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Intrinsic angular momentum in ^{3}He-A

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It is argued that the intrinsic angular momentum of ^{3}He-A should be taken as zero within the BCS model.

It is now fairly well established that, in superfluid ^{3}He, a BCS pairing of helium atoms occurs. In the A phase of superfluid ^{3}He, only atoms with parallel nuclear spin form pairs, and the pair wave function is p wave and proportional to the spherical harmonics Y_1^1 if the anisotropy axis $\hat{l}$ is along the z direction. Therefore, the pairs form with a relative angular momentum $\hbar$. This raises an important question: Does this relative angular momentum imply that the whole system itself has an angular momentum?

Here we want to consider only the angular momentum arising from this p-wave symmetry of the pair wave function, and we are therefore only interested in a (would-be) homogeneous system with a constant order parameter all over the sample. We call "intrinsic" the corresponding angular momentum. In addition to its basic interest, this intrinsic angular momentum is likely to play a very important role in orbital dynamics.[1-3]

Naturally, if we consider a real sample of ^{3}He, the order parameter will not be constant, in order to satisfy the boundary conditions.[4] The inhomogeneities will give rise to supercurrents resulting in an angular momentum which we call "extrinsic." This has been considered for example by Mermin and Ho.[5] The total angular momentum of a real sample will be the sum of the intrinsic plus the extrinsic angular momentum.

The extrinsic angular momentum arises from macroscopic supercurrents. It is not easy to figure out what are the currents giving rise to the intrinsic angular momentum. They can be thought of as microscopic currents on the scale of a pair radius, resulting from the internal structure of the pair. This is very similar to the case of diatomic molecules in a p-wave state, where the rotation of the atoms around each other in a single molecule gives rise to local currents (averaging to zero), and to nonzero angular momentum.

A classical difficulty is that, for an homogeneous sample, the currents average to zero everywhere except at the surface, and therefore the angular momentum comes ultimately from the surface currents. This problem is common to ^{3}He-A or to an assembly of diatomic molecules. In the last case, the problem is solved by finding first the angular momentum due to a single molecule, which is done by looking at the symmetry of the molecular wave function, and then adding up the angular momentum of all the molecules. For ^{3}He-A, this method does not work as such because the pairs are highly correlated and one cannot consider an isolated pair. However, in order to find the intrinsic angular momentum one can still look at the symmetry of the total ^{3}He-A wave function, and this is the method that we use here within the BCS model.

Already in the literature there are various answers for the intrinsic angular momentum L_0. In their early work, Anderson and Morel[6] gave $L_0 \sim N\hbar T_c/E_F$, where N is the number of particles in the system. Recently, Cross[3,7] and Volovik[8] have proposed $L_0 \sim N\hbar(T_c/E_F)^2$, while according to Ishikawa,[9] one should take $L_0 = \frac{1}{2}N\hbar$ at $T=0$. Here we want to show that the result is actually $L_0 = 0$ within the BCS model.

A major difficulty in the angular momentum problem is the choice of the BCS wave function. For many purposes, the physical answers are the same no matter what form of the BCS wave function is chosen. But this is not so for the angular momentum. For example, the difference between Ishikawa's result and ours can be considered as due to a different choice of the BCS wave function. Ishikawa's choice is obtained by creating in the *vacuum* $\frac{1}{2}N$ pairs of ^{3}He atoms, each pair being in a p-wave state with angular momentum $\hbar$ with the natural result that the total angular momentum is $\frac{1}{2}N\hbar$. This state is a sum of states like

$$\exp[i(\phi_{k_1} + \phi_{k_2} + \cdots + \phi_{k_{N/2}})] a^\dagger_{k_1} a^\dagger_{-k_1} \cdots a^\dagger_{k_{N/2}} a^\dagger_{-k_{N/2}} |0\rangle , \tag{1}$$

where ϕ_{k_1} is the azimuthal angle of $\vec{k}$, with respect to the z direction.

Our choice is obtained from the *normal state* by creating a given number of pairs of particles, and the same number of pairs of holes, each pair again being p wave. As a result, no net angular momentum is due to the pairs and since the normal state itself has no angular momentum, the total angular momentum is zero. Our state is a sum of states like

$$\exp[i(\phi_{k_1'}+\cdots+\phi_{k'_{n/2}})-i(\phi_{k_1''}+\cdots+\phi_{k''_{n/2}})] \times a^\dagger_{k_1'}\, a^\dagger_{-k_1'}\ \ldots\ a_{k''_{n/2}} a_{-k''_{n/2}}|N\rangle\,, \tag{2}$$

where $|N\rangle$ is the normal state

$$|N\rangle=\prod_{k<k_F} a^\dagger_k a^\dagger_{-k}|0\rangle\,. \tag{3}$$

We believe[13] that Ishikawa's wave function is not the correct choice. The essential difference between his wave function and ours lies in the fact that the former, when written in the k-space representation, contains an extra overall multiplicative phase factor $\prod_{k<k_F} e^{i\phi_k}$; it is precisely this factor which is responsible for his result $\frac{1}{2}N\hbar$ for the angular momentum. Now, such an overall phase factor would normally be regarded as physically meaningless, and indeed in our case it appears that it affects no physical quantity other than the angular momentum itself (for all other purposes, it is only the relative phase between the various components of the wave function which is meaningful). At first sight, therefore, there is no particular reason either to include or to exclude it. However, we now observe that if it is included, then it persists in the limit $\Delta\to 0$ and the resultant state still has angular momentum $\frac{1}{2}N\hbar$. Since the normal state should have zero angular momentum, we conclude that the physically correct choice is to exclude the phase factor, i.e, to adopt our wave function. Ishikawa counters this argument with the observation that the limiting procedure is ill-defined, and in particular the results may depend on whether or not one takes the limits $N\to\infty$, $\Delta\to 0$ in such a way that the coherence length is small compared to the sample size. This is clearly an arguable point and tied up with the general question of the validity of k-space arguments.[11] On the other hand, if our own proposed wave function is used the difficulty regarding the normal state does not even arise, while all other predictions of the usual BCS theory are unaffected; for this reason we believe it is the correct choice.

To summarize, if the angular momentum is calculated from the symmetry of the wave function in the k-space representation, one always finds zero. It has been argued by Mermin[11] that calculations using a k-space representation may be dangerous, because one needs to consider k as a continuous variable. We do not think that this is a real problem because the angular momentum is obtained from the basic symmetry of the BCS wave function, which should be independent of this kind of mathematical consideration. Anyway, one can escape the problem by translating our wave function in r-space representation. The angular momentum is then calculated without any problem and found to be zero.

INTRINSIC ANGULAR MOMENTUM OF ^{3}He-A

As already mentioned, in order to calculate the intrinsic angular momentum, we go back to its definition. It is the generator of the rotation of the system. If we perform a rotation of angle θ around the anisotropy axis $\hat{l}$, the ground state wave function will be multiplied by $e^{iL_0\theta/\hbar}$ since the angular momentum is obviously along $\hat{l}$. The standard BCS wave function is[12]

$$|\psi_0\rangle=\prod_k(u_k+v_k a^\dagger_k a^\dagger_{-k})|0\rangle\,. \tag{4}$$

We do not need to care about spin indices since, for the present problem, the up- and down-spin populations can be considered independent. In Eq. (1), u_k and v_k are given by

$$\begin{aligned}&|u_k|^2+|v_k|^2=1\\&2u_k^* v_k=\Delta_k/E_k\\&\Delta_k=\delta(\hat{k}_x+i\hat{k}_y)=\delta e^{i\phi_k}\,,\end{aligned} \tag{5}$$

where Δ_k is the gap and $E_k=(\xi_k^2+|\Delta_k|^2)^{1/2}$, where ξ_k is the normal particle energy measured from the Fermi surface. The standard choice for u_k and v_k is

$$\begin{aligned}&u_k=[\tfrac{1}{2}(1+\xi_k/E_k)]^{1/2}\,,\\&v_k=[\tfrac{1}{2}(1-\xi_k/E_k)]^{1/2}e^{i\phi_k}\,.\end{aligned} \tag{6}$$

We will see, however, that the particle nonconserving form Eq. (4) of the BCS wave function gives a misleading result, and we have to use the number-of-particles conserving form obtained by selecting out the terms in Eq. (4) corresponding to N particles

$$|\psi_0\rangle=\sum_{\{k_i\}}\left(\prod_{k_i} v_{k_i} a^\dagger_{k_i}a^\dagger_{-k_i}|0\rangle\right)\prod_{k_j\neq\{k_i\}} u_{k_j}\,, \tag{7}$$

where $\{k_i\}$ contains $\frac{1}{2}N$ different k_i. Now we can gather all the states $\prod_{k_i} a^\dagger_{k_i}a^\dagger_{-k_i}|0\rangle$ which can be deduced one from the other by any rotation R around $\hat{l}$. For all these states,

$$\prod_{k_i}|v_{k_i}|\prod_{k_j\neq\{k_i\}} u_{k_j}$$

will be the same. Let us call $R\{k_i\}$ such a set of states and use the notation $\{k_i\}/R$ for the various sets of this kind. Then

$$|\psi_0\rangle = \sum_{\{k_i\}/R}\left(\prod_{k_i}|v_{k_i}|\right)\left(\prod_{k_j\neq\{k_i\}} u_{k_j}\right) \times\left(\sum_{R\{k_i\}}\prod_{k_i} e^{i\phi_{k_i}}a^\dagger_{k_i}a^\dagger_{-k_i}|0\rangle\right). \tag{8}$$

Now if we compare the set $\{k_i\}$ with the set corresponding to the normal ground state ($|k_i|\leq k_F$), we have $n\leq\frac{1}{2}N$ values of k_i with $|k_i|>k_F$ corresponding to pairs above the Fermi surface. Let us call these values k_i^p. Similarly, we have n values of k_i, with $|k_i|<k_F$, which are in the normal ground state but not in the set $\{k_i\}$. These are pairs of holes, which we call k_i^h. Therefore, we can write

$$\sum_{R\{k_i\}}\prod_{k_i} e^{i\phi_{k_i}}a^\dagger_{k_i}a^\dagger_{-k_i}|0\rangle = \sum_{R\{k_i^p,k_i^h\}}\exp\, i\left(\sum_{k_i^p}\phi_{k_i^p} - i\sum_{k_i^h}\phi_{k_i^h}\right) \times\left(\prod_{k_i^p} a^\dagger_{k_i^p}a^\dagger_{-k_i^p}\right)\left(\prod_{k_i^h} a_{-k_i^h}a_{k_i^h}\right)|\psi_0\rangle_N, \tag{9}$$

where

$$|\psi_0\rangle_N = \prod_{k<k_F} e^{i\phi_k}a^\dagger_k a^\dagger_{-k}|0\rangle = \left(\prod_{k<k_F} e^{i\phi_k}\right)|N\rangle \tag{10}$$

is the normal state, except for an irrelevant phase factor. Since $i\sum_{k_i^p}\phi_{k_i^p} - i\sum_{k_i^h}\phi_{k_i^h}$ is clearly invariant under rotation and $|\psi_0\rangle_N$ is also invariant, we conclude that the left-hand side of Eq. (9) is invariant, and from Eq. (8) we obtain $L_0=0$ in the superfluid ground state.

We note that the phase factor $\prod_{k<k_F} e^{i\phi_k}$ in Eq. (10) is just the one we were referring to in the introduction. It appears in the BCS wave function as well as in the normal-state limit. If one uses r-space representation, this phase factor systematically gives a contribution $\frac{1}{2}N\hbar$ to the angular momentum.[13] But this phase factor has no physical meaning. Only the relative phase between the pairs, or more precisely the relative phase of the sets $R\{k_i\}$ is meaningful. If one discards this phase factor as it should be and then uses r-space representation, one naturally obtains a wave function with zero angular momentum. This is done as follows. After elimination of the phase factor, we can rewrite $|\psi_0\rangle$ as

$$|\psi_0\rangle = \sum_{\{k_i^p\}_n,\{k_j^h\}_n}\prod_{k_i^p}\frac{v_{k_i^p}}{u_{k_i^p}}a^\dagger_{k_i^p}a^\dagger_{-k_i^p} \times\prod_{k_j^h}\frac{u_{k_j^h}}{v_{k_j^h}}a_{-k_j^h}a_{k_j^h}|N\rangle, \tag{11}$$

where i or j go from 1 to n, and n goes from zero to $\frac{1}{2}N$. In Eq. (11) we have not written a multiplicative constant $(\prod_{k_i>k_F}u_k)(\prod_{k_i<k_F}|v_{k_i}|)$ which is unimportant, since we have not normalized $|\psi_0\rangle$. If we define

$$p(k)=(v_k/u_k)\Theta(k-k_F),\quad t(k)=(u_k/v_k)\Theta(k_F-k), \tag{12}$$

we obtain

$$|\psi_0\rangle = \sum_{\{k_i\}_n,\{k_j\}_n}\left(\prod_{k_i}p(k_i)a^\dagger_{k_i}a^\dagger_{-k_i}\right) \times\left(\prod_{k_j}t(k_j)a_{-k_j}a_{k_j}\right)|N\rangle, \tag{13}$$

where k_i or k_j are no longer restricted above or below the Fermi surface. Using now r-space representation, we finally have

$$|\psi_0\rangle = \sum_{n=0}^{N/2}|\phi_n\rangle, \tag{14}$$

where

$$|\phi_n\rangle = \int\prod_{i,j} dr_i\,dr_i'\,dr_j\,dr_j'\,p(r_i-r_i') \times\psi^\dagger(r_i)\psi^\dagger(r_i')t(r_j-r_j')\psi(r_j)\psi(r_j')|N\rangle, \tag{15}$$

and $p(r)$ and $t(r)$ are the Fourier transform of $p(k)$ and $t(k)$. Each of the $|\phi_n\rangle$ has a clear physical meaning. They represent a state obtained from the normal state by creating n pairs of particles in the p-wave state $p(r)$ and the same number of pairs of holes in the state $t(r)$. The wave function $p(r)$ appears well behaved. We note that, although $|\phi_n\rangle$ in Eq. (14) is the sum of all the $|\psi_0\rangle$ with n going from zero to $\frac{1}{2}N$, it is likely that one obtains as good a description of the superfluid state by picking out only $|\phi_{n_0}\rangle$, where n_0 is just the actual number of pairs created in the system, which is of order NT_C/E_F. Now if we use the same kind of arguments as Mermin,[11] we obtain the obvious result that each $|\phi_n\rangle$ has zero angular momentum because $|N\rangle$ has the same property, and therefore $|\psi_0\rangle$ has also zero angular momentum.

Until now we have worked with the number-of-particles conserving form, Eq. (7). But we know that the form Eq. (4) is in general much less cumbersome to handle. Let us see what happens when we try to calculate the angular momentum from Eq. (4). If we perform a rotation around $\hat{l}$ such that $r\to R_\theta(\vec{r})$, any state $|\vec{k}\rangle$ is transformed into $|R_{-\theta}(\vec{r})\rangle$. Therefore, $|\psi_0\rangle$ is transformed into

$$R_\theta|\psi_0\rangle = \prod_k (u_k + v_k a^\dagger_{R_{-\theta}(\vec{k})} a^\dagger_{R_{-\theta}(-\vec{k})})|0\rangle$$

$$= \prod_k (u_{R_\theta(\vec{k})} + v_{R_\theta(\vec{k})} a^\dagger_{\vec{k}} a^\dagger_{(-\vec{k})})|0\rangle. \qquad (16)$$

If θ is small, one has

$$\langle\psi_0|R_\theta|\psi_0\rangle \simeq \prod [|u_k|^2 + (1+i\theta)|v_k|^2] = 1 + \tfrac{1}{2}i\theta N, \qquad (17)$$

because $N = 2\sum_k |v_k|^2$ at $T=0$. Therefore, we apparently obtain $L_0 = \frac{1}{2}N\hbar$ from Eq. (17). This is not surprising since by using the form Eq. (4) of the BCS wave function, we have no way to eliminate the phase factor [see Eq. (18)] which produces systematically the result $\frac{1}{2}N\hbar$. However, once it is realized that the phase factor is responsible for the $\frac{1}{2}N\hbar$, it is easy to get rid of it by saying that only the difference in the angular momentum between the normal state and the superfluid state has a physical meaning. In this way one obtains again $L_0 = 0$ in the superfluid state.

There is even a more convenient way to derive L_0 from Eq. (4). Let us perform the canonical transformation defined by $a_k = e^{i\phi_k/2} b_k$. In such a transformation, a state with angular momentum $m\hbar$ like $\int d\phi_k e^{im\phi_k} a^\dagger_k|0\rangle$ is going into $\int d\phi_k e^{i(m-1/2)\phi_k} b^\dagger_k|0\rangle$ which has an angular momentum $(m-\frac{1}{2})\hbar$. This transformation is very similar to a Galilean transformation where $\psi(x) = e^{ik_0x}\phi(x)$ and in which state of momentum p goes into a state $p - \hbar k_0$. In this case, the total momentum of the system is decreased by an amount $-N\hbar k_0$. In the same way, in our transformation the angular momentum is decreased by $-\frac{1}{2}N\hbar$. Now, in the new representation, Eq. (4) reads

$$|\psi_0\rangle = \prod_k (|u_k| + |v_k| b^\dagger_k b^\dagger_{-k}|0\rangle), \qquad (18)$$

which has clearly no angular momentum. We conclude that, in the original representation, the angular momentum was $\frac{1}{2}N\hbar$. Naturally, we can apply exactly the same argument in the normal-state limit and therefore L_0 must be taken as zero as explained before.

This argument can easily be generalized to the case $T \neq 0$. To work at finite temperature, we have to find the correct phase of the excited states with respect to the ground state. To do this, we require that when we go to the normal state by letting $\delta \to 0$, we find the excited states corresponding to the normal state Eq. (6). This implies that the excited states, corresponding to the ground pair states[12] $(u_k + |v_k| e^{i\phi_k} a^\dagger_k a^\dagger_{-k})|0\rangle$ are $e^{i\phi_k/2} a^\dagger_k|0\rangle$ and $e^{i\phi_k/2} a^\dagger_{-k}|0\rangle$, while the excited pair state is $(-|v_k| + u_k e^{i\phi_k} a^\dagger_k a^\dagger_{-k})|0\rangle$. Performing the same transformation as before, we go again to a representation where there is obviously no angular momentum, and we conclude again that the physical L_0 must be zero.

Finally, let us comment briefly on the results of other calculations. The difficulties arising in relating the Anderson-Morel result to the total angular momentum have been discussed in Ref. 12. Regarding the calculations of Cross and Volovik, we believe that they deal with the extrinsic rather than the intrinsic angular momentum, since they consider inhomogeneous situations. The fact that their results are modified by Fermi liquid effects seems to support this view. Finally, we have extensively discussed why Ishikawa's result $\frac{1}{2}N\hbar$ should not be taken as the physical answer for L_0.

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[1]M. C. Cross and P. W. Anderson, *Proceedings of the Fourteenth International Conference on Low Temperature Physics*, edited by M. Krusius and M. Vuorio (North-Holland, Amsterdam, 1975), Vol. I, p. 17; M. C. Cross (unpublished).

[2]A. J. Leggett and S. Takagi, Phys. Rev. Lett. 36, 1379 (1976).

[3]R. Combescot, Phys. Rev. B (to be published).

[4]V. Ambegaokar, P. G. de Gennes, and D. Rainer, Phys. Rev. A 9, 2676 (1974).

[5]N. D. Mermin and T. L. Ho, Phys. Rev. Lett. 36, 832 (1976).

[6]P. W. Anderson and P. Morel, Phys. Rev. 123, 1911 (1961).

[7]M. C. Cross, D. Phil. thesis (Cambridge, England, 1975) (unpublished).

[8]G. E. Volovik, JETP Lett. 22, 108 (1975).

[9]M. Ishikawa, Prog. Theor. Phys. (Jpn.) 57, 1836 (1977).

[10]The readers can take two extreme positions with respect to the arguments presented in this paper: (i) He does not believe at all any k-space argument. In this case, he can consider that we propose, in Eqs. (14) and (15), a BCS wave function which seems as good as Ishikawa's choice and has a zero angular mo-

mentum. (ii) He does believe in k-space arguments. In this case, we show that the standard BCS wave function has zero angular momentum [Eqs. (4)–(10)]. Now if the reader believes somewhat in k-space arguments (and we think he has to in order to know the physical nature of the A phase), our point is that Ishikawa's choice for the BCS wave function in r space is *unnatural* because it is the Fourier transform of a k-space wave function containing a phase factor which is both unnecessary (in the sense that it affects no physical quantities other than the angular momentum) and leads to apparent difficulties in the normal limit.

[11] N. D. Mermin, Physica (Utrecht) 90B, 1 (1977).

[12] See A. J. Leggett [Rev. Mod. Phys. 47, 331 (1975)] for a review paper on the theory of superfluid ^{3}He, with also a discussion of the Anderson-Morel result for L_o.

[13] Actually the r-space calculation is somewhat equivalent to saying, in k space, that a rotation θ makes ϕ_k go into $\phi_k + \theta$. Therefore, $\Pi_{k<k_F} e^{i\phi_k}$ becomes $(\Pi_{k<k_F} e^{i\phi_k})\, e^{iN/2\,\theta}$, which gives the result $\frac{1}{2}N\hbar$. This argument appears naturally as an artefact of a continuous k-space representation, together with the fact that ϕ_k is not a single defined function. But when one goes in r-space representation, it does not appear so any more.