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Variational methods in the hydrodynamic theory of liquid ^{4}He

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This paper presents a derivation of the dissipationless two-fluid equations of motion for liquid ^{4}He based on a variational procedure that is closely related to one formulated originally by Zilsel. The Lagrangian that appears in this treatment has been derived before by applying statistical-mechanical methods to an elementary-excitation description of the flowing liquid; the main steps are recounted here. An important new development is that for uniform flow, the Lagrangian is shown to be a Legendre transform of the internal energy. It is a particular thermodynamic potential for which the primary, independent variables are clearly exhibited. Identification of these variables makes it possible to avoid certain steps in Zilsel's procedure which have been criticized by several workers, while one arrives at the same equations of motion. Furthermore, it is shown, by example, that the Lagrangian density postulated by Zilsel is exactly the same as that assumed by Lhuillier, Francois, and Karatchentzeff (LFK), but that terms have been grouped differently in the two treatments. This should eliminate apprehension expressed by LFK about the reliability of Zilsel's Lagrangian. The results derived here also bring new unity to the work of Zilsel and its extension by Jackson, and certain work of Khalatnikov and its extensions by LFK and by Geurst. Finally, a discussion is given of a proposal made by Lin for modifying Zilsel's variational treatment of liquid ^{4}He.

I. INTRODUCTION

In 1950, Zilsel[1] derived the dissipationless two-fluid equations of motion for liquid ^{4}He by applying Eckart's variational principle[2] to an action functional constructed with a phenomenological Lagrangian density. Zilsel's approach is appealing not only because it is straightforward and elegant, but also because it avoids intricate verbal arguments that constitute an essential part of the leading alternative approach, viz., that of Landau[3] and Khalatnikov,[4] which is based on Galilean invariance. Those considerations, as well as others, recommend the variational method for treating a wider class of systems, e.g., liquid mixtures of ^{3}He and ^{4}He, normal and superfluid liquid ^{3}He, superconductors, and various classical fluids, including plasmas. Dating back almost to the time of Zilsel's original publication, severe criticism has been directed at his derivation. Temperley[5] and Dingle[6] questioned the set of variables which he treated as independent; and they specifically objected to his treatment of x, defined as the ratio of normal-fluid density to total density, as an independent variable. In responding, Zilsel[7] implied that both of them had interpreted his treatment incorrectly, and that there was no valid basis for their complaints. However, the same criticism has been leveled at Zilsel's treatment recently by Lhuillier, Francois, and Karatchentzeff[8] (LFK). These latter authors further contend that Zilsel's treatment is unreliable because of the Lagrangian density which he used. The grounds for their second allegation is that Zilsel wrote the Lagrangian density as a difference between kinetic and internal energy terms, and that this differs from a Lagrangian density which they constructed on the basis of a detailed analysis. Lin[9] has critized Zilsel's variational calculation for another reason. He claims that Zilsel erroneously omitted a certain constraint, which according to Lin accounts for the conservation of the identity of (fluid) particles, and which he believes is essential to a correct treatment. It is relevant to note that Lin[9] has also criticized Herivel's[10] variational treatment of a single component, classical fluid on the same grounds.

In a recent paper,[11] it has been shown that Zilsel's Lagrangian density for liquid ^{4}He can be represented explicitly with formulas derived from a microscopic theory. Furthermore, that theory has been extended[12] to deal with dilute solutions of ^{3}He in liquid ^{4}He. The first of these developments gives considerable support to the idea that Zilsel's approach is basically sound, and it offers a means for settling at least some of the controversial issues. The need for clarifying those issues is increased by the second developement, i.e., extension of the theory to ^{3}He-^{4}He mixtures. In this paper the primary, independent variables of the Lagrangian density for liquid ^{4}He are deduced for the first time, and it is shown that when they are used with Eckart's principle, they lead to precisely the same equations of motion as those derived originally by Zilsel. Viewing the matter in this way, one can see that although there may have been valid grounds for skepticism about Zilsel's procedure, there is no error in his results arising from lack of independence of the variables. Next it is shown that the Lagrangian

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density constructed by LFK is exactly the same as Zilsel's Lagrangian density, the only distinction being that terms are grouped differently in the two cases. As a byproduct of these developments, new unity is exhibited in the work of Zilsel[1] and an extension of it by Jackson,[11] and the work of Khalatnikov[4] and recent extensions of it by LFK[8] and by Geurst.[13] Finally, some comments are made about Lin's suggested additional constraint.

II. MICROSCOPIC THEORY OF THE LAGRANGIAN DENSITY FOR LIQUID ^{4}He

Consider a system of N ^{4}He atoms with volume V in which there is a uniform field of flow of superfluid with velocity $\vec{v}_s$. The method of correlated wave functions has been used before to establish that the diagonal matrix elements of the Hamiltonian within the context of a noninteracting elementary excitation model can be written as

$$E' = E_0 + \tfrac{1}{2}Nmv_s^2 + \vec{v}_s \cdot \sum n_i p_i + \sum n_i \epsilon_i \ . \tag{1}$$

In the approximation where off-diagonal matrix elements are neglected, E' may be regarded as an energy eigenvalue. We note that this formula is consistent with a well-known result deduced by Landau, who used an argument based on Galilean invariance. In Eq. (1), E_0 is the ground-state energy of the liquid at rest, m is the mass of a ^{4}He atom, and n_i is the occupation number for an elementary excitation state having momentum $\vec{p}_i$. The excitations are assumed to obey Bose statistics. It is known from earlier work[11] that the type of treatment being discussed here can also accommodate models in which a fairly wide class of interactions among excitations are present. For simplicity, we shall consider only noninteracting excitations. The momentum eigenvalues for the liquid when the superfluid is flowing are given by the formula

$$\vec{P}' = Nm\vec{v}_s + \sum n_i \vec{p}_i \ . \tag{2}$$

The entropy of a set of independent bosons can be found from elementary counting considerations, and is given by

$$S = k\sum \left[(1+n_i)\ln(1+n_i) - n_i \ln n_i\right] \ . \tag{3}$$

For a fixed value of $\vec{v}_s$, one can find conditions of thermodynamic equilibrium by maximizing the entropy over the manifold of states having fixed values of total energy and total momentum. These constraints may be taken into account by introducing Lagrange multipliers β and $\vec{v}_n$, and requiring that for each value of i,

$$\frac{\partial}{\partial n_i}\left(\frac{1}{k}S - \beta E' + \beta \vec{v}_n \cdot \vec{P}'\right) = 0 \ . \tag{4}$$

This gives

$$\ln\frac{1+n_i}{n_i} = \beta\left(\epsilon_i + \vec{p}_i \cdot (\vec{v}_s - \vec{v}_n)\right) \ , \tag{5}$$

and

$$n_i = \frac{1}{e^{\beta[\epsilon_i + \vec{p}_i \cdot (\vec{v}_s - \vec{v}_n)]} - 1} \ . \tag{6}$$

It can be shown that $\beta = (1/kT)$ where T is the temperature; k is, of course, the Boltzmann constant. The Lagrange multiplier $\vec{v}_n$ may be treated consistently as the normal-fluid velocity in what follows. One can see from Eq. (4) that the procedure used is equivalent to minimizing the function W at constant $\vec{v}_s$, $\vec{v}_n$, and T, where W is defined by

$$W = E' - TS - \vec{v}_n \cdot \vec{P}' \ . \tag{7}$$

By carrying out a simple rearrangement in Eq. (3) and using Eq. (5), one finds that the thermodynamic entropy can be written as

$$TS = kT\sum \ln(1+n_i) + \sum n_i \left[\epsilon_i + \vec{p}_i \cdot (\vec{v}_s - \vec{v}_n)\right] \ . \tag{8}$$

Combining Eqs. (1), (2), and (8) with (7), one finds

$$W = E_0 + \tfrac{1}{2}Nmv_s^2 - Nm\vec{v}_s \cdot \vec{v}_n - kT\sum \ln(1+n_i) \ . \tag{9}$$

A basic postulate of the theory is that $W = W(\vec{v}_s, \vec{v}_n\ V, T, N)$ is a fundamental relation for the system. It has been noted in Ref. 11 that there are two similar, but nonequivalent, theories that are consistent with this postulate. In one, $\vec{v}_s$ is treated as an intensive thermodynamic variable, and it is conjugate to an extensive variable that we shall call $\vec{Q}$. In the second form of the theory, $\vec{v}_s$ is not a thermodynamic variable itself, but it is related to an extensive thermodynamic variable $\vec{P}_0$ by

$$\vec{P}_0 = Nm\vec{v}_s \ . \tag{10}$$

In this case, the intensive variable conjugate to $\vec{P}_0$ will be called $\vec{u}$. We shall treat these two theories separately, starting with the first, involving $\vec{v}_s$ and $\vec{Q}$. In this case, the differential dW must be of the following form:

$$dW = -pdV - SdT - \vec{P}' \cdot d\vec{v}_n + \mu dN - \vec{Q} \cdot d\vec{v}_s \ . \tag{11}$$

A point which should be emphasized here is that W is assumed to be the free-energy function for which the primary variables are those that appear explicitly in Eq. (9) when n_i is taken from Eq. (6). (In connection with that statement, one should note that the ground-state energy E_0 is a function of N and V, and the elementary excitation energy is a function of only the density of particles. That is $\epsilon_i = \epsilon_i(\rho)$, where $\rho = Nm/V$. The volume V also enters Eq. (9) through the density of states associated with the sum over i.) This circumstance makes it possible for one to take partial derivatives easily and to check that the coefficients S and $\vec{P}'$ assumed in Eq. (11) agree with the formulas given in Eqs. (3) and (2), respectively. The other coefficients of differentials may be regarded as defining the pressure p, the chemical potential per particle μ, and the variable $\vec{Q}$. Explicit formulas for these variables are given in Ref. 11. For present purposes, one need only observe that

$$\vec{Q} = M_s(\vec{v}_n - \vec{v}_s) \ , \tag{12}$$

where M_s is the superfluid mass defined by the following relations:

$$M_s = M - M_n \ , \tag{13a}$$

$$M = Nm \ , \tag{13b}$$

$$M_n(\vec{v}_n - \vec{v}_s) = \sum n_i \vec{p}_i \ . \tag{14}$$

In Eq. (14), n_i has the equilibrium value given by Eq. (6). Solving Eq. (7) for E' and using the result in combination with Eq. (11), one finds the following expression for dE':

$$dE' = -p dV + T dS + \vec{v}_n \cdot d\vec{P}' + \mu dN - Q \cdot d\vec{v}_s \ . \tag{15}$$

The primary, independent variables in the internal energy must all be extensive, and one can see from Eq. (15), that E' does not satisfy this requirement. The internal energy, call it U, is related to E' by a Legendre transformation. Specifically,

$$U = E' + \vec{Q} \cdot \vec{v}_s \ , \tag{16}$$

and

$$dU = -p dV + T dS + \vec{v}_n \cdot d\vec{P}' + \mu dN + \vec{v}_s \cdot d\vec{Q} \ . \tag{17}$$

As we shall see shortly, there are good reasons to identify the following (negative) Legendre transform of the internal energy U as the Lagrangian:

$$L = \vec{P}' \cdot \vec{v}_n - U \ . \tag{18}$$

The differential dL can be found from Eqs. (17) and (18):

$$dL = p dV - T dS + \vec{P}' \cdot d\vec{v}_n - \mu dN - \vec{v}_s \cdot d\vec{Q} \ . \tag{19}$$

Having been derived by Legendre transforming the internal energy, the function $L = L(V, S, \vec{v}_n, N, \vec{Q})$ represents a thermodynamic potential whose primary, independent variables are those indicated. If one assumes all of the results derived so far are valid, even when the liquid is only in local thermodynamic equilibrium, they will lead to an explicit representative for Zilsel's Lagrangian density. In preparation for applying the variational principle, it is useful to restate some of the foregoing results in terms of densities. Toward that end, let us introduce the following notation:

$$l = L/V = \rho \bar{l} \ , \tag{20a}$$

$$\bar{l} = L/M \ , \tag{20b}$$

$$\rho = M/V \ , \tag{20c}$$

$$s = S/M \ , \tag{20d}$$

$$\bar{\mu} = \mu/m \ , \tag{20e}$$

$$\vec{j} = \vec{P}'/M \ , \tag{20f}$$

$$\vec{q} = \vec{Q}/M \ . \tag{20g}$$

Then the following differential relations may be inferred from Eqs. (19) and (20):

$$dl = \bar{l} d\rho + \rho d\bar{l} ; \tag{21a}$$

$$d\bar{l} = -(p/\rho^2) d\rho - T ds + \vec{j} \cdot d\vec{v}_n - \vec{v}_s \cdot d\vec{q} \ . \tag{21b}$$

Equations (21a) and (21b) will be used in Sec. III when Eckart's variational principle is applied.

Next let us discuss the justificiation for identifying the function in Eq. (18) as the Lagrangian. First combine Eqs. (16) and (18), and get

$$L = \vec{P}' \cdot \vec{v}_n - \vec{Q} \cdot \vec{v}_s - E' \ . \tag{22}$$

Note that Eqs. (2), (12), (13), and (14) imply that

$$\vec{P}' \cdot \vec{v}_n - \vec{Q} \cdot \vec{v}_s = M_s v_s^2 + M_n v_n^2 \ . \tag{23}$$

Let $\bar{E}$ be defined by the relation

$$E' = \tfrac{1}{2} M_s v_s^2 + \tfrac{1}{2} M_n v_n^2 + \bar{E} \ . \tag{24}$$

Combining Eqs. (22)–(24), one gets

$$L = \tfrac{1}{2} M_s v_s^2 + \tfrac{1}{2} M_n v_n^2 - \bar{E} \ . \tag{25}$$

Properties of $\bar{E}$ have been worked out in Ref. II. We shall review briefly the method and results. From Eqs. (1) and (24), one finds

$$\bar{E} = E_0 - \tfrac{1}{2} M_n (\vec{v}_n - \vec{v}_s)^2 + \sum n_i \epsilon_i \ . \tag{26}$$

To derive the differential of $\bar{E}$, we shall introduce the function

$$Y = \bar{E} - TS - \tfrac{1}{2} M_n (\vec{v}_n - \vec{v}_s)^2 \ . \tag{27}$$

Substituting from Eqs. (8), (14), and (26) into (27), one finds

$$Y = E_0 - kT \sum \ln(1 + n_i) \ . \tag{28}$$

With the aid of Eq. (28) one can readily show that

$$dY = -p\,dV - S\,dT - M_n d\tfrac{1}{2}(\vec{v}_n - \vec{v}_s)^2 + z\,dM \ , \tag{29}$$

where

$$z = \bar{\mu} + \vec{v}_s \cdot \vec{v}_n - \tfrac{1}{2}v_s^2 \ . \tag{30}$$

The defining expressions for p, S, M_n, and $\bar{\mu}$ are those specified earlier in connection with work involving the free energy W. The easiest way to show that Eq. (29) is correct is to obtain explicit formulas for the partial derivatives with the aid of Eq. (28) and to compare them with formulas found with the aid of Eqs. (9) and (11). Solving Eq. (27) for $\bar{E}$, one gets

$$\bar{E} = Y + TS + \tfrac{1}{2}M_n(\vec{v}_n - \vec{v}_s)^2 \ . \tag{31}$$

Then using Eqs. (29) and (31), one finds

$$d\bar{E} = -p\,dV + T\,dS + \tfrac{1}{2}(\vec{v}_n - \vec{v}_s)^2 dM_n + z\,dM \ . \tag{32}$$

From Eq. (26) one can infer that $\bar{E}$ is an extensive variable, and from Eq. (32) one can see that the primary, independent variables of $\bar{E}$ are all extensive. Therefore, $\bar{E}$ is a homogeneous function of first degree in its primary variables, and Eq. (32) can be readily integrated. The result is

$$\bar{E} = -pV + TS + \tfrac{1}{2}(\vec{v}_n - \vec{v}_s)^2 M_n + zM \ . \tag{33}$$

We shall call $\bar{E}$ the intrinsic internal energy. The reason for the qualifier "intrinsic" can be understood by referring to Eq. (24), where one can see that $\bar{E}$ does not include the kinetic energy. The label "internal energy" is consistent with the extensive nature of its primary variables. In retrospect, one can see why the function Y in Eq. (27) was introduced, for it is a Legendre transform of $\bar{E}$ and its primary, independent variables are those that appear explicitly in Eq. (28), so that it is easy to calculate its partial derivatives. A straightforward explanation of the logic involved in my procedure is given in Ref. 11. The variable z is frequently called the chemical potential in the literature, but we have reserved that term for μ earlier in our discussion, so we shall call it the intrinsic chemical potential.

If we let

$$\bar{e} = \bar{E}/M \ , \tag{34}$$

then one can deduce from Eq. (32) that

$$d\bar{e} = T\,ds + (p/\rho^2)d\rho + \tfrac{1}{2}(\vec{v}_n - \vec{v}_s)^2 dx \ , \tag{35}$$

where

$$x = \rho_n/\rho = M_n/M \ , \tag{36}$$

and s is given by Eq. (20d). The function $\bar{e}$ has exactly the same partial derivatives that Zilsel[1] used for the intrinsic internal energy density with which his phenomenological Lagrangian density was constructed. Referring to Eqs. (13), (20), (25), (34), and (36), one can derive the following formula for the Lagrangian density:

$$l = \rho[\tfrac{1}{2}(1 - x)v_s^2 + \tfrac{1}{2}xv_n^2 - \bar{e}] \ . \tag{37}$$

This coincides with the formula given by Zilsel. One can now understand why we have called the function L in Eq. (18) the Lagrangian; for it is the difference between the kinetic and intrinsic internal energy, and from the work of Zilsel and our work here, it is known that it leads to two-fluid equations that are compatible, for the most part, with those derived by other methods.[3,4] To be accurate, we should call the first two terms in Eq. (37) the apparent kinetic energy, because if $x > 1$, the first term will be negative. That point has been emphasized in Ref. 11, but having recognized this condition, we shall drop the modifier apparent in the discussion here.

Next let us develop the parallel formulas for the theory in which $\vec{P}_0$, in the form given by Eq. (10), is a thermodynamic variable, and its conjugate is $\vec{u}$. In this case, a fundamental relation is represented by $W = W(P_0, v_n, V, T, N)$ and a transcription of Eq. (9) gives

$$W = E_0 + \frac{P_0^2}{2Nm} - \vec{P}_0 \cdot \vec{v}_n - kT \sum \ln(1 + n_i) \ . \tag{38}$$

The differential of W is

$$dW = -p\,dV - S\,dT - \vec{P}' \circ d\vec{v}_n + \mu\,dN + \vec{u} \cdot d\vec{P}_0 \ . \tag{39}$$

The variables p, S, and $\vec{P}'$ are given by the same formulas and have the same meanings as in the version of the theory discussed earlier, but the chemical potential μ is different. The formula for μ is given in Ref. 11; it can be computed using Eqs. (38) and (39). Furthermore, the variable $\vec{u}$ can be easily computed by using Eqs. (38) and (39); it is given by the formula

$$\vec{u} = (\rho_s/\rho)(\vec{v}_s - \vec{v}_n) \ . \tag{40}$$

In terms of $\vec{P}_0$, E' from Eq. (1) takes the form

$$E' = E_0 + \frac{P_0^2}{2Nm} + \frac{1}{Nm}\vec{P}_0 \cdot \sum n_i \vec{p}_i + \sum n_i \epsilon_i \ . \tag{41}$$

The function W in Eq. (38) is related to E' by Eq. (7). The differential dE' found with the aid of Eqs. (39) and (7) is

$$dE' = -p\,dV + T\,dS + \vec{v}_n \cdot d\vec{P}' + \mu\,dN + \vec{u} \cdot d\vec{P}_0 \ . \tag{42}$$

$E' = E'(\vec{P}_0, \vec{P}', V, S, N)$ represents a fundamental relation in which the primary, independent variables are all extensive, and so E' is the internal energy. As we shall soon see, there are good reasons to identify the following (negative) double

Legendre transform of the internal energh E' as the Lagrangian:

$$L=\vec{u}\cdot\vec{P}_0+\vec{v}_n\cdot\vec{P}'-E' \ . \tag{43}$$

The differential of L that follows from Eqs. (42) and (43) is

$$dL=pdV-TdS+\vec{P}'\cdot d\vec{v}_n-\mu dN+\vec{P}_0\cdot d\vec{u} \ . \tag{44}$$

Having been derived by Legendre transforming the internal energy, the function $L=L(V,S,\vec{v}_n,N,\vec{u})$ represents a thermodynamic potential whose primary independent variables are those indicated. Once again we shall construct a Lagrangian density and make the assumption that the formulas are valid even when the liquid is only in local thermodynamic equilibrium. Equations (20a)–(20f) are still applicable, but (20g) is replaced by a result deduced from Eq. (10), viz., $\vec{v}_s=\vec{P}_0/M$. Equation (21a) still holds, but instead of (21b), we have

$$d\bar{l}=-(p/\rho^2)d\rho-Tds+\vec{j}\cdot d\vec{v}_n+\vec{v}_s\cdot d\vec{u} \ . \tag{45}$$

Equations (21a) and (45) will be used in Sec. III when Eckart's variational principle is applied.

Next let us discuss the justification for identifying the function L in Eqs. (43) as the Lagrangian. Equations (2), (10), (13), and (40) imply

$$\vec{u}\cdot\vec{P}_0+\vec{v}_n\cdot\vec{P}'=M_s v_s^2+M_n v_n^2 \ . \tag{46a}$$

Equations (24)–(37), with the exception of Eq. (30), are all still applicable. In Ref. 11 it is shown that z and $\bar{\mu}$ are now related by

$$z=\bar{\mu}+\tfrac{1}{2}v_s^2+(\rho_n/\rho)(\vec{v}_n\cdot\vec{v}_s-v_s^2) \ . \tag{46b}$$

With these results at hand, the argument for calling L the Lagrangian can be taken over verbatim from the discussion of the first version of the theory.

Although the two models discussed so far are described by the same Lagrangian, and in fact obey the same hydrodynamic equations of motion, they are not completely equivalent. The stability properties of the system as judged by ordinary thermodynamic criteria are different for these two models. This matter is discussed in Ref. 11, where it is pointed out that which of the models is more suitable for describing liquid ^{4}He has not yet been determined.

Out next task is to show that the Lagrangian density used by LFK is the same as that used by Zilsel. Lhuillier and collaborators have chosen a set of independent variables that is different from those in the two previous models; so we must retreat to a convenient starting point for taking this into account. Let us introduce a variable $\vec{G}$ through the equation

$$\vec{G}=\sum n_i\vec{p}_i \ . \tag{47}$$

The Hamiltonian operator in the models that we are considering and the operator $\vec{G}$ are both diagonal in the occupation number representation for the excitations. The energy eigenvalues E' are still given by Eq. (1). If we so choose, a constraint can be imposed on $\vec{G}$ instead of $\vec{P}'$ when the entropy is maximized to determine the thermodynamic properties of the system. For S given by Eq. (3), let us again introduce two Lagrange multipliers β and $\vec{v}_n$ and then require

$$\frac{\partial}{\partial n_i}\left(\frac{1}{k}S-\beta E'+\beta\vec{v}_n\cdot\vec{G}\right)=0 \ . \tag{48}$$

It turns out that $\vec{v}_n$ can still be treated consistently as the normal-fluid velocity. From Eq. (48) one finds that Eqs. (5), (6), and (8) still hold. Maximizing S subject to the constraints indicated is equivalent to minimizing X at constant $\vec{v}_s$, $\vec{v}_n$, and T, where X is given by

$$X=E'-TS-\vec{v}_n\cdot\vec{G} \ . \tag{49}$$

Substituting from Eqs. (1), (8), and (47) into Eq. (49), one finds

$$X=E_0+\tfrac{1}{2}Nmv_s^2-kT\sum\ln(1+n_i) \ . \tag{50}$$

We shall postulate that $X=X(\vec{v}_s,\vec{v}_n,V,T,N)$ is a fundamental relation for the system. There are, in fact, two similar, but nonequivalent, theories that are consistent with this postulate. They are closely connected with those discussed earlier, where W is the free energy. We shall treat only one of them here, the one where $\vec{v}_s$ itself, as distinguished from $\vec{P}_0$ of Eq. (10), is a thermodynamic variable. The differential of X is

$$dX=-pdV-SdT-\vec{G}\cdot d\vec{v}_n+\mu dN+\vec{P}'\cdot d\vec{v}_s \ . \tag{51}$$

With the aid of Eq. (50) one can compute explicit formulas for the partial derivatives. In that way, one can check to see that the coefficients $\vec{P}'$, S, and $\vec{G}$ agree with Eqs. (2), (3), and (47). Equation (51) may be regarded as defining the pressure p and the chemical potential μ. The explicit formula for p is the same as that derived using the free energy W of Eq. (9). The formula for μ is not the same as that computed with W; now μ is given by

$$\mu=m\left(e_0+\rho\frac{\partial e_0}{\partial\rho}+\tfrac{1}{2}v_s^2+\frac{1}{V}\sum n_i\frac{\partial\epsilon_i}{\partial\rho}\right) , \tag{52}$$

where e_0 is defined by

$$E_0=Me_0 \ . \tag{53}$$

Solving Eq. (49) for E' and using that result in combination with Eq. (51), one gets

$$dE' = -pdV + TdS + \vec{v}_n \cdot d\vec{G} + \mu dN + \vec{P}' \cdot d\vec{v}_s \ . \quad (54)$$

The internal energy U' is given by

$$U' = E' - \vec{v}_s \cdot \vec{P}'; \quad (55)$$

and so

$$dU' = -pdV + TdS + v_n \cdot d\vec{G} + \mu dN - \vec{v}_s \cdot d\vec{P}' \ . \quad (56)$$

The Lagrangian is

$$L = \vec{G} \cdot \vec{v}_n - U' \quad (57a)$$

$$= \vec{G} \cdot \vec{v}_n + \vec{P}' \cdot \vec{v}_s - E' \ . \quad (57b)$$

With the aid of Eqs. (2), (13), and (47) one can show that

$$\vec{G} \cdot \vec{v}_n + \vec{P}' \cdot \vec{v}_s = M_s v_s^2 + M_n v_n^2 \ . \quad (58)$$

E' is the same as in the earlier version of the theory, and so Eq. (24) still holds. Combining Eqs. (57), (58), and (24), one again arrives at the Lagrangian given by Eq. (25). However, now the differential of L in terms of its primary variables is

$$dL = pdV - TdS + \vec{G} \cdot d\vec{v}_n - \mu dN + \vec{v}_s \cdot d\vec{P}' \ . \quad (59)$$

Using the notation of Eqs. (20) and (36), we find that Eq. (21a) still holds for the Lagrangian density, but instead of Eq. (21b), we now have

$$d\bar{l} = -\frac{p}{\rho^2} d\rho - Tds + \vec{j}_0 \cdot d\vec{v}_n + \vec{v}_s \cdot d\vec{j} \ , \quad (60)$$

where $\vec{j}_0$ satisfies

$$\vec{j}_0 = x(\vec{v}_n - \vec{v}_s) \ . \quad (61)$$

The intrinsic internal energy used by Lhuillier and collaborators[8] is not the same as $\bar{E}$ defined by Eq. (24). Rather, they have used $\bar{E}'$ defined by

$$E' = \tfrac{1}{2} M v_s^2 + M_n(\vec{v}_n - \vec{v}_s) \cdot \vec{v}_s + \bar{E}' \ , \quad (62)$$

as one can see in Eq. (7) of their paper. Taking into account Eq. (14) and referring to Eq. (1), one can infer that

$$\bar{E}' = E_0 + \sum n_i \epsilon_i \ . \quad (63)$$

Comparing this with Eq. (26), we see that

$$\bar{E}' = \bar{E} + \tfrac{1}{2} M_n (\vec{v}_n - \vec{v}_s)^2 \ . \quad (64)$$

To calculate the differential of $\bar{E}'$, we shall introduce the function $\bar{Y}$ defined by

$$\bar{Y} = \bar{E}' - TS - \vec{G} \cdot \vec{w} \ , \quad (65)$$

where

$$\vec{w} \equiv \vec{v}_n - \vec{v}_s \ . \quad (66)$$

Combining Eqs. (3), (47), (63), and (65), one gets

$$\bar{Y} = E_0 - kT \sum \ln(1 + n_i) \ . \quad (67)$$

With the help of Eq. (67), one can easily verify the following relation:

$$d\bar{Y} = -pdV - SdT + zdM - \vec{G} \cdot d\vec{w} \ , \quad (68a)$$

where the explicit formulas for p, S, and $\vec{G}$ are the same as those derived using the free energy X. Also the formulas for p and S are the same as those based on the free energy $W(\vec{v}_s, \vec{v}_n, V, T, N)$. The formula for z is the same as that based on the intrinsic free energy Y in Eqs. (28) and (29). The intrinsic chemical potential z of Eq. (68a) is related to the chemical potential μ of Eqs. (51) and (52) by

$$z = \mu/m - \tfrac{1}{2} v_s^2 \ . \quad (68b)$$

Solving Eq. (65) for $\bar{E}'$ and using the result in combination with Eq. (68a), one finds

$$d\bar{E}' = -pdV + TdS + zdM + \vec{w} \cdot d\vec{G} \ . \quad (69)$$

In Eq. (8) of Ref. 8, the differential of the intrinsic internal energy is written for constant volume. Except for trivial changes in notation, one can see that the differential there is the same as that in Eq. (69) when V is held constant. Therefore, the function $\bar{E}'$ that is defined by Eq. (62) and which satisfies Eqs. (63) and (64) in my model also furnishes a representative for the intrinsic internal energy postulated by LFK. Furthermore, combining Eqs. (57b), (58), and (62), one finds after an elementary calculation that the Lagrangian may be written as follows:

$$L = \tfrac{1}{2} M v_s^2 + M_n(\vec{v}_n - \vec{v}_s) \cdot v_n - \bar{E}' \ .$$

When this is expressed as a Lagrangian density referred to unit volume, it gives the formula contained in Eq. (35) of the paper by LFK.[8] We have noted before, immediately following Eq. (58), that this Lagrangian may be expressed as in Eq. (25), which produces a Lagrangian density having the properties assumed in Zilsel's theory. The two formulas for the Lagrangian look different because a term $\frac{1}{2} M_n (\vec{v}_n - \vec{v}_s)^2$ is added to the intrinsic internal energy used by Zilsel to obtain the corresponding function used by LFK. This term is just compensated in the Lagrangian by adding a term $\frac{1}{2} M_n (\vec{v}_n - \vec{v}_s)^2$ to Zilsel's kinetic energy to get a related quantity in the theory of LFK. Hence the formulas for the Lagrangian differ only in the way that terms are grouped. The differentials of the Lagrangian density appear to differ further because Zilsel's was constructed by first working with the Lagrangian per unit mass; whereas that of LFK was constructed by using the Lagrangian per unit volume directly. It is straightforward to show that the differentials for the Lagrangian per unit volume are equivalent in the two cases.

Formulas for the energy in the presence of flow and for the intrinsic internal energy assumed by LFK are the same as those used by Khalatnikov[4] and by Geurst[13] except for trivial changes in notation. Therefore the analysis that I have given exhibits new unity between their work and that of Zilsel[1] and its extension by Jackson.[11] In part, this unification has been achieved earlier in certain work of Clark[14] and of Uhlenbeck.[15] Using only macroscopic theory and thermodynamic considerations, both of them were led to consider two intrinsic internal energy functions that are related as in Eq. (64), and they wrote down formulas for their differentials that are compatible with Eqs. (32) and (69).

III. APPLICATION OF ECKART'S VARIATIONAL PRINCIPLE

In Eckart's variational principle, one associates a Lagrangian density and the variables appearing in it with fixed spatial points, and carries out the variations as in a field theory but subject to constraints. If a variable is a velocity in some particular case, it is still treated as a field amplitude. This is to be contrasted with many calculations in hydrodynamics where velocity is treated directly as the time derivative of a position vector and the position vectors are regarded as basic coordinates for the system. In the case of superfluid ^{4}He when dissipation is neglected, Zilsel pointed out that the relevant constraints are embodied in the continuity equation and the equation expressing conservation of entropy. Explicitly, the constraint equations are

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho_s\vec{v}_s+\rho_n\vec{v}_n)=0 \tag{70a}$$

and

$$\frac{\partial(\rho s)}{\partial t}+\vec{\nabla}\cdot(\rho s\vec{v}_n)=0\ . \tag{70b}$$

The constraints can be imposed with the aid of Lagrange multipliers, which we shall call α and β. Following Zilsel, we shall rewrite Eq. (70a) as

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\{\rho[(1-x)\vec{v}_s+x\vec{v}_n]\}=0\ . \tag{71}$$

Eckart's variational principle, in the form assumed by Zilsel, states that

$$0=\delta\int_{t_0}^{t_1}dt\int_V d^3r\left[l-\alpha\left(\frac{\partial\rho}{\partial t}+\vec{\nabla}\circ(\rho_s\vec{v}_s+\rho_n\vec{v}_n)\right)-\beta\left(\frac{\partial(\rho s)}{\partial t}+\vec{\nabla}\cdot(\rho s\vec{v}_n)\right)\right]. \tag{72}$$

The volume of integration is fixed in space, and the variations in all quantities are specified to vanish on the boundary of the space and time regions appearing in the integral. With the continuity equation written as in Eq. (71), and with l given by Eq. (37), Zilsel varied ρ, s, x, $\vec{v}_n$, and $\vec{v}_s$, treating them as independent variables. From thermodynamic considerations, Zilsel deduced the first two terms in Eq. (35) for the differential $d\bar{e}$; he was able to derive $(\partial\bar{e}/\partial x)_{\rho,s}=\frac{1}{2}(\vec{v}_n-\vec{v}_x)^2$ from the varitaional calculation. We will not follow his steps exactly, but rather we will use Eq. (35) in making the variations since it is available from the microscopic theory. This does not generate any important deviation from Zilsel's method but it will give some insight into the reason why his procedure is successful. The Euler variational equations of Eq. (72) are

$$\delta\rho:\ \tfrac{1}{2}(1-x)v_s^2+\tfrac{1}{2}v_n^2-\bar{e}-\frac{1}{\rho}\,p+\frac{\partial\alpha}{\partial t}+\left((1-x)\vec{v}_s+x\vec{v}_n\right)\cdot\vec{\nabla}\alpha+s\left(\frac{\partial\beta}{\partial t}+\vec{v}_n\cdot\vec{\nabla}\beta\right)=0\ , \tag{73a}$$

$$\delta s:\ -T+\frac{\partial\beta}{\partial t}+v_n\cdot\nabla\beta=0\ , \tag{73b}$$

$$\delta x:\ \tfrac{1}{2}(v_n^2-v_s^2)-\tfrac{1}{2}(\vec{v}_n-\vec{v}_s)^2+(\vec{v}_n-\vec{v}_s)\cdot\vec{\nabla}\alpha=0\ , \tag{73c}$$

$$\delta\vec{v}_n:\ x(\vec{v}_n+\vec{\nabla}\alpha)+s\vec{\nabla}\beta=0\ , \tag{73d}$$

$$\delta\vec{v}_s:\ \vec{v}_s+\vec{\nabla}\alpha=0\ . \tag{73e}$$

Equation (73e) immediately yields the result that $\vec{\nabla}\times\vec{v}_s=0$, a condition that was postulated by Landau. The Lagrange multipliers can be eliminated from Eqs. (73a)–(73e). The main steps in this process have been described by Zilsel.[1] The equations of motion that emerge are

$$\frac{D_s\vec{v}_s}{Dt}=-\frac{1}{\rho}\vec{\nabla}p+s\vec{\nabla}T+\tfrac{1}{2}x\vec{\nabla}|\vec{v}_n-\vec{v}_s|^2\ , \tag{74a}$$

$$=-\vec{\nabla}z\ , \tag{74b}$$

and

$$\frac{D_n\vec{v}_n}{Dt}=-\frac{1}{\rho}\vec{\nabla}p-\frac{1-x}{x}s\vec{\nabla}T-\frac{1-x}{2}\vec{\nabla}|\vec{v}_n-\vec{v}_s|^2-(\vec{v}_n-\vec{v}_s)\frac{\Gamma}{\rho x}\ . \tag{75}$$

In these equations the following notation has been used for convective derivatives:

$$\frac{D_s}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}_s \cdot \vec{\nabla} \ , \tag{76a}$$

and

$$\frac{D_n}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}_n \cdot \vec{\nabla} \ . \tag{76b}$$

The result in Eq. (74b) is derived in Ref. 11 with the aid of some relations written down earlier by Zilsel.[1] In Eq. (75), Γ represents a source density of the normal fluid (rate of production of normal-fluid mass per cm^3). It equals a sink density of the superfluid. For details, one can see Ref. 1.

It was noted in Sec. I that one complaint which has been repeatedly registered[5,6,8] against Zilsel's procedure is that the variable x is not independent of the others treated as independent by Zilsel. Temperley and Dingle argued, in effect, that because Zilsel arrived at the form for $d\bar{e}$ given by Eq. (35), and in particular because $(\partial\bar{e}/\partial x) = \frac{1}{2}(\vec{v}_n - \vec{v}_s)^2$, then Zilsel should have additional terms in the Euler equations found by varying $\vec{v}_s$ and $\vec{v}_n$. In some sense, Eq. (35) seems to support Zilsel's view, because there one can see that s, ρ, and x are the primary, independent variables of $\bar{e}$. However, the results in Sec. II indicate that Zilsel's set of variables contains one more element than occurs in a set of primary independent variables of the Lagrangian density. This statement can be understood clearly by referring to Eqs. (21a), (21b), (45), and (60). Therefore in the light of information that was deduced from the microscopic theory, there seem to be valid grounds for criticizing Zilsel's procedure. However, anticipating results that we shall derive shortly, I should emphasize that Zilsel's procedure leads to the same equations of motion as those derived by varying only the primary, independent variables of the Lagrangian density for each of the three theories that are treated in this paper.

In a certain sense, the success of Zilsel's method seems to be partly fortuitous. The basis for this statement can be understood by examining Eq. (73c). When the value of $\vec{\nabla}\alpha$ inferred from Eq. (73e) is substituted into (73c), the latter is reduced to a trivial identity and consequently it is not essential to the derivation of the equations of motion. It appears as an identity only because the differential $d\bar{e}$ is known completely from microscopic calculations, which were not available to Zilsel. In Zilsel's own treatment, the second term in Eq. (73c) was written as $\partial\bar{e}/\partial x$, and instead of an identity, he found an equation that he used to determine $\partial\bar{e}/\partial x$. With $d\bar{e}$ completely known as in the present treatment one might try to proceed by regarding x as a dependent variable in making the variations. However, then there would be extra terms in some of the other Euler equations that would complicate the calculation substantially; so that scheme does not provde a simple explanation for the success of Zilsel's method. Because Zilsel's procedure does give correct results at least for several different models treated in this paper, we should not overlook the possibility that there is some deep principle responsible for its success that has not been recognized yet.

Now let us apply the variational principle indicated in Eq. (72) to each of the three models considered earlier. First, we will treat the theory in which $\vec{v}_s$ is a thermodynamic variable, $\vec{Q}$ is the variable conjugate to it, and W is the free energy. The differential of the Lagrangian density is given by Eqs. (21a) and (21b). With the aid of Eqs. (12) and (20g). we shall rewrite the continuity equation, Eq. (70a), as

$$\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot [\rho(\vec{v}_n - \vec{q})] = 0 \ , \tag{77}$$

and use this form of it in applying the variational principle. There is no need to rewrite the entropy conservation equation, Eq. (70b). The form given in that equation is suitable for direct use in the variational procedure for all three of the models we will consider. The independent variables are ρ, s, $\vec{v}_n$, and $\vec{q}$. The Euler equations of Eq. (72) are

$$\delta\rho:\ \bar{l} - \frac{p}{\rho} + \frac{\partial\alpha}{\partial t} + (\vec{v}_n - \vec{q}) \cdot \vec{\nabla}\alpha + s\left(\frac{\partial\beta}{\partial t} + \vec{v}_n \cdot \vec{\nabla}\beta\right) = 0 \ , \tag{78a}$$

$$\delta s:\ -T + \frac{\partial\beta}{\partial t} + \vec{v}_n \cdot \vec{\nabla}\beta = 0 \ , \tag{78b}$$

$$\delta\vec{v}_n:\ \vec{j} + \vec{\nabla}\alpha + s\vec{\nabla}\beta = 0 \ , \tag{78c}$$

$$\delta\vec{q}:\ \vec{v}_s + \vec{\nabla}\alpha = 0 \ . \tag{78d}$$

Equation (78a) can be readily simplified if one first carries out two preliminary steps. First, combine Eqs. (20a) and (37) to find a formula for $\bar{l}$. Next, combine Eqs. (12), (13), (20g), and (36), and get

$$\vec{q} = (1 - x)(\vec{v}_n - \vec{v}_s) \ . \tag{79}$$

Substituting these results into Eq. (78a), one finds that it reduces to Eq. (73a) that was derived by using Zilsel's procedure. Furthermore, it is clear that Eqs. (78b) and (78d) are the same as Eqs. (73b) and (73e). If one further notes that the relation

$$\vec{j} = (1 - x)\vec{v}_s + \vec{v}_n \tag{80}$$

follows from Eqs. (13), (14), (20f), and (36), then it is easy to see that Eq. (78c) is the same as Eq. (73d) when (73e) is taken into account. Hence the four relations in Eqs. (78a)–(78d) are completely equivalent to those used in Zilsel's procedure to derive the equations of motion. The extra equation (73c) has no counterpart here.

Next, consider the model in which $\vec{P}_0$ and $\vec{u}$ are conjugate variables and W is the free energy. In this case, the differential of the Lagrangian density to be used in Eq. (72) is given by Eqs. (21a) and (45). The continuity equation can be written in terms of the primary, independent variables of that Lagrangian density, as follows:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot [\rho(\vec{u} + \vec{v}_n)] = 0 \ . \tag{81}$$

This equation can be inferred from Eqs. (13), (40), and (70a). The Euler equations of Eq. (72) found by varying the primary independent variables of the Lagrangian density are

$$\delta\rho\!: \ \bar{l} - \frac{p}{\rho} + \frac{\partial \alpha}{\partial t} + (\vec{u} + \vec{v}_n) \cdot \vec{\nabla}\alpha + s\left(\frac{\partial \beta}{\partial t} + \vec{v}_n \cdot \vec{\nabla}\beta\right) = 0 \ , \tag{82a}$$

$$\delta s\!: \ -T + \frac{\partial \beta}{\partial t} + \vec{v}_n \cdot \vec{\nabla}\beta = 0 \ , \tag{82b}$$

$$\delta \vec{v}_n\!: \ \vec{j} + \vec{\nabla}\alpha + s\vec{\nabla}\beta = 0 \ , \tag{82c}$$

$$\delta \vec{u}\!: \ \vec{v}_s + \vec{\nabla}\alpha = 0 \ . \tag{82d}$$

With the aid of Eqs. (20a), (37), and (40), one can rewrite Eq. (82a) in a form that is identical with Eq. (73a) which was found using Zilsel's procedure. It is easy to see that Eqs. (82b)–(82d) are equivalent to Eqs. (73b), (73d), and (73e). As noted before, this set of four equations are sufficient to determine the equations of motion, i.e., Eqs. (74a) and (75).

Now consider the third model, in which $\vec{v}_s$ and $\vec{Q}$ are conjugate variables, and X is the free energy. The differential of the Lagrangian density to be used in conjunction with Eq. (72) is given by Eqs. (21a) and (60). The continuity equation can be written in terms of the primary independent variables of $\bar{l}$ as follows:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{j}) = 0 \ . \tag{83}$$

The Euler variational equations of Eq. (72) are

$$\delta\rho\!: \ \bar{l} - \frac{p}{\rho} + \frac{\partial \alpha}{\partial t} + \vec{j} \cdot \vec{\nabla}\alpha + s\left(\frac{\partial \beta}{\partial t} + \vec{v}_n \cdot \vec{\nabla}\beta\right) = 0 \ , \tag{84a}$$

$$\delta s\!: \ -T + \frac{\partial \beta}{\partial t} + \vec{v}_n \cdot \nabla\beta = 0 \ , \tag{84b}$$

$$\delta \vec{v}_n\!: \ \vec{j}_0 + s\vec{\nabla}\beta = 0 \ , \tag{84c}$$

$$\delta \vec{j}\!: \ \vec{v}_s + \vec{\nabla}\alpha = 0 \ . \tag{84d}$$

Using a process that is by now familiar, one can show that this set of equations is equivalent to the set (73a), (73b), (73d), (73e) that are sufficient to determine the equations of motion found by Zilsel. Thus we see that working only with the primary independent variables in the Lagrangian density for each of the three theories, one can derive the same equations of motion as those found by Zilsel.

It is remarkable that Zilsel's procedure automatically yields the condition $\vec{\nabla} \times \vec{v}_s = 0$, a result that was mentioned eariler. Landau found it necessary to introduce this condition as an independent postulate. According to Lin,[9] the condition $\vec{\nabla} \times \vec{v}_s = 0$ was incorporated in Zilsel's theory unintentionally, and Lin has suggested that this result is actually due to a defect in Zilsel's variational method. The following review of certain aspects of the theory of an ordinary fluid will elucidate the basis for Lin's contention.

Herivel[10] was the first to give a satisfactory treatment of a single-component ideal classical fluid within the context of a general variational theory. Applying Hamilton's principle to such a system, he found the familiar equation of motion,

$$\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\,\tfrac{1}{2}v^2 - \vec{v} \times (\vec{\nabla} \times \vec{v}) = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}V \ . \tag{85}$$

In Eq. (85), $\vec{v}$ is the local velocity of the fluid, ρ is the mass density, p is the pressure, and V is the potential energy of external forces per unit mass. Herivel obtained this same equation of motion by two different variational procedures. In one, the Lagrangian approach, the independent variables included the position vectors of the fluid particles. In the other, the Eulerian approach, instead of the position vectors, the independent variables included the velocities of the fluid particles. The second of these procedures is the one used in Eckart's variational principle, the principle that was adapted by Zilsel to treat liquid ^{4}He. In this second procedure, there is an intermediate step in the derivation in which one finds that the velocity $\vec{v}$ can be written as

$$\vec{v} + \vec{\nabla}\alpha = \beta\vec{\nabla}s \ . \tag{86}$$

Here s is the entropy per unit mass, and α and β are functions of space and time that enter the calculation as Lagrange multipliers. From Eq. (86) one can see that if the entropy per unit mass were constant, i.e., if the liquid were isentropic, then the solutions to Eq. (85) would be limited to irrotational flow. Lin considers this to be a difficulty with the theory because this restriction does not arise in the first version of the theory where the particle coordinates are varied, and because it does not arise in the equations that are derived from mechanical and thermodynamic considerations without using a variational procedure.

Lin noticed that this restriction could be avoided in the Eulerian version of Herivel's variational procedure if in addition to the constraint equations for mass and entropy imposed by Herivel, one introduced a third (vector) constraint equation of the form

$$\frac{\partial \vec{X}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{X} = 0 . \qquad (87)$$

$\vec{X}$ is a position vector of a fluid particle at one definite time. The components of $\vec{X}(\vec{x}, t)$ are the Lagrange coordinates of a particle in the Eulerian description of the fluid, i.e., the description in which the coordinates $\vec{x}$ refer to fixed points in space. In describing Eq. (87), Uhlenbeck[15] states: "This condition says that as you move with a particle, its position is unchanged." Lin refers to this condition as the conservation of identity of particles.

Lin's belief that the equations derived from Herivel's second method are less fundamental than those in the other two methods mentioned earlier is clearly at the root of his proposal for introducing Eq. (87) as a constraint. Lin's view has also been taken in Serrin's[16] discussion of the theory of a classical one-component fluid, and conclusions reached by Seliger and Whitham[17] are consistent with this view. In the words of these latter authors: "Lin's device still remains somewhat mysterious from a mathematical viewpoint, but the necessity for it seems to be firmly established as we proceed." They also state: "In some cases, such as flow behind a curved shock, vorticity is associated with entropy gradients. However, in simple incompressible flow, for example, it should be possible to have rotational flows." In the context of their discussion, it is clear that the last sentence refers to isentropic conditions. Uhlenbeck[15] also appears to concur with Lin on this matter. The preponderance of opinion seems to support Lin's contention that Herivel's second method, the Eulerian approach to the variational problem, is defective unless it is supplemented by an additional constraint.

Turning to the theory of superfluid ^{4}He, Lin suggested that the constraint in Eq. (87) should also be incorporated in Zilsel's variational procedure by taking $\vec{v}$ to be defined by

$$\rho\vec{v} = \rho_s\vec{v}_s + \rho_n\vec{v}_n . \qquad (88)$$

He states that if one had introduced this constraint in Zilsel's derivation, the equations obtained would have been identical with equations given in Lin's own treatment that do not contain the restriction $\vec{\nabla} \times \vec{v}_s = 0$. For a certain class of solutions of Lin's equations, the condition $\vec{\nabla} \times \vec{v}_s = 0$ does hold, and for that class, the equations of motion reduce to those found using Zilsel's original procedure, i.e., Eqs. (74a) and (75), except that the term involving Γ in Eq. (75) is missing. Lin agrees that in some situations the superfluid is not rotating, i.e., $\vec{\nabla} \times \vec{v}_s = 0$; but according to him, it is not that it cannot rotate, but rather that it does not rotate because of special conditions that occur in certain experimental situations. It is noteworthy that Lhuillier, Francois, and Karatchentzeff incorporated Lin's constraint and a generalization of it in parts of their theory.

Applying Lin's constraint in the theory of liquid ^{4}He seems to be much more speculative than applying it in the theory of a single-component, classical fluid. If one adopts a literal interpretation of the two-fluid model for the purpose of analysis, then it is clear from Eqs. (87) and (88) that $\vec{X}$ does not fix the location of any fluid particle, and it does not seem to have any clearly identifiable physical significance. The consequences of introducing two constraint equations, one for the superfluid and one for the normal fluid, have been discussed by Clark.[14] He indicates that they are unsatisfactory for several reasons. Perhaps the most obvious of these is that difficulties are encountered in situations where normal fluid is converted into superfluid, or vice versa. Nevertheless, one can introduce a constraint formally, in the manner proposed by Lin, and study the results while relying on experiment to test the validity of the method.

Such a test could focus on the term involving Γ that appears in the equation of motion for the normal fluid, Eq. (75), derived by Zilsel's original procedure. According to Lin, that term is absent when his constraint is imposed. (Clark,[14] incidentally, derived both Lin's and Zilsel's equations from nonvariational procedures based on assumptions that were slightly different in the two cases. Clark also discusses problems of uniqueness in some of Lin's derivations.) If experiment should indicate that the term involving Γ is present, then one could conclude that Lin's constraint should not be used with Zilsel's procedure. Finally, it is noteworthy that after discussing Lin's constraint, Uhlenbeck[15] derived the two-fluid equations for liquid ^{4}He by a variational method without using that constraint.

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