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Calculation of the electromagnetic coherence length in superconductors with magnetic impurities

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We calculate the electromagnetic coherence length ξ for superconductors with magnetic impurities by using the theory of Shiba and of Rusinov. The expression for ξ is analytically evaluated in several limits, and it is numerically evaluated for a range of values of the variables on which ξ depends. It is found that, over the range of values of the variables for which ξ is calculated, ξ can be fitted with a simple empirical function to an accuracy that is never worse than 6%, and is typically better than 2%, whereas a naive estimate of ξ can be as much as 80% higher than the correct value. We show how to calculate the penetration depth in the London limit and the extreme anomalous limit from the results for ξ and the already calculated local-limit penetration depth.

I. INTRODUCTION

Studies of the effects of magnetic impurities in superconductors illuminate the phenomenon of superconductivity and provide information concerning the interaction between magnetic spins and conduction electrons in metals. For example, gapless superconductivity was first proposed in the original work of Abrikosov and Gor'kov[1] (AG) on the effects of magnetic impurities on superconductivity. In this theory the interaction between individual spins and the electrons is treated by perturbation theory. The results of the theory for a variety of phenomena have been reviewed by Maki.[2]

The work of AG has been succeeded by extensions of the theory to take into account the interaction between the individual impurity spins and the conduction electrons beyond perturbation theory. This is not simply a matter of quantitative interest. In normal metals it is known that this extension leads to an understanding of the Kondo effect. A treatment of the thermodynamic properties of superconductors that takes some account of the Kondo effect has been given by Müller-Hartmann and Zittartz.[3-6] This treatment appears to be moderately successful, albeit with some modification of the theoretical parameters,[7] for alloys with $T_K/T_{c0} \leqslant 1$. A self-consistent treatment of the thermodynamic properties of superconductors with magnetic impurities has recently been given by Schuh and Müller-Hartmann.[8] It has not yet been fully compared with experimental results. It is not easy to extend these treatments to cover the effects of external fields and of irreversible phenomena.

An alternative extension of the theory of AG has been provided by Shiba[9,10] and Rusinov.[11] In this theory the impurity spins are treated classically, but otherwise their interaction with the electrons is calculated exactly. This treatment should therefore be valid for large impurity spins, and provides, in any case, a useful model. One striking new qualitative feature of the model is the existence of bound states in the energy gap. Reviews of the comparison of observations with some of the consequences of the model have been provided by Ginsberg[12] and by Takayanagi and Sugawara.[7] More recently,[13,14] good agreement has been found between this theory and measurements of the electronic thermal conductivity of Pb-Mn, In-Mn, and In-Cr alloy films. This success is an encouragement to explore further consequences of the theory.

In the past, the electromagnetic properties of superconductors have proved useful in elucidating the properties of impurities in superconductors. However, the electromagnetic coherence length has not been calculated for superconductors containing magnetic impurities, even in the framework of the AG theory. We have therefore begun a program to extend the theory in this direction by computing the electromagnetic coherence length of superconductors containing magnetic impurities which conform to Shiba's model. Since it seems that Shiba's form for the single-particle Green's function is a good approximation to that obtained by Müller-Hartmann and Zittartz, provided that Shiba's parameter ϵ_0 (see Sec. II) is given an appropriate temperature dependence[3,5,6] the present calculation may also be applicable to superconductors for which that theory is a suitable model.

We derive an analytic expression for the electromagnetic coherence length ξ by using the Shiba theory. We numerically evaluate this expression for ξ, using physically reasonable values for the variables on which ξ depends. We fit the calculated values of ξ with a simple empirical function. These three steps are described in detail in Sec. II, III, and IV, respectively. Sec. V is a summary. In Sec. VI we add a few remarks about the electromagnetic penetration depth.

II. THEORY

The point of departure for this work is the theory of Shiba[9] and Rusinov[11] for the Green's functions for electrons in a superconductor with magnetic impurities. Assuming s-wave scattering by the impurities, and using a classical treatment of the spins, one finds that the single-particle temperature-dependent Green's function $G(\vec{k}, i\omega_n)$ is, for a temperature T, given by

$$G(\vec{k}, i\omega_n) = 1/(i\tilde{\omega}_n - \epsilon_{\vec{k}}\rho_3 - \tilde{\Delta}_n\rho_1\sigma_2)\,, \tag{1}$$

$$\omega_n = 2\pi T(n + \tfrac{1}{2})\,. \tag{2}$$

There, the 4×4 matrix notation (see Maki[2]) is used, and the matrices σ_i and ρ_i are Pauli spin matrices, the former being an operator in spin space, the latter in the space of electrons and holes. In Eq. (1), $\epsilon_{\vec{k}}$ is the energy of an electron in the normal state, and $\tilde{\omega}$ and $\tilde{\Delta}$ are related to ω_n and the gap parameter Δ through the equations

$$\tilde{\omega}_n = \omega_n + \frac{1}{2}\left(\frac{\hbar}{\tau_1} + \alpha\right)\frac{u_n(u_n^2+1)^{1/2}}{u_n^2+\epsilon_0^2} + \frac{\hbar}{2\tau'}\frac{u_n}{(u_n^2+1)^{1/2}}\,, \tag{3}$$

$$\tilde{\Delta}_n = \Delta + \frac{1}{2}\left(\frac{\hbar}{\tau_1} - \alpha\right)\frac{(u_n^2+1)^{1/2}}{u_n^2+\epsilon_0^2} + \frac{\hbar}{2\tau'}\frac{1}{(u_n^2+1)^{1/2}}\,, \tag{4}$$

and

$$u_n = \tilde{\omega}_n/\tilde{\Delta}_n\,. \tag{5}$$

The parameters τ', τ_1, α, and ϵ_0 are related to the scattering of electrons by the impurities; τ'^{-1} is the scattering rate from any nonmagnetic impurities or defects which may be present, τ_1^{-1} is the non-spin-flip scattering rate from magnetic impurities, $\alpha/\hbar$ is the spin-flip scattering rate from magnetic impurities, and ϵ_0 is Shiba's parameter giving the energy (normalized to Δ) of the bound state that lies in the gap and is due to the magnetic impurities. For a given host metal and impurity, these parameters can, in principle, be calculated, but in this paper they are treated as parameters. Equations (3) and (4) can be combined to yield a single equation for u_n,

$$\omega_n/\Delta = u_n[1 - (\alpha/\Delta)(u_n^2+1)^{1/2}/(u_n^2+\epsilon_0^2)]\,. \tag{6}$$

To complete the determination of the Green's function it is necessary to add the gap equation that determines $\Delta(\alpha, T)$ as a function of α and temperature T. According to Lo and Nagi,[15] this equation can be written

$$\ln t = \frac{2\gamma t}{\delta(p,t)}\sum_{n\geq 0}\left((1+u_n^2)^{-1/2} - \frac{\delta(p,t)}{2\gamma t(n+\frac{1}{2}+p/4\gamma t)}\right) + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2}+\frac{p}{4\gamma t}\right)\,, \tag{7}$$

where

$$\begin{aligned} &t \equiv T/T_{c0}, \quad \delta(p,t) \equiv \Delta(\alpha,t)/\Delta(0,0)\,,\\ &p \equiv 2\alpha/\Delta(0,0), \quad \gamma \equiv \pi T_{c0}/\Delta(0,0) \approx 1.781\,, \end{aligned} \tag{8}$$

and T_{c0} is the transition temperature of the host material. The function $\psi(x)$ is the digamma function. When $t = T_c/T_{c0}$, the ratio of the transition temperatures of the host with and without magnetic impurities, the sum in Eq. (7) is identically zero, and the resulting expression is used to determine p for any value of T_c/T_{c0}. It can be shown that when $T_c/T_{c0} = 0$, then $p = 1$. Note that neither u_n nor δ is a function of τ_1 or τ'. Equations (1)–(8) are sufficient to determine the single-particle Green's function as a function of ω_n and T. For certain values of the parameters these equations have been solved by Lo and Nagi.[15]

As we have said, Shiba's treatment of the scattering of electrons by a single impurity goes beyond perturbation theory. Perturbation theory can, however, be recovered by letting ϵ_0 tend to unity. This yields the theory of AG. Deviations of ϵ_0 from unity thus represent deviations from perturbation theory.

The calculation of the electromagnetic properties of the system very closely follows the similar calculation for the case dealt with by AG (see Maki[2]). For the Meissner effect, one requires the response of the system to a static magnetic field, represented by the vector potential $\vec{A}(\vec{r})$. The current $\vec{j}(\vec{q})$ induced by a single Fourier component $\vec{a}(\vec{q})$ of the vector potential can be written

$$j_\mu(\vec{q}) = -\frac{c}{4\pi}\sum_\nu K_{\mu\nu}(\vec{q})\,a_\nu(\vec{q})\,, \tag{9}$$

where $K_{\mu\nu}(\vec{q})$ can be calculated in terms of the single-particle Green's functions using standard many-body theory. To be consistent with the previous calculation of the Green's functions, we still assume only s-wave scattering by the impurities. There are then no vertex corrections to the expression for $K_{\mu\nu}(\vec{q})$ that is given by (see Maki[2])

$$\frac{c}{4\pi}K_{\mu\nu}(\vec{q})=\frac{e^2\hbar^2T}{2m^2c}\sum_{\vec{k}}k_\mu k_\nu \times\sum_{\omega}\mathrm{Tr}G(\vec{k},i\omega_n)G(\vec{k}+\vec{q},i\omega_n) - \frac{Ne^2}{mc}\delta_{\mu\nu}\,, \tag{10}$$

where N is the density of conduction electrons. For a transverse vector potential in an isotropic superconductor one finds

$$K_{\mu\nu}(\vec{q})=K(q)\delta_{\mu\nu}\,, \tag{11}$$

where

$$\frac{c}{4\pi}K(q)=\frac{3e^2TN}{16mc}\sum_{\omega_n}\int d\epsilon\int d\mu(1-\mu^2) \times\mathrm{Tr}G(\vec{k},i\omega_n)G(\vec{k}+\vec{q},i\omega_n) \tag{12}$$

and

$$\mu=\vec{k}\cdot\vec{q}/kq\,. \tag{13}$$

In Eq. (12) the integration over ϵ is to be performed before the sum over ω_n. The change in order of summation and integration between Eqs. (10) and (12) eliminates the term $(Ne^2/mc)\delta_{\mu\nu}$.

The integration over ϵ can be performed analytically, and leads to the result

$$\frac{c}{4\pi}K(q)=\frac{3e^2NT\pi}{2mc}\sum_{\omega_n}\int d\mu \times\frac{(1-\mu^2)}{(u_n^2+1)[2\tilde{\Delta}_n(u_n^2+1)^{1/2}-i\hbar qv_F\mu]}\,, \tag{14}$$

where v_F is the Fermi velocity, and it has been assumed as is usual that $q\ll k_F$. Equation (14) is identical in form with that for the theory of AG. The difference stems only from the different dependence of u_n on ω_n and Δ that comes from the deviation of ϵ_0 from unity in Eq. (6). Further integration over μ yields

$$\frac{c}{4\pi}K(q)=\frac{3e^2NT\pi}{mc\hbar v_f q}\sum_{\omega_n}\frac{1}{u_n^2+1} \times\left[(1+s_n^2)\tan^{-1}\left(\frac{1}{s_n}\right)-s_n\right], \tag{15}$$

where

$$s_n=2\tilde{\Delta}_n(u_n^2+1)^{1/2}/\hbar v_f q\,. \tag{16}$$

The electromagnetic coherence length ξ can be defined by[16]

$$\lim_{q\to\infty} qK(q,T)/K(0,T)=3\pi/4\xi\,, \tag{17}$$

where the left-hand side is finite. The definition agrees with that of BCS when no magnetic impurities are present.[17] From Eqs. (15) and (17) one finds

$$\xi=\frac{\hbar v_f}{2}\sum_{n\geq 0}\frac{1}{\tilde{\Delta}_n(1+u_n^2)^{3/2}}\Bigg/\sum_{n\geq 0}\frac{1}{(1+u_n^2)}\,. \tag{18}$$

The procedure for calculating ξ is as follows: for a given value of T_c/T_{c0}, determine p by setting $t=T_c/T_{c0}$, and solving Eq. (7); then, for a given value of T/T_{c0}, find δ by simultaneously solving Eqs. (6) and (7); finally, use Eqs. (4), (6), and (18) to calculate ξ.

In general, ξ can be found only by numerical methods, but it can be found analytically in several limits. The analytic expressions serve two purposes: firstly, they can be used to show that Eq. (18) reduces to previously calculated results in certain limits; secondly, they can be used to check the accuracy of the computer program used for the numerical calculations of ξ. We will examine four limits. In the following, we use the definition

$$l\equiv v_f\,[(1/\tau_1)+(1/\tau')]^{-1}\,. \tag{19}$$

(i) Let $T/T_c\to 0$, $T_c/T_{c0}\to 1$, and $l\to\infty$. Then $p\to 0$, $u_n\to\omega_n/\Delta(0,0)$, and $\tilde{\Delta}_n\to\Delta(0,0)$. Putting these results into Eq. (18) yields

$$\xi=\frac{\hbar v_f}{2}\sum_{n\geq 0}\frac{1}{\Delta(0,0)\{1+[\omega_n^2/\Delta(0,0)^2]\}^{3/2}} \times\left(\sum_{n\geq 0}\frac{1}{\{1+[\omega_n^2/\Delta(0,0)^2]\}}\right)^{-1}. \tag{20}$$

Since $t\to 0$, the sums can be converted into integrals, and Eq. (20) becomes

$$\xi=\hbar v_f/\pi\Delta(0,0)\equiv\xi_{00}\,. \tag{21}$$

This is the BCS[17] expression for the zero-temperature coherence length for infinite mean free path.

(ii) Let $T/T_c\to T_c/T_{c0}\to 1$ and $l\to\infty$. Then $\tilde{\Delta}_n\to 0$, $\Delta\to 0$, and $u_n\to\omega_n/\Delta\to\infty$. Putting these results into Eq. (18), we find

$$\xi=\frac{\hbar v_f}{2\pi T_{c0}}\sum_{n\geq 0}\frac{1}{(2n+1)^3}\Bigg/\sum_{n\geq 0}\frac{1}{(2n+1)^2}\,. \tag{22}$$

These sums are related to Riemann ζ functions, and their values yield

$$\xi=0.752\xi_{00}\,. \tag{23}$$

This also agrees with the BCS theory.[16,18]

(iii) Let $1/\tau'$ be large enough so that $\tilde{\Delta}_n\to\hbar/2\tau'(u_n^2+1)^{1/2}$. Then Eq. (18) indicates that

$$\xi=v_f\tau'=l\,. \tag{24}$$

This is the well-known result that the electromagnetic coherence length equals the electron mean free path when the electron mean free path is very short.

(iv) Let $T/T_c \to 0$ and $T_c/T_{c0} \to 0$. Then $p \to 1$, $\tilde{\Delta}_n \to 0$, and $u_n \to \infty$. After doing some algebra and converting sums to integrals, Eq. (18) becomes

$$\xi = \xi_{00}(\pi/b^2)[b - \ln(1+b)] , \tag{25}$$

where

$$b = \pi\xi_{00}/l - \tfrac{1}{2} . \tag{26}$$

Therefore, in this limit, ξ depends on l but not on ϵ_0. Although this limit of the theory is not usually experimentally attainable because of impurity-impurity interactions, it can still be used to check the accuracy of the computer program. Equations (25) and (26) indicate that when $l \to \infty$, then $\xi \to 2.427\xi_{00}$. The same program that we use to calculate ξ to an estimated accuracy of 0.1% reproduces this result to that accuracy.

III. NUMERICAL CALCULATION OF ξ

We calculate ξ numerically by using the procedure outlined in Sec. II. We find it convenient to introduce several quantities. First, instead of using the spin-flip $(\alpha/\hbar)$ and non-spin-flip $(1/\tau_1)$ scattering rates as independent variables, we use the ratio of these scattering rates

$$R \equiv (\alpha/\hbar)/(1/\tau_1) \tag{27}$$

and the reduced transition temperature T_c/T_{c0}. In real materials, R should perhaps be between 0.01 and 1. We define an electron mean free path limited only by non-spin-flip scattering from the magnetic impurities

$$l_1 \equiv v_f \tau_1 . \tag{28}$$

Finally, we define

$$\xi_0 \equiv \xi(T/T_c,\ l=\infty,\ R=\infty,\ \epsilon_0,\ T_c/T_{c0}) . \tag{29}$$

ξ_0 is analogous to the BCS coherence length ξ_{00}. In fact, $\xi_0 = \xi_{00}$ when $T_c/T_{c0} = 1$.

The calculation was performed for several values of the variables on which ξ depends. In particular, the calculation was done for

$$T/T_c = 0.05, 0.10, \ldots, 0.95 ,$$

$$l/\xi_0 = 10^m ,$$

where

$$\left.\begin{aligned} &m = -3, -2, -1, -\tfrac{2}{3}, -\tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{2}{3}, 1, 2, 3 , \\ &R = 0.01, 0.03, 0.1, 0.3, 1 , \\ &\epsilon_0 = 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1 , \end{aligned}\right. \tag{30}$$

and

$$T_c/T_{c0} = 0.2, 0.4, 0.6, 0.8, 0.84, 0.88, 0.92, 0.96, 1 .$$

As mentioned earlier, the computer program was designed to calculate ξ to an accuracy of about 0.1%. The program was first tested by comparing our calculated values of Δ against those obtained by Lo and Nagi.[15] Then our values of ξ were checked in the limits discussed in Sec. II.

In order to discuss the deviations from BCS-like behavior[16,17] in ξ that are induced by the magnetic impurities, we discuss ξ_0/ξ_{00} and ξ/ξ_0 separately. This procedure helps to elucidate where and how the deviations arise.

The corresponding-states prediction for ξ_0/ξ_{00} would be

$$\xi_0/\xi_{00} = T_{c0}/T_c , \tag{31}$$

but this is not correct for superconductors with magnetic impurities. In Fig. 1 we plot some values of $(\xi_0/\xi_{00})(T_c/T_{c0})$, which Eq. (31) indicates should be identically equal to 1. It is clear from Fig. 1 that the magnetic impurities cause deviations from this value.

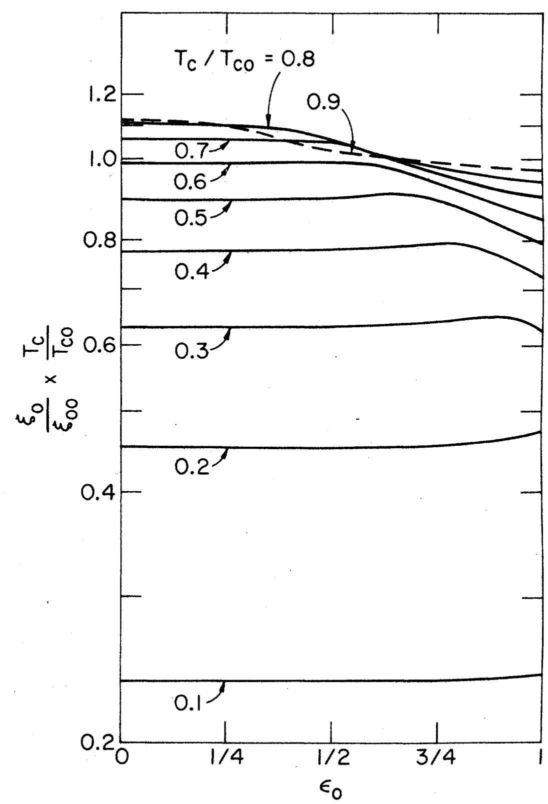

FIG. 1. Plot of $(\xi_0/\xi_{00})(T_c/T_{c0})$ vs ϵ_0 for several values of T_c/T_{c0}. For a superconductor without magnetic impurities, this quantity is identically 1.

In Figs. 2 and 3 we show representative curves of ξ/ξ_0. From these curves one can see that the effect of the magnetic impurities is to reduce ξ/ξ_0 from the BCS value. This reduction is due in part to the non-spin-flip scattering from the magnetic impurities and in part to the spin-flip scattering from the magnetic impurities. The non-spin-flip scattering, which affects ξ/ξ_0 through Eq. (4), causes ξ to depend on R. Clearly, for $\epsilon_0=1$, the terms in Eq. (4) containing $1/\tau_1$ and $1/\tau'$ are the same. However, as ϵ_0 decreases from 1, the term containing $1/\tau_1$ increases although l stays the same. This means that the effective mean free path is shorter than l. Therefore, the coherence length decreases. This effect is largest when both R is smallest and $\epsilon_0=0$. The effect disappears when either $R\to\infty$ (an experimentally unobtainable limit) or when $\epsilon_0=1$. The effect also disappears when $T/T_c\to 1$. This can be seen by noting that Eq. (6) indicates that when $T/T_c\to 1$, $u_n\to\infty$ for all n. Hence the terms in Eq. (4) that contain $1/\tau_1$ and $1/\tau'$ become identical. The reduction of ξ/ξ_0 due to the non-spin-flip scattering from the magnetic impurities can be best seen in Fig. 2. There one can see that holding all other variables constant and decreasing R (increasing $1/\tau_1$) decreases ξ/ξ_0. As T/T_c approaches 1, all of the curves come together to intersect at the same point.

In Fig. 2 one also sees that even at $T/T_c=1$, the calculated curves lie below the BCS curve. This is due to the spin-flip scattering from the magnetic impurities.

The mean-free-path dependence of ξ/ξ_0 is roughly the same as the BCS behavior; ξ/ξ_0 de-

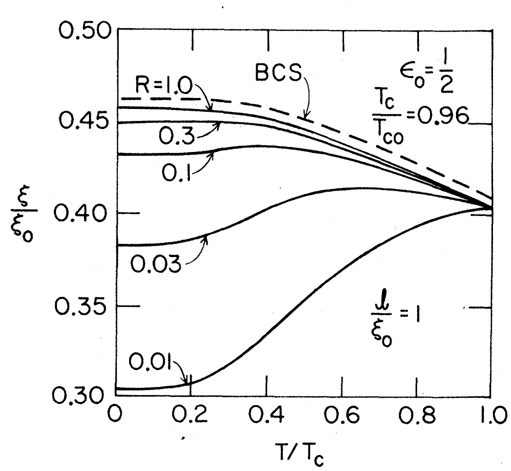

FIG. 2. Plot of ξ/ξ_0 for several values of R. The dashed line is calculated for $T_c/T_{c0}=1$.

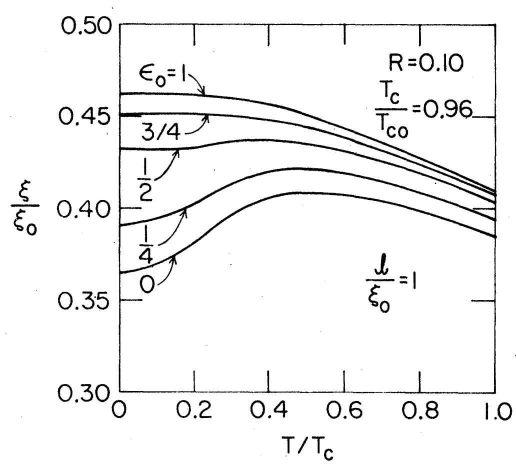

FIG. 3. Plot of ξ/ξ_0 for several values of ϵ_0. The values of ξ/ξ_0 calculated for $T_c/T_{c0}=1$ lie very slightly above the curve shown for $\epsilon_0=1$ and are too close to that curve to draw separately.

creases with decreasing mean free path, approaching the limit $\xi/\xi_0=l/\xi_0$ for very small mean free paths.

IV. EMPIRICAL FIT TO THE CALCULATED VALUES OF ξ

An empirical fit to ξ is useful because it provides a way of calculating ξ for any values of the variables which affect it. If one estimates ξ by using the usual interpolation formula[16]

$$\xi=[1/\xi_{\mathrm{BCS}}(T/T_c,\, l=\infty)+1/l]^{-1}, \tag{32}$$

one will arrive at a value which is up to 9% high for a superconductor without magnetic impurities, and up to 80% high for a superconductor with magnetic impurities in which $\epsilon_0=0$ and $R=0.01$. Our empirical fit is much better than this, as we will see below.

A. Fitting procedure

The calculated values of ξ/ξ_0 are fitted with a four-parameter function according to the following procedure.

Values of ξ/ξ_0 calculated in the unphysical limit $R=\infty$ are compared with a slightly altered version of the usual interpolation function

$$g\left(\frac{T}{T_c},\frac{l}{\xi_0},\epsilon_0,\frac{T_c}{T_{c0}}\right)\equiv\left(\frac{\xi_0}{\xi(T/T_c,\infty,\infty,\epsilon_0,T_c/T_{c0})}+\frac{\xi_0}{l}\right)^{-1}. \tag{33}$$

It is found that the fractional deviation of ξ/ξ_0 from g, as a function of l/ξ_0, can be fitted with a Gaussian function in this manner

$$\frac{\xi/\xi_0 - g}{\xi/\xi_0} = h \equiv A \exp\left(-\frac{[\ln(l/B\xi_0)]^2}{2C^2}\right). \tag{34}$$

For each set of values for T/T_c, ϵ_0, and T_c/T_{c0}, the parameters A, B, and C are independently varied to give the best fit.

To fit the values of ξ/ξ_0 calculated for finite values of R, a new parameter must be introduced to account for the enhancement of the non-spin-slip scattering from the magnetic impurities, which was discussed in Sec. III. We call this parameter D, and we use it to define an effective mean free path

$$\tilde{l} \equiv l/(1 + Dl/l_1). \tag{35}$$

Note that D is always positive, so that $\tilde{l} \leq l$; $\tilde{l} = l$ only when $\epsilon_0 = 1$ or $T/T_c = 1$. For each set of values for T/T_c, ϵ_0, R, and T_c/T_{c0}, D is chosen to give the best fit to ξ/ξ_0 when $\tilde{l}$ is substituted for l in Eq. (34).

Solving Eq. (34) for ξ/ξ_0, and explicitly noting that one should use $\tilde{l}$ in place of l, one arrives at the basic fitting function for all of the calculated values of ξ/ξ_0:

$$\xi/\xi_0 = f \equiv g(\tilde{l})/[1 + h(\tilde{l})]. \tag{36}$$

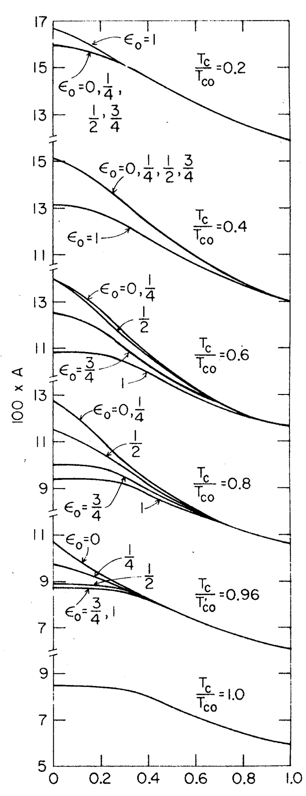

FIG. 4. Plot of the temperature dependence of the fitting parameter A for several values of ϵ_0 and T_c/T_{c0}.

B. Discussion

First, we discuss how the fitting parameters A, B, C, and D depend on the variables on which ξ/ξ_0 depends.

The parameter A represents the largest fractional deviation of ξ/ξ_0 from the usual interpolation function g. It is a function of T/T_c, ϵ_0, and T_c/T_{c0}. Figure 4 shows how A depends on these variables. The figure indicates that A varies from about 0.05 to 0.17 over the range of the variables for which ξ/ξ_0 was calculated. The accuracy of the fit is reasonably sensitive to A; a change in A of 0.01 causes a change in f of up to 1%.

The parameter B is found to be a weak function of T/T_c, ϵ_0, and T_c/T_{c0}. In fact, for $T_c/T_{c0} \geq 0.2$, the parameter B can be taken to be a constant 0.42 without degrading the fit by more than about 1%.

The parameter C is found to be a constant 1.84.

The parameter D is, by definition, the only

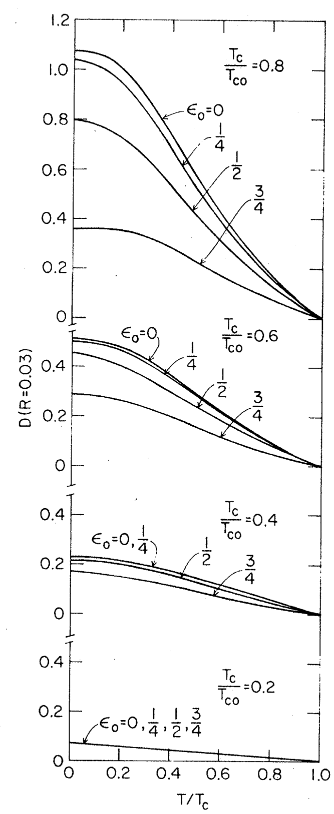

FIG. 5. Plot of the temperature dependence of the fitting parameter $D(R=0.03)$ for several values of ϵ_0 and T_c/T_{c0}.

parameter which is a function of R. It is found that the R dependence of D can be roughly approximated by the relation

$$D(R)/D(R=0.03)=1+0.22\ln(R/0.03). \qquad (37)$$

The dependence of $D(R=0.03)$ on the variables T/T_c, ϵ_0, and T_c/T_{c0} is shown in Figs. 5 and 6, which illustrate the fact that the enhancement of the effective mean free path disappears when $T/T_c=1$ or when $\epsilon_0=1$, because D goes to zero in both of these limits.

Values of $\xi(T/T_c,\ l=\infty,\ R=\infty,\ \epsilon_0,\ T_c/T_{c0})/\xi_0$, which appear in the definition of g, are presented in Fig. 7.

The four-parameter fit is accurate to 0.5% in the BCS limit, and to 1% in the AG limit, $\epsilon_0=1$. When $\epsilon_0<1$, the fit is accurate to at least 6%, and it is typically accurate to better than 2%. The largest deviations occur only for $T/T_c\leqslant 0.1$, $T_c/T_{c0}\geqslant 0.9$, $\epsilon_0<1/4$, and $l_1/10\leqslant l\leqslant l_1$. The fit would be accurate to about 2% for all variables for which ξ was calculated if one did not use the approximation given in Eq. (37) for the dependence of D on R.

V. SUMMARY

An expression for the electromagnetic coherence length ξ has been derived for superconductors with magnetic impurities by using the theory of Shiba and Rusinov. The expression has been numerically evaluated for a range of values for the relevant variables.

The results of the numerical calculation indicate that if one were to estimate ξ by naively assuming that a superconductor with magnetic impurities conforms to the BCS theory, and then applying the interpolation formula given by Eq. (32), one would obtain a value for ξ which is up to 80% higher than the correct value.

The results of the numerical calculation also indicate that for $T_c/T_{c0}\geqslant 0.2$, ξ can be fitted with

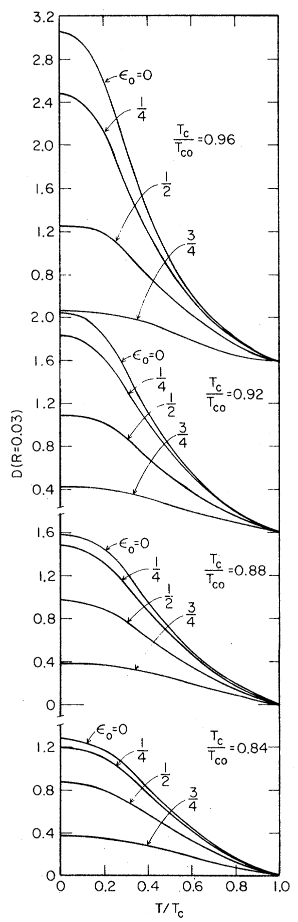

FIG. 6. Plot of the temperature dependence of the fitting parameter $D(R=0.03)$ for several values of ϵ_0 and T_c/T_{c0}.

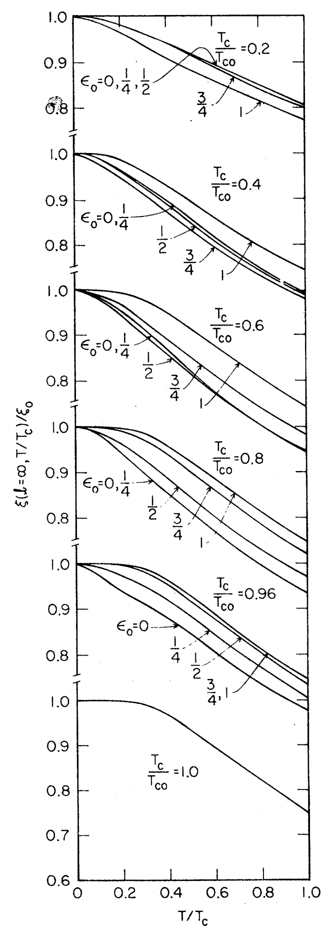

FIG. 7. Plot of $\xi\ (l=\infty,\ T/T_c)/\xi_0$ for several values of ϵ_0 and T_c/T_{c0}.

a simple function of the form given by Eq. (36). In this function, the parameters A and $D(R=0.03)$ are functions of T/T_c, ϵ_0, and T_c/T_{c0}, but not of l or R. The fit is worst when all four of the conditions $T/T_c \leq 0.1$, $T_c/T_{c0} \geq 0.9$, $l_1/10 \leq l \leq l_1$, and $\epsilon_0 < \frac{1}{4}$ are satisfied. In this case, the accuracy of the fit is not worse than about 6%. When any of the above conditions is not satisfied, the fit is accurate to about 2%.

VI. PENETRATION DEPTHS

We add here a few observations about some limiting values of the electromagnetic penetration depth λ in superconductors with magnetic impurities. These limiting values[16] are the London penetration depth λ_L, the extreme anomalous penetration depth λ_a, and the local-limit penetration depth λ_l. The size of λ approaches these limiting values in appropriate limits

$$\lambda \cong \lambda_L \text{ for } \lambda \gg \xi , \tag{38}$$

$$\lambda \cong \lambda_a \text{ for } \lambda \ll \xi , \tag{39}$$

and

$$\lambda \cong \lambda_l \text{ for } l \ll \xi_0 . \tag{40}$$

Like ξ, the London penetration depth λ_L can be calculated from the values of $\tilde{\Delta}_n$ and u_n, since

$$\lambda_L = [\lim_{q\to 0} K(q)]^{-1/2} . \tag{41}$$

By using Eq. (15), one can express λ_L in terms of λ_{L0}, the value of λ_L in the limit $T\to 0$, $l\to\infty$, and $\alpha\to 0$. The result is

$$\frac{\lambda_L^2}{\lambda_{L0}^2} = \left(2\pi T \sum_{n\geq 0} \frac{1}{\tilde{\Delta}_n (1+u_n^2)^{3/2}}\right)^{-1} . \tag{42}$$

The local-limit penetration depth λ_l has already been calculated[19]; it is given by

$$\frac{\lambda_l}{\lambda_{l0}} = \left[\lim_{T\to 0}\left(T\sum_{n\geq 0} \frac{1}{1+u_n^2}\right)\bigg/ T\sum_{n\geq 0}\frac{1}{1+u_n^2}\right]^{1/2} , \tag{43}$$

where λ_{l0} is the value of λ_l as $T\to 0$.

Equations (18), (42), and (43) can be used to express λ_L in terms of λ_l and ξ:

$$\lambda_L^2/\lambda_{L0}^2 = (\xi_{00}/\xi)\lambda_l^2/\lambda_{l0}^2 . \tag{44}$$

Similarly, we find that the extreme anomalous penetration depth λ_a is given by the relation

$$\lambda_a^3/\lambda_{a0}^3 = \lambda_l^2/\lambda_{l0}^2 , \tag{45}$$

where λ_{a0} is the value of λ_a in the limit $T\to 0$.

Equations (44) and (45) enable one to calculate λ_L and λ_a in terms of the already calculated temperature dependences of ξ and λ_l. The following equation follows from Eqs. (44) and (45):

$$\lambda_a^3/\lambda_{a0}^3 = \xi\lambda_L^2/\xi_{00}\lambda_{L0}^2 . \tag{46}$$

Equations (45) and (46) agree with results obtained previously for superconductors without magnetic impurities.[16]

Since λ_l and λ_a are independent of ξ, the penetration depth depends appreciably on ξ only when l and λ are about as large as ξ or larger.

Tables of our results may be obtained from one of the authors (DMG).

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