lambda_k)\cdots \\ \cdots\left(\dfrac{\omega}{cq_k}\alpha_3(y_k) + 3\dfrac{cq_k}{\omega\epsilon_2}\dfrac{j_3(y_k)}{y_k}\right) d_{3m}^+(\lambda_k) - \dfrac{cq_k}{\omega\bar{\epsilon}}\dfrac{j_3(y_k)}{y_k}\Delta_{3m}^+(\lambda_k)\cdots \\ \vdots \end{vmatrix}} \qquad (4.22)$$

and, for $\sigma = -$ (odd parity or magnetic) we get

$$Z_{11}^{m-} = \frac{\begin{vmatrix} \cdots - j_2(y_k) d_{1m}^-(\lambda_k) \cdots \\ \cdots \left(\frac{\omega}{cq_k} \alpha_2(y_k) + 2 \frac{cq_k}{\omega\epsilon_2} \frac{j_2(y_k)}{y_k} \right) d_{2m}^-(\lambda_k) - \frac{cq_k}{\omega\bar{\epsilon}} \frac{j_2(y_k)}{y_k} \Delta_{2m}^-(\lambda_k) \cdots \\ \cdots j_2(y_k) d_{3m}^-(\lambda_k) \cdots \\ \vdots \end{vmatrix}}{\begin{vmatrix} \cdots j_0(y_k) d_{1m}^-(\lambda_k) \cdots \\ \cdots \left(\frac{\omega}{cq_k} \alpha_2(y_k) + 2 \frac{cq_k}{\omega\epsilon_2} \frac{j_2(y_k)}{y_k} \right) d_{2m}^-(\lambda_k) - \frac{cq_k}{\omega\bar{\epsilon}} \frac{j_2(y_k)}{y_k} \Delta_{2m}^-(\lambda_k) \cdots \\ \cdots j_2(y_k) d_{3m}^-(\lambda_k) \cdots \\ \vdots \end{vmatrix}} \tag{4.23}$$

Here $y_k = q_k a$, with

$$q_k = \frac{\omega}{c} \left(\frac{\tilde{\epsilon}}{1 - i\lambda_k \bar{W}} \right)^{1/2}, \quad \mathrm{Re}\ q_k > 0, \tag{4.24}$$

In (4.22) and (4.23) the typical element in the first three rows of the numerator and denominator determinants are indicated. Note that numerator and denominator differ only in the first row.

The expressions (4.20) and (4.21) represent a complete solution in the small-ka limit. From them one can easily obtain expressions for other physically observable quantities. For example, the power absorbed from the electric and magnetic fields is

$$\begin{aligned} \mathcal{P}_{\mathrm{elec}} &= \tfrac{1}{2} \omega \,\mathrm{Im}\, \vec{\mathrm{P}} \cdot \vec{\mathrm{E}}_1^* , \\ \mathcal{P}_{\mathrm{mag}} &= \tfrac{1}{2} \omega \,\mathrm{Im}\, \vec{\mathrm{M}} \cdot \vec{\mathrm{B}}_1^* . \end{aligned} \tag{4.25}$$

We conclude with some remarks concerning the relation of our small-ka limit with the corresponding results obtained with the so-called quasistationary approximation.[19] This approximation, which is appropriate to the small-ka regime where the wavelength is long compared with the sphere radius, consists in expressing the fields outside the sphere as a superposition of multipole fields. In addition, in the case of electric excitation one neglects the magnetic multipoles and applies electric boundary conditions (continuity of $\hat{r} \cdot \vec{\mathrm{D}}$ and $\hat{r} \times \vec{\mathrm{E}}$), while in the case of magnetic excitation one neglects electric multipoles and applies magnetic boundary conditions (continuity of $\vec{\mathrm{B}}$). The fields inside the sphere are treated without approximation as in Sec. III. Unfortunately, when this approximation method is applied to the gyrotropic sphere problem it gives an incorrect answer! The quasistationary result differs from our small-ka limit only in the even rows of the determinants. In the electric case, the even rows of (4.22) are given by

$$[j_l(y_k)/y_k]\, d_{lm}^+(\lambda_k) \quad \text{(incorrect)} \tag{4.26}$$

instead of

$$j_{l-1}(y_k)\, d_{lm}^+(\lambda_k) \quad \text{(correct)}, \tag{4.27}$$

In the magnetic case, the even rows of (4.23) are given by

$$j_l(y_k)\, d_{lm}^-(\lambda_k) \quad \text{(incorrect)}, \tag{4.28}$$

instead of

$$\left(\frac{\omega}{ck} \alpha_l(y_k) + l \frac{cq_k}{\omega\epsilon_2} \frac{j_l(y_k)}{y_k} \right) d_{lm}^-(\lambda_k) - \frac{cq_k}{\omega\bar{\epsilon}} \frac{j_l(y_k)}{y_k} \Delta_{lm}^-(\lambda_k) \quad \text{(correct)}. \tag{4.29}$$

This failure of the quasistationary approximation is surprising, since it is known that it gives the correct answer for the small-ka limit of the classical Mie problem.[20] The reason is that in the gyrotropic sphere the electric and magnetic multipoles are coupled through the fields inside the sphere, and it is incorrect to neglect the magnetic multipoles when fitting boundary conditions in the electric case and vice versa.

C. Rayleigh limit

In the Rayleigh limit the wavelength inside the sphere as well as outside is large compared with the sphere radius. Thus we obtain this limit from the results in Sec. IV B for the small-ka limit and assuming

$$y \equiv qa \to 0 \ . \tag{4.30}$$

Here we will keep the lowest-order corrections to this limit. Since in Sec. IV B we neglected such corrections in forming the small-ka limit, to be consistent we must here assume

$$q \gg k \ . \tag{4.31}$$

We will need the first two terms in the expansion of the spherical Bessel functions[5]

$$j_l(y) = [y^l/(2l+1)!!]\,(1 - y^2/2(2l+3) + \cdots) \ , \tag{4.32}$$

and, using (2.38),

$$\alpha_l(y) = [(l+1)y^{l-1}/(2l+1)!!]\,(1 - y^2/(2l+3) + \cdots) \ . \tag{4.33}$$

We begin with the electric coefficients. There we will need the identity

$$\Delta^{+}_{1m}(\lambda) = -2[\,(1-m^2)\tilde{\gamma} + i(\lambda - m)\tilde{W}]\, d^{+}_{1m}(\lambda) \ , \tag{4.34}$$

which can be verified using the definition (3.21) of Δ^{σ}_{lm} and the eigenvalue equation (3.15) with $l=1$. Using this in (4.11) and (4.12) for the case $l=1$ and $\sigma=+$, inserting the above expansions, and using (3.10), we can show

$$\begin{aligned} Y^{m+}_1(\lambda) &\approx \frac{2cq}{3\omega\bar{\epsilon}}\left(1 - \frac{y^2}{10}\right)\left[(1-m^2)\tilde{\gamma} - im\tilde{W} + 1 - \frac{\bar{\epsilon}}{\epsilon_2} - i\frac{\tilde{V}}{10}\right] d^{+}_{1m}(\lambda) \ , \\ X^{m+}_1(\lambda) &\approx \frac{2cq}{3\omega\epsilon}\left(1 - \frac{y^2}{10}\right)\left[(1-m^2)\tilde{\gamma} - im\tilde{W} + 1 + \frac{\bar{\epsilon}}{2\bar{\epsilon}_2} - i\frac{\tilde{V}}{10}\right] d^{+}_{1m}(\lambda) \ , \end{aligned} \tag{4.35}$$

where quantities of order y^4 have been neglected and we have introduced the small dimensionless parameter

$$\tilde{V} = -i(1 - i\lambda\tilde{W})y^2 = -i\bar{\epsilon}(\omega a/c)^2 \ . \tag{4.36}$$

But the quantities in square brackets in (4.35) are independent of λ. This means that these quantities can be factored from the first row of the numerator and denominator determinant in (4.22), leaving two identical determinants which cancel. Hence, Z^{m+}_{11} is just the ratio of $Y^{m+}_1(\lambda)$ to $X^{m+}_1(\lambda)$ given by (4.35). Putting this in (4.20) we get the following expression for the induced electric dipole moment:

$$\vec{\mathrm{P}} = a^3 \sum_{m=-1}^{1} \frac{\bar{\epsilon} - \epsilon_2[1 + (1-m^2)\tilde{\gamma} - im\tilde{W} - \frac{1}{10}i\tilde{V}]}{\bar{\epsilon} + 2\epsilon_2[1 + (1-m^2)\tilde{\gamma} - im\tilde{W} - \frac{1}{10}i\tilde{V}]}\, \hat{e}^{*}_m \cdot \vec{\mathrm{E}}_1 \hat{e}_m \ . \tag{4.37}$$

This expression, which includes the lowest-order (in $\tilde{V}$) corrections to the Rayleigh limit, is the same as that obtained by us previously using a perturbation technique,[21] excepting only that here ϵ_2 is not taken to be unity.

To obtain the corresponding result in the magnetic case it is necessary to expand the first two rows of the determinants in (4.23). Using (4.32) and (4.33) in (4.11) and (4.12) for $l=1$, $\sigma=-$ (the elements in the first rows), we find

$$Y^{m-}_1(\lambda) \approx \tfrac{1}{15}y^2(1 - \tfrac{1}{14}y^2)d^{-}_{1m}(\lambda) \tag{4.38}$$

and

$$X^{m-}_1(\lambda) \approx (1 - \tfrac{1}{6}y^2)d^{-}_{1m}(\lambda) \ , \tag{4.39}$$

while for $l=2$, $\sigma=-$ (the elements in the second rows), we find

$$X^{m-}_2(\lambda) \approx (c/\epsilon_2\omega a)\{[\tfrac{2}{15}y^2(1 - \tfrac{1}{14}y^2) + \tfrac{1}{5}x^2(1 - \tfrac{1}{7}y^2)]d^{-}_{2m}(\lambda) - \tfrac{1}{15}x^2(1 - \tfrac{1}{14}y^2)(1 - i\lambda\tilde{W})^{-1}\Delta^{-}_{2m}(\lambda)\} \ , \tag{4.40}$$

where we have used (3.10) and (3.32). But, on account of condition (4.31), $x \ll y$, we can neglect the terms proportional to x^2 in this expression and write

$$X^{m-}_2(\lambda) \approx \frac{c}{\epsilon_2\omega a}\,\tfrac{2}{15}y^2(1 - \tfrac{1}{14}y^2)d^{-}_{2m}(\lambda) \ . \tag{4.41}$$

Using the eigenvalue equation (3.15) for $l=1$, one can verify the identity

$$d_{2m}^{-}(\lambda)=\left(\frac{5}{4-m^2}\right)^{1/2}\frac{-m^2\tilde{\gamma}+im\tilde{W}-2i\lambda\tilde{W}}{m\tilde{\gamma}+i\tilde{W}}d_{1m}^{-}(\lambda)\ . \tag{4.42}$$

We now rearrange the denominator determinant in (4.23) by multiplying the elements of the second row (4.41) by the factor

$$(3c\epsilon_2/4\omega a\tilde{\epsilon})[5(4-m^2)]^{1/2}(m\tilde{\gamma}+i\tilde{W})\ , \tag{4.43}$$

using the identity (4.42), and subtracting the result from the elements of the first row (4.39). The elements of the first row become

$$X_1^{m-}-(3c\epsilon_2/4\omega a\tilde{\epsilon})[5(4-m^2)]^{1/2}(m\tilde{\gamma}+i\tilde{W})X_2^{m-}$$
$$=\frac{1+\frac{1}{2}m^2\tilde{\gamma}-\frac{1}{2}im\tilde{W}-\frac{2}{21}i\tilde{V}}{i\tilde{V}}y^2(1-\tfrac{1}{14}y^2)d_{1m}^{-}, \tag{4.44}$$

where we have used (3.12), (3.33), and (4.36). But now the elements (4.38) in the first row of the numerator determinant differ from the elements (4.44) in the first row of the denominator determinant only by factors independent of λ. The ratio of the determinants is therefore just the ratio of these factors. Putting this in (4.21), we get the following expression for the induced magnetic dipole moment:

$$\vec{M}=ia^3\frac{\tilde{V}}{30}\sum_{m=-1}^{1}\frac{\hat{e}_m^*\cdot\vec{B}_1\hat{e}_m}{1+\frac{1}{2}m^2\tilde{\gamma}-\frac{1}{2}im\tilde{W}-i\frac{2}{21}\tilde{V}}\ . \tag{4.45}$$

This result was also obtained by us previously.[21]

We should emphasize the importance of the condition (4.31), which in effect says that the waves within the sphere must be much slower than those outside. If we relax this condition, the terms of order $\tilde{V}$ in the electric dipole formula (4.37) are no longer meaningful since we have already dropped terms of order $(ka)^2$, which would be larger. However, the lowest order result, obtained by setting $\tilde{V}=0$ in (4.37), is valid without the restriction (4.31). On the other hand, the magnetic dipole formula (4.45), since it is proportional to $\tilde{V}$, is never valid without the restriction (4.31).

V. SUMMARY AND CONCLUSIONS

In this paper we have formally solved the problem of the scattering and absorption of a plane electromagnetic wave by a gyrotropic sphere. The complexity of the problem arises from the fact that the gyrotropic wave equation [Eq. (1.4)] is not separable in spherical coordinates.

Our tenchinque for solution has involved the following steps.

(i) Expand the incoming plane electromagnetic wave in the regular vector spherical waves $\vec{A}_{lm}$ and $\vec{C}_{lm}$ [Eqs. (2.27) and (2.28)].

(ii) Expand the scattered wave in terms of outgoing vector spherical waves $\vec{A}_{lm}^{(1)}$ and $\vec{C}_{lm}^{(1)}$, with amplitude coefficients g_{lm} and f_{lm} [Eq. (2.33)]. The scattering and absorption cross sections are directly related to these coefficients [Eqs. (2.54) and (2.57)].

(iii) Inside the sphere we chose to solve for the electric displacement vector $\vec{D}$. We expanded $\vec{D}$ in terms of the vector spherical waves $\vec{A}_{lm}$ and $\vec{C}_{lm}$. [Eq. (3.3)].

(iv) We found that in order for this expansion to be a solution of the gyrotropic wave equation, special conditions on the expansion coefficients a_{lm} and c_{lm} must be satisfied. We found that these special conditions could be cast in the form of an auxiliary eigenvalue problem [Eq. (3.15)] in which the components of the eigenvectors d_{lm} were a_{lm} and c_{lm} and the eigenvalues λ determine the spectrum of allowed wavevectors q [Eq. (3.12)] inside the sphere. The solutions to this eigenvalue problem were found to be separable into results of even and odd parity ($\sigma=\pm$).

(v) Thus, for each eigenvalue λ, magnetic quantum number m, and parity σ we found a solution $\vec{D}_{\lambda}^{m\sigma}$ of the gyrotropic wave equation [Eqs. (3.18) and (3.19)]. The general solution, therefore involves a sum over all of these solutions, in which we called the expansion coefficients $G^{m\sigma}(\lambda)$.

(vi) The electric field $\vec{E}$ and the magnetic field $\vec{B}$ could then be written down using the inverse dielectric relation [Eq. (3.1)] and Faraday's law [Eq. (1.3)], respectively.

(vii) The boundary conditions were applied in Sec. IIIB. We required (a) continuity of $\vec{B}$, (b) continuity of the normal component of $\vec{D}$, (c) continuity of the tangential component of $\vec{E}$. The application of these boundary conditions led to six scalar equations for the unknown coefficients $G^{m\sigma}(\lambda)$ inside the sphere, and the unknown amplitude coefficients f_{lm} and q_{lm} outside the sphere. Only four of these six equations are independent.

(viii) These equations were, in fact, found to be matrix equations of infinite size. We outlined a numerical technique for their solution in Sec. III C, using the method of truncation. The results for f_{lm} and g_{lm} could be expressed as a single sum over Z_{ll} [Eqs. (3.45) and (3.46)]. Each element $Z_{ll'}$ could be expressed as the ratio of two $N\times N$ determinants when the matrix equations were truncated at a size $N\times N$.

In Sec. IV we examined the general solution in various limits: (i) Mie limit when $\tilde{\epsilon}\to\epsilon_1$. (ii) Small-$ka$ limit. (iii) Rayliegh limit ($qa\to 0$).

In the following paper by Dixon and Furdyna[23] two selected applications of the solutions presented in this paper are given: (i) A gyrotropic sphere made from a single-carrier semiconductor at microwave frequencies. (ii) A compensated, two-carrier magnetoplasma sphere.

Markiewicz[22] has used our results to analyze Alfvén-wave oscillations in an electron-hole droplet at microwave frequencies.

There are a number of directions where extensions of this work may prove interesting and useful:

(i) It is clear that the mathematical techniques developed here could be extended easily to include magnetic materials (paramagnetic, ferrimagnetic, and ferromagnetic) that are either conducting or insulating.

(ii) Although the theory presented in this paper is equally valid for large ka as for small ka, the phenomena and resonant structure expected at $ka \gtrsim 2$, where the scattering cross section begins to compete with the absorption cross section, remains largely unexplored. New numerical algorithms will probably have to be developed to properly examine this domain of incident wavelengths.

(iii) It is apparent that the extension of the results of this paper to slightly deformed spheres would be useful in a number of research areas. Except at very low frequencies, this appears to be a rather forbidding task.

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