# Discrete coherent states on the von Neumann lattice

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The discrete subset of coherent states on a lattice in phase space is used as an expansion basis for states and operators in quantum mechanics. The nature of the expansion as well as the uniqueness of the expansion coefficients is investigated. An explicit example is given for the harmonic-oscillator states.

## I. INTRODUCTION

The use of coherent states in quantum mechanics is by now well established, and their properties and applications have been described in many articles particularly since the early 1960's.[1] These states are often referred to as the most classical states and they are specified by the expectation values $\bar{x}$ and $\bar{p}$ of the coordinate and momentum operators. On one hand, because of their classical nature the coherent states find different applications in a variety of problems in physics and in particular in quantum optics.[1] On the other hand, these states have some unusual properties in the framework of quantum mechanics, e.g., they are nonorthogonal for different pairs of $\bar{x}$ and $\bar{p}$ and they form an overcomplete set when $\bar{x}$ and $\bar{p}$ cover all the points of the phase plane.

It is of interest to define a discrete subset of coherent states that are associated with the von Neumann lattice.[2] This subset is obtained from the full set by restricting the values of $\bar{x}$ and $\bar{p}$ to a lattice in the phase plane. The full set of coherent states has been well studied by Bargmann,[3] Glauber,[4] and Klauder,[5] and by others since, and it was shown that in spite of being nonorthogonal and overcomplete, these states nevertheless provide a good and useful basis for expansions of general states and operators. More recently Bargmann, Butera, Girardello, and Klauder[6] and Perelomov,[7] have investigated the discrete set of von Neumann, and have demonstrated the completeness of this basis as well. Their proof has lately been simplified and generalized by using the $kq$ representation.[8]

Perelomov in his paper[7] introduced the set of functions biorthogonal to the discrete set of coherent states associated with the von Neumann lattice. With their aid he derived some interesting and remarkable properties of the von Neumann set, and pointed out that they can be used as a basis for the expansion of a general state, somewhat in the spirit of Glauber's[4] approach for the full set of coherent states. However, as Perelomov notes, one has to be careful how such expansions in a discrete nonorthogonal basis are to be interpreted. We wish in this paper to go further into the nature of the expansions and of the properties discovered by Perelomov, and for this purpose the $kq$ representation proves an extremely useful tool because both the von Neumann set and the biorthogonal set there assume very simple forms. We also give new results on the expansion coefficients for the important case of the harmonic-oscillator states. In applications, the need of expanding in states on a discrete lattice arises, for example, in the Pippard network for studying Landau levels in a crystal.[9] It was shown that these lattice states correspond formally to the von Neumann discrete set.[10]

A normalized coherent state $|\alpha\rangle$ in one dimension (to which we shall restrict ourselves) is definable as an eigenstate of the annihilation operator $\hat{a}$ with the eigenvalue $\alpha$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{1}$$

where

$$\hat{a} = \frac{1}{\lambda\sqrt{2}}\left(\hat{x} + i\frac{\lambda^2}{\hbar}\hat{p}\right), \quad \alpha = \frac{1}{\lambda\sqrt{2}}\left(\bar{x} + i\frac{\lambda^2}{\hbar}\bar{p}\right). \tag{2}$$

Here $\hat{x}$ and $\hat{p}$ are the coordinate and momentum operators, and $\bar{x}$ and $\bar{p}$ can be identified as coordinate and momentum expectation values for $|\alpha\rangle$, which has minimal wave-packet form in the $x$ representation:

$$\langle x|\alpha\rangle = \left(\frac{1}{\pi\lambda^2}\right)^{1/4} \exp\left[-\frac{(x-\bar{x})^2}{2\lambda^2} + \frac{i}{\hbar}\bar{p}x - i\frac{\bar{p}\bar{x}}{2\hbar}\right]. \tag{3}$$

The number $\lambda$ ($\lambda^2 = \hbar/m\omega$) is a constant associated with the harmonic oscillator for which the ground state is $|0\rangle$. We note some important properties of the set of states $|\alpha\rangle$.[1,3-5] Every $|\alpha\rangle$ can be generated from $|0\rangle$ by translation in the phase plane $(\bar{x}, \bar{p})$,

$$|\alpha\rangle = \exp\left(\frac{i}{\hbar}(\hat{x}\bar{p} - \hat{p}\bar{x})\right)|0\rangle, \tag{4}$$

and can also be expressed in terms of the harmonic-oscillator states $|N\rangle$ ($N = 0, 1, 2, \ldots$),

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{N=0}^{\infty} \frac{\alpha^N}{(N!)^{1/2}} |N\rangle. \tag{5}$$

The states are nonorthogonal,

$$\langle\beta|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 + \alpha\beta^*), \tag{6}$$

and are complete, i.e., if $|f\rangle$ is any state, then from the condition $\langle f|\alpha\rangle = 0$ for all $\alpha$ it follows that $|f\rangle = 0$. Furthermore, they are overcomplete. However, they can still be used for expansions of states and operators because

$$\hat{I} = \frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha|, \tag{7}$$

where $\hat{I}$ is the identity operator and the integral is over the whole complex plane. This permits us to write, for any $|f\rangle$, the expansion

$$|f\rangle = \frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle \exp(-\tfrac{1}{2}|\alpha|^2) f(\alpha^*), \tag{8}$$

$$f(\alpha^*) = \exp(\tfrac{1}{2}|\alpha|^2)\langle\alpha|f\rangle = \sum_{N=0}^{\infty} \frac{\langle f|N\rangle}{(N!)^{1/2}} (\alpha^*)^N, \tag{9}$$

where (5) has been used. In this expression, $f(\alpha^*)$ is an entire analytic function of the complex conjugate $\alpha^*$ of $\alpha$; and in fact it can be shown[3] that (9) affords a one-to-one relation between the $|f\rangle$, and the entire functions $f(\alpha^*)$ that grow no faster than $\exp[\frac{1}{2}(\alpha^*)^2]$ and for which

$$\int d^2\alpha\, |f(\alpha^*)|^2 \exp(-|\alpha|^2) < \infty.$$

Furthermore, the expansion coefficients in (8) are unique so long as we keep the analyticity requirement that the function $f$ depends only on $\alpha^*$; this uniqueness is lost as soon as we allow more general functions of $\alpha$ and $\alpha^*$ (see, e.g., Ref. 4, p. 2773).

Operator expansions are written in a similar manner; thus for a general operator $T$,

$$T = \frac{1}{\pi^2}\int\int d^2\alpha\, d^2\beta\, |\alpha\rangle\langle\alpha|T|\beta\rangle\langle\beta|, \tag{10}$$

where [again using (5)] $\langle\alpha|T|\beta\rangle$ is expressed in terms of an entire fundtion of $\alpha^*$ and $\beta$, characteristic of $T$. Most of our discussion below will deal with expansions of states $|f\rangle$; operator expansions will be only briefly mentioned again at the end of the paper.

The discrete subset of coherent states $|\alpha_{mn}\rangle$ first studied by von Neumann[2] is obtained by taking the so-called von Neumann lattice of points

$$\alpha_{mn} = \frac{1}{\lambda\sqrt{2}}\left(na + i\frac{2\pi}{a}\lambda^2 m\right) \tag{11}$$

in the complex plane, where $a$ is arbitrary and the lattice cell area is $\pi$. This area corresponds to an area $h$ in the phase plane, as is seen from (2). The set of nonorthogonal $|\alpha_{mn}\rangle$ is also complete,[6-8] i.e., for any state $|f\rangle$ the vanishing of the scalar product $\langle f|\alpha_{mn}\rangle$ for all $(m,n)$ leads to $|f\rangle = 0$. It has, however, the strange property that it is overcomplete by just one member,[7,8] that is, if any one member is removed from the set, the rest are still complete, but this is not true if two are removed.[7,8] Without losing generality we remove $|\alpha_{00}\rangle$ and denote the remaining exactly complete set by $(|\alpha_{mn}\rangle)'$ with the prime. We can then construct the biorthogonal set[7] $(|\tilde{\alpha}_{mn}\rangle)'$ to $(|\alpha_{mn}\rangle)'$:

$$\langle\tilde{\alpha}_{m'n'}|\alpha_{mn}\rangle = \delta_{m'm}\delta_{n'n} \quad [(m,n),(m',n') \neq (0,0)]. \tag{12}$$

The question then arises: Can we expand any state vector $|f\rangle$ in terms of the $|\alpha_{mn}\rangle$ $[(m,n) \neq (0,0)]$, in the manner of (8) for the full set of coherent states? The answer is yes, provided we are careful about the nature of the convergence of the expansion; this remark takes emphasis from the fact that, as we shall see, there is some nonuniqueness in the expansion coefficients. It will be shown in Sec. III that any state $|f\rangle$ can be expanded in a series [where the prime excludes $(m,n) = (0,0)$]

$$|f\rangle \sim \sum_{m,n}{}' |\alpha_{mn}\rangle\langle\tilde{\alpha}_{mn}|f\rangle. \tag{13}$$

We have purposely used the sign ~ in (13) instead of the equality sign in order to point out that the series on the right-hand side does not converge to the vector $|f\rangle$ in the usual (mean) sense. Its precise meaning is given in Sec. III, where we prove that

$$\langle\varphi|f\rangle = \sum_{m,n}{}' \langle\varphi|\alpha_{mn}\rangle\langle\tilde{\alpha}_{mn}|f\rangle \tag{13'}$$

for any $|\varphi\rangle$ with a "smooth" $\langle x|\varphi\rangle$ in the sense of belonging to the test function space for (tempered) distributions. In other words the expansion (13) converges in the space of (tempered) distributions and the sign ~ means throughout the paper equivalence in this space.[22,24] At this point it is natural to question the uniqueness of the expansion for $|f\rangle$ in this sense, i.e., to ask for the general solution of

$$|f\rangle \sim \sum_{m,n}{}' c_{mn} |\alpha_{mn}\rangle$$

for the coefficients $c_{mn}$. There do exist relationships among the basis functions $|\alpha_{mn}\rangle$ relative to the equivalence ~ ; and we shall see in Sec. III that the most general relationship within the basis $(|\alpha_{mn}\rangle)'$ is

$$\sum_{mn}{}'(-1)^{m+n+mn}F(\alpha_{mn})|\alpha_{mn}\rangle \sim 0\,, \tag{14}$$

where $F(\alpha)$ is an arbitrary polynomial in $\alpha$ with $F(0)=0$. Hence the $c_{mn}$ are not uniquely determined but have the general form

$$c_{mn}=\langle\tilde{\alpha}_{mn}|f\rangle + F(\alpha_{mn})\,.$$

The choice $F(\alpha)\equiv 0$ gives the "canonical" solution.

In the relation (13′) the quantities $\langle\varphi|\alpha_{mn}\rangle$ are known from properties of coherent states[4] while the $\langle\tilde{\alpha}_{mn}|f\rangle$ are new quantities that have to be calculated. For the important special cases $|f\rangle=|\alpha\rangle$ or $|N\rangle$, Perelomov[7] provides the expression for the former whereas we show in Sec. II that the expression for the latter can be derived using classical complex analysis. We furthermore give a proof of (13′) when both $|f\rangle$ and $|\varphi\rangle$ are harmonic-oscillator states, that is, independent of the proof of Sec. III for the more general case. This result is shown to yield some interesting summation identities.

## II. EXAMPLE: HARMONIC-OSCILLATOR STATES

We first consider the coefficients $\langle\tilde{\alpha}_{mn}|\alpha\rangle$ for $|f\rangle=|\alpha\rangle$. This is just equal, of course, to the complex conjugate of the coefficient in the expansion (8) of $|\tilde{\alpha}_{mn}\rangle$ in terms of the $|\alpha\rangle$ and has been given by Perelomov,[7] from which

$$\langle\tilde{\alpha}_{mn}|\alpha\rangle=(-1)^{m+n+mn}\exp(-\tfrac{1}{2}|\alpha|^2)\frac{\alpha_{mn}}{\alpha(\alpha-\alpha_{mn})} \times\sigma(\alpha)\exp(-\nu\alpha^2)\,. \tag{15}$$

Here $\sigma(\alpha)$ is the Weierstrass $\sigma$ function of the theory of elliptic functions,[11]

$$\sigma(\alpha)=\alpha\prod_{m,n}{}'\left(1-\frac{\alpha}{\alpha_{mn}}\right)\exp\left(\frac{\alpha}{\alpha_{mn}}+\frac{\alpha^2}{2\alpha_{mn}{}^2}\right)\,, \tag{16}$$

where the prime excludes $\alpha_{00}=0$, and $\nu$ is a number depending on the lattice cell dimensions ($\nu=0$ for a square cell). We observe from (16) that $\sigma(\alpha)$ has simple zeros at the lattice points and that in (15) $\langle\tilde{\alpha}_{mn}|\alpha\rangle$ vanishes whenever $\alpha=\alpha_{m'n'}$ ($\neq\alpha_{mn}$ or $\alpha_{00}$), in accordance with (12). The verification of (12) for $\alpha=\alpha_{mn}$ proceeds via the quasiperiodicity property of $\sigma(\alpha)$ under lattice translations $\alpha\rightarrow\alpha+\alpha_{mn}$ in the complex plane.[12] Finally, we can also see directly from (15) and (16) that

$$\langle\tilde{\alpha}_{mn}|0\rangle=-(-1)^{m+n+mn}\,. \tag{17}$$

As a next illustration we calculate $\langle\tilde{\alpha}_{mn}|N\rangle$ for the expansion of $|N\rangle$, the harmonic-oscillator state. To do this we substitute the expression (5) for $|\alpha\rangle$ into (15) and equate the coefficients of $\alpha^N$ on both sides of the equation. Residue theory gives us

$$\langle\tilde{\alpha}_{mn}|N\rangle=(-1)^{m+n+mn}\frac{(N!)^{1/2}\alpha_{mn}}{2\pi i} \times\int_C\frac{\sigma(\alpha)\exp(-\nu\alpha^2)}{\alpha^{N+2}(\alpha-\alpha_{mn})}\,d\alpha \tag{18}$$

integrated along any contour enclosing the origin and avoiding $\alpha=\alpha_{mn}$. Writing formally

$$\frac{\sigma(\alpha)\exp(-\nu\alpha^2)}{\alpha}=\sum_{P=0}^{\infty}a_P\alpha^P \tag{19}$$

and taking $C$ to be a circle of radius $<|\alpha_{mn}|$, which is always possible since $\alpha_{mn}\neq 0$, we can expand $(\alpha-\alpha_{mn})^{-1}$ in powers of $\alpha/\alpha_{mn}$ and obtain

$$\langle\tilde{\alpha}_{mn}|N\rangle=-(-1)^{m+n+mn}(N!)^{1/2} \times\sum_{Q=0}^{N}a_{N-Q}\left(\frac{1}{\alpha_{mn}}\right)^Q\,. \tag{20}$$

To go further we must use the power-series expansion of $\sigma(\alpha)$,[13]

$$\sigma(\alpha)=\alpha+a\alpha^5+b\alpha^7+\cdots\,,$$

from which the first few values of the $a_P$ for $P$ even are ($a_P=0$ for $P$ odd)

$$\begin{aligned} &a_0=1\,,\quad a_2=-\nu\,,\quad a_4=a+\tfrac{1}{2}\nu^2 \\ &a_6=b-a\nu-\tfrac{1}{6}\nu^3\,. \end{aligned} \tag{21}$$

The first few terms of the coefficients $\langle\tilde{\alpha}_{mn}|N\rangle$ are then

$$\langle\tilde{\alpha}_{mn}|N\rangle=-(-1)^{m+n+mn}(N!)^{1/2} \times\left[\left(\frac{1}{\alpha_{mn}}\right)^N-\nu\left(\frac{1}{\alpha_{mn}}\right)^{N-2} +(a+\tfrac{1}{2}\nu^2)\left(\frac{1}{\alpha_{mn}}\right)^{N-4}+\cdots\right]\,. \tag{22}$$

We readily obtain from (20) the recursion relation

$$\langle\tilde{\alpha}_{mn}|N\rangle=\frac{N^{1/2}}{\alpha_{mn}}\langle\tilde{\alpha}_{mn}|N-1\rangle -(-1)^{m+n+mn}(N!)^{1/2}a_N\,,\quad N\geqslant 1\,.$$

The closure relation

$$\sum_{N=0}^{\infty} \langle\tilde{\alpha}_{mn}|N\rangle\langle N|\alpha_{m'n'}\rangle = \delta_{mm'}\delta_{nn'}$$

can be checked using (19) and the expression

$$\langle N|\alpha_{m'n'}\rangle = \exp(-\tfrac{1}{2}|\alpha_{m'n'}|^2)\frac{(\alpha_{m'n'})^N}{(N!)^{1/2}} \qquad (23)$$

derived from (5). By choosing the contour $C$ as a circle of radius greater than $|\alpha_{m'n'}|$ the sum over $N$ can be performed to give

$$\sum_{N=0}^{\infty} \langle\tilde{\alpha}_{mn}|N\rangle\langle N|\alpha_{m'n'}\rangle$$

$$= (-1)^{m+n+mn}\exp(-\tfrac{1}{2}|\alpha_{m'n'}|^2)\frac{\alpha_{mn}}{2\pi i} \times \int_C \frac{\sigma(\alpha)\exp(-\nu\alpha^2)}{\alpha(\alpha-\alpha_{mn})(\alpha-\alpha_{m'n'})}\,d\alpha\,.$$

For $\alpha_{mn} \neq \alpha_{m'n'}$ the integrand has no poles and closure is verified; if $\alpha_{mn} = \alpha_{m'n'}$ then it has a simple pole at $\alpha = \alpha_{mn}$ and its residue is evaluated again by using the quasiperiodicity of $\sigma(\alpha)$,[12] yielding the value unity for the last expression in this case.

A more interesting closure relation to check is (13′) in the case $|f\rangle = |N\rangle$, $|\varphi\rangle = |M\rangle$:

$$\sum_{mn}{}' \langle M|\alpha_{mn}\rangle\langle\tilde{\alpha}_{mn}|N\rangle = \delta_{MN}\,. \qquad (24)$$

We start by evaluating

$$\sum_{mn}{}' \langle M|\alpha_{mn}\rangle\langle\tilde{\alpha}_{mn}|\alpha\rangle$$

$$= \exp(-\tfrac{1}{2}|\alpha|^2)\frac{\sigma(\alpha)\exp(-\nu\alpha^2)}{(M!)^{1/2}\alpha} \times \sum_{mn}{}'(-1)^{m+n+mn}\frac{(\alpha_{mn})^{M+1}}{\alpha-\alpha_{mn}}\exp(-\tfrac{1}{2}|\alpha_{mn}|^2)$$

from (15) and (23). There is a well-known theorem for analytic functions with only simple poles, that equates the function to an expansion in terms of the poles and residues (see Ref. 11, p. 134). From this we obtain directly the relation

$$\frac{\alpha^{M+1}}{\sigma(\alpha)\exp(-\nu\alpha^2)} = \sum_{mn}{}'(-1)^{m+n+mn}\frac{(\alpha_{mn})^{M+1}}{\alpha-\alpha_{mn}} \times\exp(-\tfrac{1}{2}|\alpha_{mn}|^2)\,,$$

whence

$$\sum_{m,n}{}' \langle M|\alpha_{mn}\rangle\langle\tilde{\alpha}_{mn}|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\frac{\alpha^M}{(M!)^{1/2}}\,.$$

Substituting for $|\alpha\rangle$ from (5) and equating coefficients of $\alpha^M$, the closure relation (24) is immediate.

By using expressions (20) and (23) for the matrix elements in (24) some interesting identities can be obtained. If we introduce the sums

$$S_P \equiv \sum_{m,n}{}'(-1)^{m+n+mn}(\alpha_{mn})^P\exp(-\tfrac{1}{2}|\alpha_{mn}|^2) \qquad (25)$$

for $P$ integral (positive, negative, or zero) then (24) takes the compact form

$$\sum_{Q=0}^{N} a_{N-Q}S_{M-Q} = -\delta_{MN}\,. \qquad (24')$$

First, putting $N=0$ and $M=P\geq 0$ we have, using $a_0 = 1$ from (21),

$$S_P = -\delta_{P0} \quad (P\geq 0)\,, \qquad (26)$$

a result found by Perelomov.[7] Then for $N>0$ and $M\leq N$ we arrive at some apparently new identities enabling us to evaluate the sums (25) when $P = -|P| < 0$. For $P$ odd, $S_{-|P|}$ evidently vanishes (since $\alpha_{-m-n} = -\alpha_{mn}$) but the $S_{-|P|}$ for $P$ even are determined from (24′), which we write in the slightly rearranged form

$$S_{-|P|} = -\sum_{R=1}^{|P|} a_R S_{R-|P|} \quad (|P|>0)\,. \qquad (27)$$

Here the summation is cut off where the sums (26) vanish for positive integer, and we again used $a_0 = 1$. Iteration of (27) yields the negative-even-integer sums

$$S_{-2} = a_2\,, \quad S_{-4} = a_4 - a_2^{\,2}\,,$$
$$S_{-6} = a_6 - 2a_4a_2 + a_2^{\,3}\,,\ \ldots\,,$$

where the $a_R$ are known from (21). In this way we obtain for example the identities

$$S_{-2} \equiv \sum_{m,n}{}'(-1)^{m+n+mn}\frac{1}{(\alpha_{mn})^2}\exp(-\tfrac{1}{2}|\alpha_{mn}|^2) = -\nu\,,$$

$$S_{-4} \equiv \sum_{m,n}{}'(-1)^{m+n+mn}\frac{1}{(\alpha_{mn})^4}\exp(-\tfrac{1}{2}|\alpha_{mn}|^2) = a - \tfrac{1}{2}\nu^2\,.$$

Equation (27) can equally be used to express the $a_P$ iteratively in terms of the sums $S_{-|P|}$. The case of a square lattice, with $\alpha_{mn} = \sqrt{\pi}(m+in)$ and $\nu = 0$, is a particularly simple one where from (21) the $a_P$ reduce to the expansion coefficients of $\sigma(\alpha)/\alpha$. It was shown in Ref. 7 that the identities (26) for the $S_P$ $(P\geq 0)$ lead to interesting relations between $\Theta$ functions. We now find that the identities (27) for the $S_P$ $(P<0)$ seem to give an alternative way for calculating the expansion coefficients of $\sigma(\alpha)$.

Further discussion of these relationships is to be published elsewhere.[23]

We close the section with a few remarks on the expansions

$$|N\rangle\sim\sum_{m,n}{}'\langle\tilde{\alpha}_{mn}|N\rangle\,|\alpha_{mn}\rangle \tag{28}$$

for the harmonic-oscillator states. We assume here the results of Sec. III on the meaning and properties of the equivalence, already summarized at the end of Sec. I. For the special case $N=0$ we have using (17), the relationship

$$|0\rangle\sim-\sum_{m,n}{}'(-1)^{m+n+mn}|\alpha_{mn}\rangle \tag{28$'$}$$

among the von Neumann set discovered by Perelomov.[7] Our first remark is that repeated application of the annihilation operator $\hat{a}$ on (28′) (legitimate in distribution space) gives, using (1), Perelomov's other relations

$$\sum_{m,n}{}'(-1)^{m+n+mn}(\alpha_{mn})^M|\alpha_{mn}\rangle\sim 0\quad(M\geq 1). \tag{14$'$}$$

This essentially proves (14), except for the statement that the relationships (14′) are exhaustive (Sec. III).

The second remark is that we can obtain considerable information on the coefficients $\langle\tilde{\alpha}_{mn}|N\rangle$ in (28) without doing the exact calculation leading to (20). Operating on (28) with $(\hat{a})^N$ and recalling that $\hat{a}|N\rangle=N^{1/2}|N-1\rangle$ gives

$$|0\rangle\sim(N!)^{-1/2}\sum_{m,n}{}'\langle\tilde{\alpha}_{mn}|N\rangle(\alpha_{mn})^N|\alpha_{mn}\rangle. \tag{28$''$}$$

Thus comparing the two expansions (28′) and (28″) for $|0\rangle$, and using the freedom expressed by (14) or (14′) we get

$$\langle\tilde{\alpha}_{mn}|N\rangle=-\frac{(-1)^{m+n+mn}(N!)^{1/2}}{(\alpha_{mn})^N}f(\alpha_{mn}),$$

$$f(\alpha_{mn})=\sum_{P=0}\tilde{a}_P(\alpha_{mn})^P\quad(\tilde{a}_0=1),$$

where the coefficients $\tilde{a}_P$ are to be determined. Putting these expressions for $\langle\tilde{\alpha}_{mn}|N\rangle$ into the closure relation (24) together with (23) for the $\langle M|\alpha_{mn}\rangle$, and using (26), we recover (27) with $a_R$ replaced by $\tilde{a}_R$. It is then an easy argument from (26) and (27) that the $\tilde{a}_P$ are completely determined for $P\leq N$ to be $\tilde{a}_P=a_P$, and that they are completely undetermined for $P>N$. We therefore get complete agreement with (22) for $\langle\tilde{\alpha}_{mn}|N\rangle$ if we arbitrarily put the undeterminate coefficients equal to zero. The justification for doing so, however, depends on referring to the full calculation (19) of $\langle\tilde{\alpha}_{mn}|N\rangle$.

It is worth remarking that all the results of this section hold, with only trivial modifications, if we choose a lattice of points $\alpha_{mn}=2mw_1+2nw_2$, where $w_1$ and $w_2$ are any two noncollinear complex numbers chosen so that the lattice cell area is $\pi$.

## III. EXPANSIONS OF STATES AND OPERATORS

To clarify the nature of our expansion (13) it is very convenient to go over to the $kq$ representation,[14] in which the basis states $|\alpha_{mn}\rangle$ take a particularly simple form.[8] We recall that the general transformation for a state $|f\rangle$ from the $x$ representation to the $kq$ representation is

$$\langle kq|f\rangle=\left(\frac{a}{2\pi}\right)^{1/2}\sum_{l=-\infty}^{\infty}\exp(ikal)\langle q-la|f\rangle,$$

where the real number $a$ (lattice constant) may be arbitrary but is here chosen to be the same quantity as in (11). The coordinates $k,q$ run over a basic cell $-\pi/a\leq k\leq\pi/a$, $-\frac{1}{2}a\leq q\leq\frac{1}{2}a$, and the functions in the $kq$ representation have the characteristic periodicity

$$\begin{aligned}&\langle k+2\pi/a,q|f\rangle=\langle kq|f\rangle,\\&\langle k,q+a|f\rangle=\exp(ika)\langle kq|f\rangle.\end{aligned} \tag{29}$$

The $|\alpha_{mn}\rangle$, which from (4), (2), and (11), are generated from $|\alpha_{00}\rangle$ by discrete displacements in the phase plane, have the $kq$ representation

$$\langle kq|\alpha_{mn}\rangle=(-1)^{mn}\exp\left(i\frac{2\pi}{a}qm-iakn\right)\langle kq|0\rangle, \tag{30}$$

where

$$\langle kq|0\rangle=\left(\frac{a}{2\pi^{3/2}\lambda}\right)^{1/2}\sum_{l=-\infty}^{\infty}\exp\left(ikal-\frac{(q-la)^2}{2\lambda^2}\right).$$

The simple form of (30) is due to the fact that for the chosen area in the phase plane the momentum and position displacements commute.[8] The function $\langle kq|0\rangle$ has the important property that it has just one zero in the $kq$ cell, a simple zero at $k=\pi/a$, $q=\frac{1}{2}a$. It is convenient to observe that it can be written

$$\langle kq|0\rangle=\left(\frac{a}{2\pi^{3/2}\lambda}\right)^{1/2}\exp\left(-\frac{q^2}{2\lambda^2}\right)\Theta_3(z) \tag{31}$$

with

$$z=\tfrac{1}{2}ka-\frac{i}{2}\frac{aq}{\lambda^2}.$$

In (31) $\Theta_3(z)$ is the $\Theta$ function which is well known to be an entire complex function with only one (simple) zero in the complex cell defined by the $kq$ cell.[11] For use below, we note the form of the annihilation operator in this representation,

$$\hat{a}=\frac{1}{\lambda\sqrt{2}}\left(q+ia\frac{\partial}{\partial z^*}\right). \tag{32}$$

The biorthogonal set $|\tilde{\alpha}_{mn}\rangle$ also takes a relatively simple form in the $kq$ representation,

$$\langle kq|\tilde{\alpha}_{mn}\rangle=\frac{(-1)^{mn}}{2\pi}\frac{\exp[i(2\pi/a)qm-iakn]-(-1)^{m+n}}{\langle 0|kq\rangle}. \tag{33}$$

The orthogonality relations (12) can be checked directly; it can be seen, moreover, that the singularity in the denominator of (33) at $k=\pi/a$, $q=\frac{1}{2}a$ is removed by the numerator.

The relation (13′), which is to prove, now reads

$$\int\langle\varphi|kq\rangle\langle kq|f\rangle dkdq = \sum_{m,n}{}'(-1)^{mn}\langle\tilde{\alpha}_{mn}|f\rangle\int\exp\left(i\frac{2\pi}{a}qm-ikan\right)\times\langle\varphi|kq\rangle\langle kq|0\rangle dkdq, \tag{34}$$

with

$$\langle\tilde{\alpha}_{mn}|f\rangle=\int\frac{\exp[ikan-i(2\pi/a)qm]-(-1)^{m+n}}{\langle kq|0\rangle}\times\langle kq|f\rangle dkdq,$$

where the integrations go over the $kq$ cell. Here $|\varphi\rangle$ is any function for which $\langle kq|\varphi\rangle$ is smooth(i.e., continuously differentiable to all orders), so that the $|\varphi\rangle$ form the space of test functions for tempered distributions.[15] Note that whereas the $|\alpha_{mn}\rangle$ are smooth, the $|\tilde{\alpha}_{mn}\rangle$ are not, due to a discontinuity at $k=\pi/a$, $q=\frac{1}{2}a$. Proving (34) then tells us that the sum on the right-hand side of (13) converges to $|f\rangle$ in the space of tempered distributions, and we give the sign ~ a corresponding interpretation. Proceeding with the proof, consider the sum

$$\sum_{m,n}{}'\langle\tilde{\alpha}_{mn}|k'q'\rangle\int\exp\left(i\frac{2\pi}{a}qm-iakn\right)\times\langle\varphi|kq\rangle\langle kq|0\rangle dkdq. \tag{35}$$

A study of (33) shows us that

$$|\langle\tilde{\alpha}_{mn}|k'q'\rangle|<\text{const}\left[|n|a+\frac{2\pi\lambda^2}{a}|m|\right];$$

further, the integrals are the Fourier sum coefficients of the periodic function $\langle\varphi|kq\rangle\langle kq|0\rangle$ [periodic from (29)] and consequently decrease faster than any powers of $|m|$ and $|n|$. Hence (35) converges (almost everywhere uniformly) to an integrable function $\chi(k'q')$; substituting (33) in (35) we have

$$\chi(k'q')\langle k'q'|0\rangle=\langle\varphi|k'q'\rangle\langle k'q'|0\rangle-\left\langle\varphi\left|\frac{\pi}{a}\frac{a}{2}\right.\right\rangle\left\langle\frac{\pi}{a}\frac{a}{2}\left|0\right.\right\rangle=\langle\varphi|k'q'\rangle\langle k'q'|0\rangle.$$

Thus $\chi(k'q')=\langle\varphi|k'q'\rangle$, since $\langle k'q'|0\rangle$ has the one isolated zero. We used above the facts that a periodic smooth function equals everywhere its Fourier sum, and that $\langle(\pi/a)\frac{1}{2}a|0\rangle=0$. The sum (35), therefore, equals $\langle\varphi|k'q'\rangle$ and (34) follows by integration with $\langle kq|f\rangle$ [uniform convergence of (35) allows term-by-term integration].

Another way of writing (13), in view of the above proof, is

$$\langle kq|f\rangle\sim D_f(kq)\langle kq|0\rangle, \qquad D_f(kq)=\sum_{mn}{}'(-1)^{mn}\langle\tilde{\alpha}_{mn}|f\rangle\exp\left(i\frac{2\pi}{a}qm-ikan\right), \tag{36}$$

where $D_f$ is now a periodic distribution on $kq$ space (for a discussion of periodic distributions, see Refs. 16 and 17, Vol. II).[18] For example when $|f\rangle=|0\rangle$, then using (17)

$$D_0(kq)=\sum_{mn}{}'(-1)^{m+n}\exp\left(i\frac{2\pi}{a}qm-ikan\right)=1-2\pi\Delta\left(k-\frac{\pi}{a}\right)\Delta\left(q-\frac{a}{2}\right). \tag{37}$$

Here $\Delta$ is the $\delta$ function on the space of the unit cell. The expansion (36) for $|0\rangle$ is then

$$\langle kq|0\rangle\sim\left[1-2\pi\Delta\left(k-\frac{\pi}{a}\right)\Delta\left(q-\frac{a}{2}\right)\right]\langle kq|0\rangle,$$

and this equivalence of distributions is the precise meaning of the expansion (28′) for $|0\rangle$. [The equivalence is evident because $\langle(\pi/a)\frac{1}{2}a|0\rangle=0$.] We remark that in (36) for $|f\rangle=|0\rangle$, each term in the expansion has the norm $\|\langle kq|0\rangle\|=1$, so the convergence is certainly not in the usual mean sense. The result we have says that closure (using ~ here to relate operators)

$$\sum_{mn}{}'|\alpha_{mn}\rangle\langle\tilde{\alpha}_{mn}|\sim\tilde{I}$$

is correct for any $|f\rangle$ on the right and any smooth $|\varphi\rangle$ on the left. It is always true therefore between two smooth states, such as for $|M\rangle$ and $|N\rangle$ in (24), because

$$\langle kq|N\rangle=(N!)^{-1/2}(\hat{a}^\dagger)^N\langle kq|0\rangle$$

is smooth. Here the creation operation $\hat{a}^\dagger$ is the Hermitian conjugate of (32).

Finally we come to the problem of nonuniqueness in the expansions, discussed at the end of Sec. I. Thus, in terms of the $kq$ representation, we ask: In the equivalence

$$\langle kq|f\rangle \sim D(kq)\langle kq|0\rangle , \tag{38}$$

what is the freedom in $D(kq)$, where $D$ is a periodic distribution? Evidently, any solution of (38) differs from the "canonical" one $D_f(kq)$ of (36) by a solution of

$$\bar{D}(kq)\langle kq|0\rangle \sim 0 , \tag{39}$$

with

$$\bar{D}(kq)=\sum_{mn}{}' c_{mn}\exp\left(i\frac{2\pi}{a}qm-ikan\right). \tag{40}$$

There exist nontrivial distributions satisfying this equation. Because of the isolated zero of $\langle kq|0\rangle$ we know from general distribution theory (see Ref. 17, Vol. I, p. 100) that $\bar{D}$ is a linear combination of derivates of $\delta$ function $\Delta(k-\pi/a)\Delta(q-\frac{1}{2}a)$. Indeed, if we first solve (39) without the condition (40) which restricts us to distributions whose Fourier series expansions have $c_{00}=0$, we first note that

$$\Delta\left(k-\frac{\pi}{a}\right)\Delta\left(q-\frac{a}{2}\right)\langle kq|0\rangle \sim 0 , \tag{41}$$

with

$$\Delta\left(k-\frac{\pi}{a}\right)\Delta\left(q-\frac{a}{2}\right)=\frac{1}{2\pi}\sum_{mn}(-1)^{m+n}\times\exp\left(i\frac{2\pi}{a}qm-ikan\right). \tag{42}$$

By applying the annihilation operator (32), with the property $\hat{a}\langle kq|0\rangle=0$ for the ground state, we obtain from repeated operation

$$\left\{\left(\frac{\partial}{\partial z^*}\right)^M\left[\Delta\left(k-\frac{\pi}{a}\right)\Delta\left(q-\frac{a}{2}\right)\right]\right\}\langle kq|0\rangle\sim 0 \quad (M\geq 1), \tag{43}$$

with

$$\left(\frac{\partial}{\partial z^*}\right)^M\left[\Delta\left(k-\frac{\pi}{a}\right)\Delta\left(q-\frac{a}{2}\right)\right]=\frac{1}{2\pi}\left(\frac{1}{ia}\right)^M\sum_{mn}{}'(-1)^{m+n}\left(na+i\frac{2\pi\lambda^2}{a}m\right)^M\times\exp\left(i\frac{2\pi}{a}qm-ikan\right). \tag{44}$$

This is legitimate, since tempered distributions are infinitely differentiable. Near the zero at $k=\pi/a$, $q=\frac{1}{2}a$, an analysis of (31) shows that $\langle kq|0\rangle$ behaves like $(z-z_0)$ where $z_0=\frac{1}{2}\pi-ia^2/4\lambda^2$. Since therefore

$$\frac{\partial}{\partial z}\langle kq|0\rangle\Big|_{z=z_0}\neq 0 ,$$

it follows from the properties of distributions that the most general derivative $(\partial/\partial z^*)^M(\partial/\partial z)^N$ of the $\delta$ function would not satisfy a relation such as (43) or (41) if $N\neq 0$. We conclude that $\bar{D}(kq)$ must be an arbitrary linear combination of the distributions (44) for $M\geq 1$; $M=0$ is excluded because from (40) the coefficient $c_{00}=0$ in $\bar{D}$. This is then the degree of nonuniqueness in the solution of (38). Equations (43) are just the equations (14′) in the $kq$ representation and therefore we have proved the statement in Sec. I that (14) is the most general relationship among the $|\alpha_{mn}\rangle$ $[(m,n)\neq(0,0)]$.

Corresponding to the expansion (13) for states there is an expansion

$$T\sim\sum_{mn}{}'\sum_{m'n'}{}'|\alpha_{mn}\rangle\langle\tilde{\alpha}_{mn}|T|\alpha_{m'n'}\rangle\langle\tilde{\alpha}_{m'n'}|$$

for operators. The equivalence again means the equality is true for all matrix elements between $|f\rangle$ on the right and smooth $|\varphi\rangle$ on the left; in other words

$$T|f\rangle\sim\sum_{mn}{}'\sum_{m'n'}|\alpha_{mn}\rangle\langle\tilde{\alpha}_{mn}|T|\alpha_{m'n'}\rangle\langle\alpha_{m'n'}|f\rangle$$

is a distribution equality. There is of course the same nonuniqueness here as for state expansions.

## IV. CONCLUSIONS

We have shown that the discrete coherent states on a von Neumann lattice in phase space form a basis for expanding states and operators in quantum mechanics. The expansion in the discrete states is somewhat in the spirit of Glauber's approach for the full set of coherent states. Because of the particular structure of the von Neumann lattice (one state corresponds to a phase-space cell of area $h$) the discrete coherent states seem to have some special physical significance. Intriguing is, however, the fact that one state can be removed (but not more than one) and the system still remains complete.[7,8] This apparently has to do with the nonorthogonality of the coherent states. It is clear that for each state of the von Neumann subset the minimal uncertainty is preserved and from this point of view the classical nature of the discrete coherent states is unchanged. However, if one tries to build any orthogonal set of states on the von Neumann lattice it becomes impossible to preserve their coherence property.[8,19] The usefulness of the full set of coherent states in different applications in physics is quite well established.[1] Discrete sets of states on a von Neumann lattice have been used in a number of physical problems[9,10,20,21] and we hope that this paper will further stimulate the application of discrete coherent states.


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[10]M. H. Boon, Helv. Phys. Acta 48, 551 (1975); *Lecture Notes in Physics* (Springer, Berlin, 1976), Vol. 50, p. 282.

[11]E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, Cambridge, 1950), p. 447. (Note that $\sigma(\alpha)$ is defined for general $\alpha_{mn} = 2mw_1 + 2nw_2$ where the cell area is not restricted to $\pi$.) The constant $\nu = (i/\pi)\ (\eta + w_2^* - \eta_2 w_1^*)$ contains the standard elliptic function constants $\eta_1$, $\eta_2$ defined on p. 446.

[12]See Ref. 11, p. 448, Example 1. The identity of §20.411 must also be used.

[13]M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), p. 635.

[14]J. Zak, in *Solid State Physics*, edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1972), Vol. 27.

[15]The proof of this is obtained by using the transformation relating the $\langle kq | \varphi \rangle$ to the $\langle x | \varphi \rangle$ and its inverse, together with the periodicity (29), and showing that indeed the corresponding $\langle x | \varphi \rangle$ are those functions that (a) are continuously differentiable in $x$ to all orders; (b) have all derivatives decreasing faster than any power of $x$ at infinity. For the properties of tempered distributions we refer to the books of Refs. 16 and 17.

[16]I. Schwartz, *Mathematics for the Physical Sciences* (Addison-Wesley, Reading, Mass., 1966).

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[18]The coefficients in the Fourier sum (36) satisfy $|\langle \tilde{\alpha}_{nm} | f \rangle| \leqslant \text{const} \times \|f\|\ [\,|n|\,a + (2\pi\lambda^2/a)\,|m|\,]$ in accordance with Ref. 17, Vol. II, Chap. VII, where we learn that the coefficients of the Fourier series for distributions are required to rise slower than arbitrary powers of $|m|$ and $|n|$. This property classifies the periodic distributions.

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[22]It is important to remark that Perelomov (Ref. 7) gives a different definition of the equivalence in (13). Perelomov uses the Bargmann form of Hilbert space (Ref. 3) in which states are represented by entire functions, and proves that the expansion converges uniformly in the complex plane. This convergence is to be compared with the convergence in the space of tempered distributions, which can also be defined in the Bargmann representation (Ref. 3). We do not discuss the relation between Perelomov's and our definitions of equivalence further here.

[23]M. Boon and J. Zak, J. Math. Phys. (to be published).

[24]We recall that Hilbert-space states can be regarded as distributions by the standard embedding of the Hilbert space in distribution space (Refs. 16 and 17), wherein each state is identified with the distribution defined as its scalar product with the functions of the test space.