Collective oscillations in a simple metal. II. Electrical conductivity

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We calculate the conductivity tensor for a metal with an isotropic Fermi surface, including scattering and allowing arbitrary frequency, wave number, and uniform static magnetic field (within the limits of Landau's theory). We discuss the present state of the theory of excitation of cyclotron waves, and explain its importance to the analysis of observations. We show that, although the Poynting vector for a cyclotron wave is antiparallel to the group velocity and to the damping direction, inclusion of quasiparticle energy flow leads to a net energy flow in the expected direction.

I. INTRODUCTION

We shall describe an effective method for calculating the complex conductivity tensor $\vec{\sigma}(\vec{q}, \omega, \vec{H})$ for an isotropic charged Fermi liquid in a uniform static magnetic field. The calculation includes quasiparticle scattering, an arbitrary wave vector and frequency (provided only that $q \ll k_F$ and $\omega \ll \epsilon_F/\hbar$, so that Landau's theory applies; for the same reason we require that $\hbar\omega_c \ll k_B T$, where ω_c is the cyclotron frequency eH/m^*c), and a finite number L of Fermi-liquid parameters A_l (L can be as large as one likes). The conductivity can be used to calculate the complex dispersion relation, $\omega = \Omega(\vec{q})$, for electromagnetic waves: Form the permittivity tensor, $\vec{\epsilon}(\vec{q}, \omega, \vec{H}) = \vec{1} + (4\pi i/\omega)\vec{\sigma}(\vec{q}, \omega, \vec{H})$ and determine ω so that the equations $q^2\vec{E} - \vec{q}\,\vec{q}\cdot\vec{E} = (\omega^2/c^2)\vec{\epsilon}\cdot\vec{E}$ have a solution. $\vec{E}$ is the electric field of the corresponding wave.

This program was carried out for cyclotron waves ($\vec{q}\cdot\vec{H} = 0$, $\omega \lesssim n\omega_c$) many years ago by Platzman *et al.*,[1,2] omitting quasiparticle scattering, with special emphasis on the effect of electron correlations (the parameters A_l of Landau's theory) on the dispersion relation for $v_F q/\omega \ll 1$. They compared their theoretical results with experimental data on the reflection of microwaves from alkali-metal foils, and so estimated A_2 and A_3 in sodium and potassium. Microwave-transmission data[3] were analyzed using the results of the present computational method,[4,5] which is really the same as that used for analysis of spin-wave data in the alkali metals.[4,6,7] We are now giving a detailed presentation of our method to complement the preceding paper.[3]

The inclusion of scattering, with moderate values of $\omega\tau$ (~10–20) modifies the cyclotron-wave dispersion relation most significantly at the ends of the finite-frequency intervals for which the waves exist at all in the absence of scattering.[3-5,8] With no scattering, cyclotron waves are evanescent for $\omega > (1+A_2)\omega_c$, and the wave number approaches zero as ω approaches $(1+A_2)\omega_c$. (In this discussion we are focusing our attention on waves near the fundamental cyclotron resonance at $\omega = \omega_c$; a similar story applies for the waves near any harmonic $\omega = n\omega_c$, except that different Landau parameters A_l are relevant.) In the presence of scattering, there is no sharp cutoff for wave propagation. For real ω, the complex wave number q varies smoothly as ω is increased through $(1+A_2)\omega_c$, with Imq growing rapidly and exceeding Req somewhere near $(1+A_2)\omega_c$. For moderate values of $\omega\tau$, $v_F|q|/\omega$ is never small enough for a series solution to be valid. At the low-frequency end, ω_L, of the no-scattering pass band, nothing remarkable at all happens to the complex wave number. As ω is lowered, Imq gradually increases and eventually, well below ω_L for moderate $\omega\tau$, Imq rises rapidly and exceeds Req. In other words, for finite $\omega\tau$ the cyclotron wave band is significantly extended on the low-frequency (high-field) side. For no scattering and $\omega \gtrsim \omega_L$, there are several possible wave numbers. With scattering, these branches of the dispersion relation also extend to $\omega < \omega_L$. As a computational check, one can make $\omega\tau$ very large ($\gtrsim 100$) and recover the no-scattering results as closely as desired.

The modification of the cyclotron-wave dispersion relation by scattering implies serious difficulties in the interpretation of data when $\omega\tau$ is not very high. We could hope that the magnetic field dependence of the transmitted signal is predominantly given by a propagation factor $\exp(iqL)$, where L is the sample thickness (or perhaps by $(\sin qL)^{-1}$ to account for multiple passes of the waves through the sample, but Im$qL \gg 1$ in practice, so we can ignore this refinement). This hope is in vain. It is true that the oscillations of the signal are related to those of $e^{i\,\mathrm{Re}qL}$, but the relation is indirect.

Even more troublesome, the first few oscillations near the threshold $\omega=(1+A_2)\omega_c$ occur where Re$qL \ll 1$, so they come entirely from the phase variation with magnetic field of the prefactor of e^{iqL}. This means that serious analysis of data for moderate $\omega\tau$ cannot be done using only the dispersion relation or even, presumably, the conductivity $\vec{\sigma}(\vec{q}, \omega, H)$ which applies in an infinite medium. It is essential to solve a boundary value problem. This we have not been able to do. In desperation, several formulas for the prefactor of e^{iqL} were generated by intuition and wishful thinking, and their application is described in the immediately preceding paper.

The first attempt was the equivalent current sheet method,[9] which gave poor results. For orientation, not for actual computation, we point out that for no scattering and $A_2=0$, the current sheet method gives a prefactor, for the ordinary wave ($\vec{E}_{rf}$ parallel to $\vec{H}_{dc}$), $(\partial\sigma_{33}/\partial q)^{-1} \propto (\omega_c-\omega)^{1/2}$ for ω_c near ω.

The second attempt was an impedance matching argument. The cyclotron wave impedance, relative to the vacuum, is $E/H=\omega/cq$. At the surface of the sample where the cyclotron wave is generated, H is equal to the applied microwave cavity field, which is relatively independent of H_{dc}, so the electric field which is detected is $H(\omega/cq)e^{iqL}$. This gives a prefactor (omitting factors which are field independent) q^{-1}, a guess which works quite well. To contrast this with the current sheet method, note that, for no scattering and $A_2=0$, $q^{-1}\propto(\omega_c-\omega)^{-1/2}$ for ω_c near ω. The phase variation of q^{-1} with field does seem to account for the signal oscillations near threshold.

The third attempt, least rational but most successful, was to compare the first two attempts and notice that for weak scattering and near threshold, q^{-1} is proportional to $\partial\sigma_{33}/\partial q$. Accordingly, the expression $(\partial\sigma_{33}/\partial q)e^{iqL}$ for the transmitted signal was tried, and explaining its success[3] remains a theoretical challenge.

In Sec. II we derive our algorithm for calculation of the conductivity tensor. Although the preceding discussion has dealt with the case $\vec{q}\cdot\vec{H}=0$, we shall not make any such restrictions hereafter. In Sec. III we discuss energy transport for electromagnetic waves in the presence of spatial dispersion (dependence of σ on q). The issue is that for waves propagating in the $+x$ direction we must have Im$q>0$. For cyclotron waves, this implies that the group velocity $d\omega/dq>0$, which is fine, but Re$q<0$, so that the real part of the wave impedance is negative and both the phase velocity and the Poynting vector are directed antiparallel to the direction of wave propagation. We show that energy is transported in the proper direction, with velocity equal to the group velocity, so there is no paradox, but energy transport is predominantly by the quasiparticle system with back flow in the electromagentic field.

II. CALCULATION OF THE CONDUCTIVITY

A. Formal solution of the kinetic equation

The Landau kinetic equation is simply the Boltzmann equation with a self-consistent field $\delta\epsilon(\vec{k}, \vec{r}, t)$, and this is most simply written as an equation for $\Psi(\hat{k}, \vec{r}, t)$, defined in terms of the quasiparticle distribution $f(\vec{k}, \vec{r}, t)$ by

$$f(\vec{k}, \vec{r}, t)=\eta(\epsilon_F-\epsilon_k-\delta\epsilon(\vec{k}, \vec{r}, t)) + \delta(\epsilon_F-\epsilon_k)\Psi(\hat{k}, \vec{r}, t), \tag{1}$$

where $\eta(x)$ is the unit step function, ϵ_k is the quasiparticle energy, and ϵ_F is the Fermi energy. The kinetic equation is

$$\frac{\partial}{\partial t}(\Psi-\delta\epsilon)+v_F\hat{k}\cdot\frac{\partial}{\partial\vec{r}}\Psi-\frac{e}{c}v_F\hat{k}\times\vec{H}\cdot\frac{\partial}{\partial\vec{k}}\Psi+ev_F\vec{E}\cdot\hat{k} = \int d^2k'\, w(\hat{k}\cdot\hat{k}')[\Psi(\hat{k}')-\Psi(\hat{k})]. \tag{2}$$

The right-hand side is the collision term; $w(\hat{k}\cdot\hat{k}')$ is the equilibrium scattering rate for scattering on the Fermi surface (smeared by k_BT) from $\hat{k}$ to $\hat{k}'$. The self-consistent field, for $|\vec{k}|=k_F$, is

$$\delta\epsilon(\vec{k}, \vec{r}, t)=\int d^2k'\,\tilde{A}(\hat{k}\cdot\hat{k}')\Psi(\hat{k}', \vec{r}, t), \tag{3}$$

where $\tilde{A}(\hat{k}\cdot\hat{k}')$ is related to the usual Landau parameters by

$$\tilde{A}(\hat{k}\cdot\hat{k}')=\frac{1}{4\pi}\sum_l(2l+1)\frac{A_l}{1+A_l}P_l(\hat{k}\cdot\hat{k}'). \tag{4}$$

We make a similar expansion of $w(\hat{k}\cdot\hat{k}')$,

$$w(\hat{k}\cdot\hat{k}')=\frac{1}{4\pi}\sum_l(2l+1)w_lP_l(\hat{k}\cdot\hat{k}'), \tag{5}$$

assume that everything depends on $\vec{r}$ and t as exp $[i(\vec{q}\cdot\vec{r}-\omega t)]$, and we have

$$(w_0+iv_F\vec{q}\cdot\hat{k}-i\omega)\Psi(\hat{k})-\frac{e}{c}v_F\hat{k}\times\vec{H}\cdot\frac{\partial}{\partial\vec{k}}\Psi(\hat{k}) = -ev_F\vec{E}\cdot\hat{k}-i\omega\int d^2k'\,[\tilde{A}(\hat{k}\cdot\hat{k}') + \frac{i}{\omega}w(\hat{k}\cdot\hat{k}')]\Psi(\hat{k}'). \tag{6}$$

If, for the moment, we pretend that the right-hand side of (6) is known, we can solve (6) easily by noticing that the characteristics are just the particle orbits in $\vec{k}$ space in the dc field $\vec{H}$, which are known. (They are circles on the Fermi sphere in planes perpendicular to $\vec{H}$, traced out with constant angular velocity ω_c.) In particular, we consider the equation

$$(w_0 + iv_F\vec{q}\cdot\hat{k} - i\omega)\Psi(\hat{k}) - \frac{e}{c}v_F\hat{k}\times\vec{H}\cdot\frac{\partial}{\partial\vec{k}}\Psi(\hat{k}) = -i\omega Y_{lm}(\hat{k}), \quad (7)$$

where $Y_{lm}(\hat{k})$ is a standard spherical harmonic[10] (we choose the x_3 axis along $\vec{H}$), and write the solution of (7) in the form

$$\Psi(\hat{k}) = \sum_{l',m'} Y_{l'm'}(\hat{k})\langle l'm'|K|lm\rangle. \quad (8)$$

The matrix K is defined by (7) and (8).

We now write the solution of (6), our actual kinetic equation, in the form

$$\Psi(\hat{k}) = \sum_{l,m}\Psi_{lm}Y_{lm}(\hat{k}). \quad (9)$$

The Ψ_{lm} then satisfy the equations

$$\Psi_{lm} = \sum_{l'm'}\langle lm|K|l'm'\rangle\left[\delta_{l'1}\vec{S}_{m'}\cdot\vec{E} + \left(\frac{A_{l'}}{1+A_{l'}} + \frac{iw_{l'}}{\omega}\right)\Psi_{l'm'}\right], \quad (10)$$

where

$$\vec{S}_m = -\frac{iev_F}{\omega}\int d^2k\,\hat{k}Y^*_{lm}(\hat{k}). \quad (11)$$

B. Calculation of the conductivity

The electric current density is expressed most simply in terms of $\Psi(\hat{k})$ because of the renormalization of the current vertex:

$$\vec{J} = -\nu_F ev_F\int\frac{d^2k}{4\pi}\hat{k}\Psi(\hat{k}) = +\frac{i\omega\nu_F}{4\pi}\sum_m\Psi_{lm}\vec{S}^*_m, \quad (12)$$

where ν_F is the density of states at the Fermi energy. The vectors $\vec{S}_m$ are

$$\vec{S}_{\pm1} = \pm(\tfrac{2}{3}\pi)^{1/2}(iev_F/\omega)(\hat{x}_1 \mp i\hat{x}_2), \qquad \vec{S}_0 = -(\tfrac{4}{3}\pi)^{1/2}(iev_F/\omega)\hat{x}_3. \quad (13)$$

The scheme for calculating the conductivity is to truncate the equations (10) by setting A_l and w_l equal to zero for $l \geq L$, solve the $L^2\times L^2$ system of linear algebraic equations (10) for Ψ_{lm}, and then use (12) to express $\vec{J}$ in terms of Ψ_{lm}. In practice, for the interesting cases $\vec{q}$ parallel or perpendicular to $\vec{H}$ selection rules operate to reduce the dimension of the truncated system (10) substantially.

At a test case for our scheme, let us calculate $\overleftrightarrow{\sigma}$ when $\vec{q}=0$. From (7) and (8) we find directly that

$$\langle lm|K(\vec{q}=0)|l'm'\rangle = \frac{-i\omega}{w_0 - i\omega + im\omega_c}\delta_{ll'}\delta_{mm'}, \quad (14)$$

where $\omega_c = eH/m^*c$, $m^* = k_F/v_F$. From (10) we find that

$$\Psi_{lm} = \frac{-i\omega}{w_0 - w_1 - i\omega/(1+A_1) + im\omega_c}\vec{S}_m\cdot\vec{E}, \quad (15)$$

and then from (12),

$$\overleftrightarrow{\sigma} = \frac{\nu_F\omega^2}{4\pi}\sum_m\frac{\vec{S}^*_m\vec{S}_m}{(w_0 - w_1 - i\omega/(1+A_1) + im\omega_c)}. \quad (16)$$

Substituting (13) for $\vec{S}_m$ leads to the usual formula for $\overleftrightarrow{\sigma}$.

In the Sec. II c we calculate the matrix K exactly. Therefore, the only approximation we have made, other than the use of Landau kinetic theory, is the neglect of A_l and W_l for $l \geq L$.

C. Calculation of the K matrix

We must solve (7) in order to find $\langle lm|K|l'm'\rangle$. This task has already been accomplished,[4,6] but we do it here to save the reader the trouble of translating notation. We choose coordinates so that $\vec{H} = H\hat{x}_3$, $\vec{q} = q(\sin\Delta\hat{x}_1 + \cos\Delta\hat{x}_3)$, introduce standard polar coordinates θ, φ on the sphere, and note that $\hat{x}_3\cdot\vec{k}\times(\partial/\partial\vec{k}) = \partial/\partial\varphi$. Define

$$X(\theta) = (v_Fq/\omega_c)\sin\Delta\sin\theta, \quad (17)$$

$$Y(\theta) = (\omega + iw_0 - v_Fq\cos\Delta\cos\theta)/\omega_c. \quad (18)$$

Then (7) becomes

$$Y(\theta)\Psi - X(\theta)\cos\varphi\Psi + i\frac{\partial\Psi}{\partial\varphi} = \frac{\omega}{\omega_c}Y_{lm}(\theta,\varphi), \quad (19)$$

whose periodic solution is

$$\Psi(\theta,\varphi) = -\frac{i\omega}{\omega_c}\int_0^\infty d\Phi\,Y_{lm}(\theta,\varphi-\Phi)e^{iY(\theta)\Phi} \times\exp\{iX(\theta)[\sin(\varphi-\Phi)-\sin\varphi]\}. \quad (20)$$

If (20) is substituted into

$$\langle l'm'|K|lm\rangle = \int_0^{2\pi}d\varphi\int_0^\pi d\theta\sin\theta\,Y^*_{l'm'}(\theta,\varphi)\Psi(\theta,\varphi), \quad (21)$$

we have the first equation of Appendix A of Ref. 6. We can, therefore, borrow the results derived there, and we have, with $C_{lm}(\theta) = Y_{lm}(\theta,\varphi)e^{-im\varphi}$ and $m_>$ ($m_<$) the greater (lesser) of m and m',

$$\langle l'm'|K|lm\rangle = (-)^{m_>}\frac{2\pi^2\omega}{\omega_c}\times\int_0^\pi d\theta\sin\theta\,C_{lm}(\theta)C_{l'm'}(\theta)\times\frac{J_{Y(\theta)-m_<}(X(\theta))J_{-[Y(\theta)-m_>]}(X(\theta))}{\sin\pi Y(\theta)}. \quad (22)$$

The skeptical reader can check that in the limit $q \to 0$, so that $X \to 0$, (22) gives us the simple result (14). In case X or Y is independent of θ ($\Delta = 0$ or $\Delta = \pi/2$), the integral (22) can be carried out analytically.[4]

III. ENERGY FLOW IN SYSTEMS WITH SPATIAL DISPERSION

When we are asked to solve an initial value problem for the electromagnetic field in a medium with permittivity $\overleftrightarrow{\epsilon}(\vec{q}, \omega) = \overleftrightarrow{1} + (4\pi i/\omega)\overleftrightarrow{\sigma}(\vec{q}, \omega)$, we find the dispersion relation $\omega = \Omega(\vec{q})$ for waves in the medium, decompose the initial field into plane-wave components with real wave vectors $\vec{q}$, give each component a phase $\exp[-i\Omega(\vec{q})t]$, and add up the pieces. Because of dissipative processes, each component is damped (assuming that the medium was in thermal equilibrium before the initial field was introduced), so $\mathrm{Im}\Omega(\vec{q}) < 0$. The procedure for solution of a boundary value problem is different. Consider a semi-infinite sample occupying the region $x_1 > 0$. The field is given at $x_1 = 0$, and in practical cases is monochromatic with real frequency ω. We suppose the excitation to be independent of x_2 and x_3 (normal incidence) for simplicity. In the medium, far from the boundary, a wave is propagated with wave number q: $\vec{E} \propto e^{iqx_1}$. The wave number q is a solution of $\Omega(q\hat{x}_1) = \omega$ with real ω determined by the signal generator, and q is complex with $\mathrm{Im}q > 0$.

If the damping of the wave is weak, we have $|\mathrm{Im}\Omega(q)| \ll |\mathrm{Re}\Omega(q)|$ for real q. [We are writing $\Omega(q)$ as an abbreviation for $\Omega(q\hat{x}_1)$.] We can solve $\Omega(q) = \omega$, ω real, by first choosing real q_0 such that $\mathrm{Re}\Omega(q_0) = \omega$, and then writing $q = q_0 + \delta q$, with $|\delta q| \ll |q_0|$ so that

$$\omega = \Omega(q_0 + \delta q) = \Omega(q_0) + \delta q \left(\frac{d\Omega(q)}{dq}\right)_{q_0},$$

$$\delta q = -\mathrm{Im}\Omega(q_0) \left(\frac{d\Omega(q)}{dq}\right)_{q_0}.$$

If we neglect $\mathrm{Im}[d\Omega(q)/dq]$, we have

$$\mathrm{Im}q = -\mathrm{Im}\Omega(q_0)/v_g(q_0),$$

where $v_g = \mathrm{Re}(d\Omega/dq)$ is the group velocity of the wave. We know that $\mathrm{Im}\Omega(q_0) < 0$ for a stable medium, and we want $\mathrm{Im}q > 0$, so we must have $v_g > 0$, which is sensible.

When this standard program is carried out for cyclotron waves with ω near $n\omega_c$, we find that we must choose $\mathrm{Re}q < 0$. The wave impedance is $E/H = \omega/cq$, which has a negative real part. This makes it clear that we cannot possibly identify the cyclotron wave impedance with the surface impedance of the sample, even though that identification is fairly successful in fitting microwave transmission data. Moreover, the x_1 component of the (time averaged) Poynting vector is, for

$$\vec{E} = \vec{E}_0 \exp[i(qx_1 - \omega t)] + \vec{E}_0^* \exp[-i(qx_1 - \omega t)],$$

$$(c^2q/2\pi\omega)\,|\vec{E}_0|^2 < 0,$$

so the Poynting vector cannot describe the energy flow associated with the wave.

The paradox is resolved by imitating the discussion of electromagnetic wave energy in a dispersive medium.[11] Let the envelope $\vec{E}_0$ vary slowly in time and space. From Maxwell's equations,

$$\frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{H}}{\partial t}\right) + \nabla\cdot\frac{c}{4\pi}\vec{E}\times\vec{H} = 0,$$

neglecting magnetic phenomena (actually magnetic phenomena cannot be distinguished from electric phenomena macroscopically when spatial dispersion is admitted). We have[11]

$$\frac{\partial\vec{D}}{\partial t} = \exp i(\vec{q}\cdot\vec{x} - \omega t)\left(-i\omega + \frac{\partial}{\partial t}\right) \times\overleftrightarrow{\epsilon}\left(\vec{q} - i\frac{\partial}{\partial\vec{r}}, \omega + i\frac{\partial}{\partial t}\right)\cdot\vec{E}_0 + \mathrm{c.c.}$$

$$= \exp i(\vec{q}\cdot\vec{x} - \omega t)\left(-i\omega\overleftrightarrow{\epsilon}(\vec{q}, \omega)\cdot\vec{E}_0 + \frac{\partial\omega\overleftrightarrow{\epsilon}}{\partial\omega}\cdot\frac{\partial\vec{E}_0}{\partial t} - \omega\frac{\partial\overleftrightarrow{\epsilon}}{\partial q_i}\cdot\frac{\partial\vec{E}_0}{\partial x_i}\right),$$

so, averaging in time and space,

$$\frac{1}{2\pi}\left[\mathrm{Im}(\omega\vec{E}_0^*\cdot\overleftrightarrow{\epsilon}\cdot\vec{E}_0) + \mathrm{Re}\left(\vec{E}_0^*\cdot\frac{\partial\omega\overleftrightarrow{\epsilon}}{\partial t}\cdot\frac{\partial\vec{E}_0}{\partial t}\right) - \mathrm{Re}\left(\omega\vec{E}_0^*\cdot\frac{\partial\overleftrightarrow{\epsilon}}{\partial q_i}\cdot\frac{\partial\vec{E}_0}{\partial x_i}\right) + \vec{H}_0\cdot\frac{\partial\vec{H}_0}{\partial t}\right] + \nabla\cdot\frac{c}{2\pi}\mathrm{Re}(\vec{E}_0^*\times\vec{H}) = 0.$$

The first term is the power dissipation per unit volume. If this is negligible, so that $\overleftrightarrow{\epsilon}$ is approximately Hermitian, we have

$$\frac{\partial U}{\partial t} + \nabla\cdot\vec{S} = 0, \tag{23}$$

with energy density

$$U = \frac{1}{4\pi}\left(\vec{E}_0^*\cdot\frac{\partial\omega\overleftrightarrow{\epsilon}}{\partial\omega}\cdot\vec{E}_0 + \vec{H}_0^*\cdot\vec{H}_0\right) \tag{24}$$

and energy flow

$$\vec{S} = \frac{c}{2\pi}\mathrm{Re}(\vec{E}_0^*\times\vec{H}_0) - \frac{\omega}{4\pi}\left(\vec{E}_0^*\cdot\frac{\partial\overleftrightarrow{\epsilon}}{\partial q_i}\cdot\vec{E}_0\right)\hat{x}_i. \tag{25}$$

The form (24) of U is standard,[11] and the second term in (25) describes energy flow in the medium.

We can relate U and $\vec{S}$, defined by (24) and (25), to the group velocity $\vec{v}_g = \partial\Omega/\partial\vec{q}$. By Faraday's law, $\vec{H}_0 = (c/\omega)\vec{q}\times\vec{E}_0$, and therefore

$$U = \frac{1}{4\pi\omega} \vec{E}_0^* \cdot \frac{\partial \omega^2 \vec{\epsilon}}{\partial \omega} \cdot \vec{E}_0 , \tag{26}$$

$$\vec{S} = \frac{c^2}{4\pi\omega}(2\vec{E}_0^* \cdot \vec{E}_0 \vec{q} - \vec{q} \cdot \vec{E}_0^* \vec{E} - \vec{q} \cdot \vec{E}_0 \vec{E}_0^*) - \frac{\omega}{4\pi} \vec{E}_0^* \cdot \frac{\partial \vec{\epsilon}}{\partial q_l} \cdot \vec{E}_0 \hat{x}_l . \tag{27}$$

If we vary $\vec{q}$, $\vec{\omega}$, and $\vec{E}_0$ in such a way as to always have

$$q^2 \vec{E}_0 - \vec{q}\,\vec{q} \cdot \vec{E}_0 = (\omega^2/c^2) \vec{\epsilon} \cdot \vec{E}_0 , \tag{28}$$

we find

$$(1/c^2)\delta(\omega^2 \vec{\epsilon}) \cdot \vec{E}_0 - (2\vec{E}_0 \vec{q} - \vec{q} \cdot \vec{E}_0 \vec{1} - \vec{q}\,\vec{E}_0) \cdot \delta\vec{q} = q^2 \delta\vec{E}_0 - \vec{q}\,\vec{q} \cdot \delta\vec{E}_0 - (\omega^2/c^2)\vec{\epsilon} \cdot \delta\vec{E}_0 . \tag{29}$$

Scalar multiplication of (29) by $(c^2/4\pi\omega)\vec{E}_0^*$, and use of (26)–(28), yields

$$U\delta\omega - \vec{S} \cdot \delta\vec{q} = 0 , \tag{30}$$

or

$$\vec{v}_g = \frac{\partial \omega}{\partial \vec{q}} = \frac{\vec{S}}{U} . \tag{31}$$

Inclusion of the energy flow in the medium, Eq. (25), is just what is needed to avoid any paradox about the direction of wave propogation in a medium with spatial dispersion, even when the phase and group velocities are in opposite directions. The direction of wave damping, the direction of wave packet propagation, and the direction of energy flow for a plane wave all coincide.

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[1] P. M. Platzman, W. M. Walsh, and E-Ni Foo, Phys. Rev. 172, 689 (1968).

[2] P. M. Platzman and P. A. Wolff, *Waves and Interactions in Solid State Plasmas* (Academic, New York, 1973) Chaps. VIII and X, and references cited therein.

[3] D. Pinkel, G. Dunifer, and S. Schultz, immediately preceding paper, Phys. Rev. B 18, 6658 (1978).

[4] A. Wilson, Ph.D., thesis, (Oxford University, 1968), (unpublished), Chap. IV.

[5] A. R. Wilson and D. R. Fredkin, Proceedings of the Eleventh International Conference on Low Temperature Physics, St. Andrews, 1968, (unpublished), Chap. 6.2; *Physics (LT11)* edited by J. Allen, D Finlayson, and D. McCall (Univ. of St. Andrews, Scotland, 1968), Vol. 2, p. 1188.

[6] Andrew Wilson and D. R. Fredkin, Phys. Rev. B 2, 4656 (1970).

[7] G. L. Dunifer, D. Pinkel, and S. Schultz, Phys. Rev. B 10, 3159 (1974).

[8] J. B. Frandsen and R. A. Gordon, Phys. Rev. B 14, 4342 (1976).

[9] Reference 2, Sec. 49.

[10] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge U. P., Cambridge, 1963).

[11] L. D. Landau and E. M. Lifschitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1960), pp. 253–256.